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THE RELATION OF THE CLASSICAL ORTHOGONAL POLYNOMIALS TO THE POLYNOMIALS OF APPELL.

By J. Shohat.

Introduction. The classical orthogonal polynomials ($\equiv COP$) of Jacobi (J), Laguerre (L) and Hermite (H) all satisfy a differential equation of the following type:[1]

$$A(x)y'' + B(x)y' + C_n y = 0,$$

(J) in $(0,1)$:
$$x^{\alpha-1}(1-x)^{\beta-1}T_n'' + [\alpha - (\alpha+\beta)x]T_n' + n(n+\alpha+\beta-1)T_n = 0,$$
$$T_n \equiv T_n(x;\alpha,\beta),\quad \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}T_mT_ndx = 0;$$

(L)
$$xL_n'' + (\alpha - x)L_n' + nL_n = 0,$$
$$L_n \equiv L_n(x;\alpha),\quad \int_0^\infty x^{\alpha-1}e^{-x}L_mL_ndx = 0;$$

(H)
$$H_n'' - 2xH_n' + 2nH_n = 0,\quad H_n \equiv H_n(x),$$
$$\int_{-\infty}^{\infty} e^{-x^2}H_mH_ndx = 0;$$
$$(\alpha,\beta > 0;\ m,n = 0,1,\cdots;\ m \neq n).$$

They all enjoy the property that their derivatives again form orthogonal systems of polynomials, with the new weight-function

$$(2)\qquad p_1(x) = A(x)p(x);\ p(x) = x^{\alpha-1}(1-x)^{\beta-1}(J),\ x^{\alpha-1}e^{-x}(L),\ e^{-x^2}(H).$$

Thus, taking the coefficient of x^n equal to unity and using for orthogonal polynomials ($\equiv OP$) with the weight-function $p(x)$ the notation

$$(3)\qquad \Phi_n(x;p) \equiv \Phi_n(x) = x^n - S_nx^{n-1} + d_{n,n-2}x^{n-2} + \cdots \qquad (n = 0,1,\cdots),$$

we have for COP

$$(4)\qquad \Phi'_n(x;p) = n\Phi_{n-1}(x;Ap) \qquad (n = 0,1,\cdots).$$

For Hermite polynomials (4) takes the simplest form

$$(5)\qquad \Phi'_n(x) = n\Phi_{n-1}(x) \qquad (n = 0,1,\cdots).$$

[1] The notations here employed are the same as those in my work: "Théorie générale des polynomes orthogonaux de Tchebichef," *Mémorial des Sciences Mathématiques*, Fasc. 66. The reader is referred to this work for further details concerning theorems and formulae.

(4, 5) naturally lead to the study of the relation between *COP* and Appell[2] polynomials ($\equiv AP$) for which the general definition is

$$A'_n(x) = nA_{n-1}(x) \tag{6}$$

($A_n(x)$, a polynomial of degree $n = 0, 1, \cdots$; thus $A_n(x) = x^n + \cdots$).

This study is the main object of the present paper. It is based upon the difference equation *characteristic for OP* (see Theorem I below):

$$\Phi_n(x) = (x - c_n)\Phi_{n-1}(x) - \lambda_n\Phi_{n-2}(x) \tag{7}$$

$$(n \geqq 2;\ \Phi_0 = 1,\ \Phi_1(x) = x - c_1;\ c_n,\ \lambda_n\text{-const.}).$$

In this connection we give some interesting properties of the constants c_n, λ_n in (7), also some limitations of the zeros of $\Phi_n(x)$. We conclude with a new simple proof that Hermite polynomials form the only sequence of polynomials which are at the same time *AP* and *OP*.

I. *The difference equation for OP.*

THEOREM I. *A necessary and sufficient condition that the sequence of polynomials*

$$\Phi_0(x) = 1,\ \ \Phi_1(x) = x - c_1,\ \ \Phi_n(x) = x^n - S_nx^{n-1} + d_{n,n-2}x^{n-2} + \cdots \quad (n = 2, 3, \cdots)$$

form a sequence of OP is that they satisfy a difference equation of the form

$$\Phi_n(x) = (x - c_n)\Phi_{n-1}(x) - \lambda_n\Phi_{n-2}(x) \tag{7}$$

$$(n \geqq 2;\ c_n, \lambda_n \text{ are constants})$$

with positive λ_n.

Proof. The necessity of (7) is well known. Its sufficiency, in view of the importance of the theorem, will be proved in extenso as follows.[3] (7) implies that $\{\Phi_n\}$ are denominators of the successive convergents $\Omega_n(x)/\Phi_n(x)$ $(n = 0, 1, \cdots)$ of the continued fraction

$$F(x) \equiv \frac{\lambda_1/}{/x - c_1} - \frac{\lambda_2/}{/x - c_2} - \cdots - \frac{\lambda_n/}{/x - c_n} - \cdots \tag{8}$$

$$(\lambda_1 > 0, \text{ arbitrary}),$$

whence the characteristic property:

---

[2] Appell, "Sur une classe de polynomes," *Ann. Ec. Norm.*, 2, vol. 9 (1880).

[3] We have been in possession of this proof for several years. Recently J. Favard published an identical proof in the *Comptes Rendus* ("Sur les polynomes de Tchebicheff," *Comptes Rendus*, vol. 200 (1935)). Cf. also, O. Perron, *Die Lehre von den Kettenbrüchen*, 2 ed., 377 ff., and J. Sherman, "On the numerators of the convergents of the Stieltjes continued fractions," *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 64-87.

$$(9) \qquad F(x) - \frac{\Omega_n(x)}{\Phi_n(x)} = \frac{\alpha'}{x^{2n+1}} + \cdots \equiv \left(\frac{1}{x^{2n+1}}\right) \qquad (\alpha', \cdots \text{ constants}).$$

It follows that

$$(9 \text{ bis}) \qquad \begin{aligned} \frac{\Omega_n(x)}{\Phi_n(x)} &= \frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \cdots + \frac{\alpha_{2n}}{x^{2n}} + \frac{\beta}{x^{2n+1}} + \cdots \\ & \qquad\qquad\qquad\qquad (\alpha_0 = \lambda_1), \\ \frac{\Omega_{n+1}(x)}{\Phi_{n+1}(x)} &= \frac{\alpha_0}{x} + \cdots + \frac{\alpha_{2n}}{x^{2n}} + \frac{\alpha_{2n+1}}{x^{2n+1}} + \frac{\alpha_{2n+2}}{x^{2n+2}} + \frac{\gamma}{x^{2n+3}} + \cdots, \end{aligned}$$

and the formal expansion of $F(x)$ in ascending powers of $1/x$ is

$$F(x) = \frac{\alpha_0}{x} + \frac{\alpha_1}{x^2} + \cdots + \frac{\alpha_n}{x^n} + \cdots \qquad (\alpha_0 = \lambda_1 > 0).$$

The theory of continued fractions enables us to conclude from the existence of convergents for (8) of *all* ranks $n = 0, 1, \cdots$, that all determinants

$$(10) \qquad \Delta_n = \begin{vmatrix} \alpha_0 & \alpha_1 \cdots \alpha_{n-1} \\ \alpha_1 & \alpha_2 \cdots \alpha_n \\ \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \\ \alpha_{n-1} & \alpha_n \cdots \alpha_{2n-2} \end{vmatrix} \neq 0 \quad (n = 0, 1, \cdots; \ \Delta_0 = 1; \ \Delta_1 = \alpha_0).$$

Moreover,

$$(11) \qquad \lambda_n = \frac{\Delta_{n-2}\Delta_n}{\Delta^2_{n-1}} \qquad (n \geqq 2; \ \lambda_1 = \alpha_0 = \Delta_1).$$

Since, by hypothesis, all $\lambda_n > 0$ $(n \geqq 2)$ and λ_1 $(\equiv \Delta_1)$ is also chosen positive, we see at once that *all* Δ_n *are positive* $(n = 0, 1, \cdots)$. But this is Hamburger's[4] condition for the existence of (at least) one monotonic bounded function $\psi(x)$, non-decreasing in $(-\infty, \infty)$, a solution of the Moments Problem

$$(12) \qquad \alpha_n \ (\equiv \text{moment of order } n) = \int_{-\infty}^{\infty} x^n d\psi(x) \qquad (n = 0, 1, 2, \cdots).$$

(9), rewritten as

$$(13) \qquad \begin{gathered} \Phi_n(x)F(x) = \Omega_n(x) + (1/x^{n+1}) \\ (\Phi_n(x) = x^n + f_1 x^{n-1} + \cdots + f_n), \end{gathered}$$

leads to the system of equations

$$f_n\alpha_i + f_{n-1}\alpha_{i+1} + \cdots + f_1\alpha_{i+n-1} + \alpha_{i+n} = 0 \qquad (i = 0, 1, \cdots, n-1),$$

which, by virtue of (12), can be written in integral form as

[4] H. Hamburger, "Ueber eine Erweiterung des Stieltjesschen Momentenproblem, I, II, III," *Mathematische Annalen*, vols. 81, 82 (1920-1921), pp. 235-319, 120-164, 168-187.

$$(14) \qquad \int_{-\infty}^{\infty} \Phi_n(x) x^i d\psi(x) = 0 \qquad (0 \leqq i < n = 1, 2, \cdots),$$

which is one of the expressions of the orthogonality property of $\{\Phi_n(x)\}$.

Remarks. (i) The interval $(-\infty, \infty)$ in (14) may be reduced to a subinterval (a, b), if $\psi(x)$ remains constant outside (a, b). The "true" interval of orthogonality is given by the limiting values (finite or infinite, but always existing) of the two extreme zeros of $\Phi_n(x)$, as $n \to \infty$. Hereafter, "interval of orthogonality" means the "true" interval. (ii) The Moments Problem (12) may have more than one solution. All such solutions evidently generate the same sequence $\{\Phi_n(x)\}$ of *OP*, so that from this standpoint they are all equivalent. We shall write

$$(15) \qquad \Phi_n(x; d\psi) \equiv \Phi_n(x) \equiv \Phi_n.$$

2. *Symmetric OP.* By this we mean $\{\Phi_n(x)\}$ with the property

$$(16) \qquad \Phi_n(-x) \equiv (-1)^n \Phi_n(x) \qquad (n = 1, 2, \cdots).$$

(Legendre or Hermite polynomials are illustrations.) We have the following obviously necessary and sufficient condition that the sequence $\{\Phi_n(x)\}$ be symmetric, namely: *all* c_n $(n \geqq 2)$ *in the recurrence relation* (7), *also* c_1, *vanish*, and this is equivalent to the vanishing of S_n— sum of the zeros of $\{\Phi_n(x)\}$ $(n = 1, 2, \cdots)$. In fact,

$$(17) \qquad S_n = c_1 + c_2 + \cdots + c_n \qquad (S_n - S_{n-1} = c_n).$$

The following remark will be used later: if in (7)

$$(18) \qquad c_1 = c_2 = \cdots = c_n = \cdots = c,$$

then $\{\Phi_n(x)\}$ is reducible to a symmetric sequence by the linear substitution $(x - c) | x$. It is of interest to interpret the symmetric property (16) in terms of the moments $\{\alpha_n\}$. This is given in

Theorem II. *The sequence* $\{\Phi_n\}$ *is symmetric if and only if all odd moments* $\alpha_1, \alpha_3, \cdots$ *vanish. It is generated by* $\psi(x)$ *with the property:* $\psi(-x) \equiv -\psi(x)$ *in* $(-\infty, \infty)$.

Proof. Assuming (16), we conclude that the continued fraction (8), for which Ω_n/Φ_n is the $(n+1)$-st convergent, takes the form

$$\frac{\lambda_1/}{/x} - \frac{\lambda_2/}{/x} - \cdots - \frac{\lambda_n/}{/x} - \cdots,$$

so that $\Omega_n(x)$, of degree $n - 1$ (satisfying the same recurrence relation (7), with $\Omega_0 = 0$, $\Omega_1 = \lambda_1$) is also symmetric. Hence,

$$\frac{\Omega_{2n}(x)}{\Phi_{2n}(x)}=\frac{xP_{n-1}(x^2)}{Q_n(x^2)},\qquad \frac{\Omega_{2n+1}(x)}{\Phi_{2n+1}(x)}=\frac{P_n(x^2)}{xS_n(x^2)}\qquad (n=1,2,\cdots),$$

(P_i, Q_i, S_i are polynomials of degree i),

and this, by virtue of (9, 9 bis, 12), shows that

$$(19)\qquad \alpha_{2n-1}=\int_{-\infty}^{\infty}x^{2n-1}d\psi(x)=0\qquad (n=1,2,\cdots).$$

Moreover,

$$(20)\qquad \alpha_{2n}=\int_{-\infty}^{\infty}x^{2n}d\psi(x)=-\int_{-\infty}^{\infty}x^{2n}d\psi(-x);$$

$$\alpha_{2n+1}=\int_{-\infty}^{\infty}x^{2n+1}d\psi(x)=\int_{-\infty}^{\infty}x^{2n+1}d\psi(-x).$$

Now consider the function

$$\phi(x)\equiv\frac{\psi(x)-\psi(-x)}{2},$$

which is also monotonic non-decreasing in $(-\infty,\infty)$. From (20),

$$\int_{-\infty}^{\infty}x^{2n}d\phi(x)=\int_{-\infty}^{\infty}x^{2n}d\psi(x),\quad \int_{-\infty}^{\infty}x^{2n+1}d\phi(x)=0\qquad (n=0,1,\cdots).$$

It follows from (19), that $\phi(x)$ has the same moments as $\psi(x)$. Thus $\{\Phi_n(x)\}$ may be considered as $\{\Phi_n(x;d\phi)\}$, where evidently

$$(21)\qquad \phi(-x)\equiv-\phi(x)\ \text{ in }\ (-\infty,\infty),$$

and the necessity of the condition stated is established. That it is also sufficient, we prove as follows: If all $\alpha_{2n+1}=0$ $(n=0,1,\cdots)$, we conclude as above that $\Phi_n(x)\equiv\Phi_n(x;d\psi)$ with $\psi(-x)\equiv-\psi(x)$. The orthogonality property

$$\int_{-\infty}^{\infty}\Phi_n(x)G_{n-1}(x)d\psi(x)=0\qquad (n=1,2,\cdots),$$

where $G_s(x)$ generally stands for an *arbitrary* polynomial of degree $\leqq s$, gives now

$$\int_{-\infty}^{\infty}\Phi_n(-x)G_{n-1}(-x)d\psi(x)=0,$$

and this shows (in view of the uniqueness, to within constant factors, of the sequence $\{\Phi_n(x)\}$) that

$$\Phi_n(-x)\equiv\text{Const.}\cdot\Phi_n(x)\equiv(-1)^n\Phi_n(x).\text{[5]}$$

[5] Another proof of Theorem II, based on the determinant-representation of $\Phi_n(x)$ was communicated to me by Mr. G. Milgram.

3. *Some properties of the constants c_n, λ_n.* It is interesting to note that, given an arbitrary sequence of real constants $\{c_n, \lambda_n\}$ $(n = 1, 2, \cdots)$ subject to the single limitation $\lambda_n > 0$ for all n, we get, by means of (7), a sequence of *OP*. Thus we get a new approach to the theory of *OP* through the study of the sequences $\{c_n\}$, $\{\lambda_n\}$. This is clearly shown in the discussion which follows.

THEOREM III. (i) *The interval of orthogonality for $\{\Phi_n(x)\}$ is finite if and only if both sequences $\{c_n\}$, $\{\lambda_n\}$ are bounded; in other words, if any one of these sequences is unbounded, the interval of orthogonality is infinite.* (ii) *Both sequences are unbounded, if the orthogonality interval is* $(0, \infty)$. (iii) *If only one of the sequences $\{c_n\}$, $\{\lambda_n\}$ is bounded, the orthogonality interval is then* $(-\infty, \infty)$. [(a, ∞) or $(-\infty, a)$, a finite, can be reduced to $(0, \infty)$ by a linear transformation].

Proof. (i) Necessity follows from the inequalities

$$a < c_n < b; \ \lambda_n < (b-a)^2/4,$$

and sufficiency—from the fact that the zeros of the sequence $\{\Phi_n(x)\}$ all lie within finite limits. We prove the latter statement by making very simple use of the recurrence relation (7) as follows. We have

$$\Phi_1(x) = x - c_1 > 0, \text{ for } x > c_1,$$

$$\frac{\Phi_2(x)}{\Phi_1(x)} = x - c_2 - \frac{\lambda_2}{x - c_1} \geqq k_2 > 0,$$

$$\text{for } x \geqq X_2 = \frac{c_1 + c_2 + k_2 + \sqrt{(c_1 + c_2 + k_2)^2 - 4(c_1 k_2 + c_1 c_2 - \lambda_2)}}{2},$$

$$\text{and } x > c_1,$$

$$\frac{\Phi_3(x)}{\Phi_2(x)} = x - c_3 - \frac{\lambda_3}{\phi_2/\phi_1} > x - c_3 - \frac{\lambda_3}{k_2} \geqq k_3 > 0,$$

$$\text{for } x \geqq c_3 + \frac{\lambda_3}{k_2} + k_3,$$

$$\text{and } x > c_1,\ x \geqq X_2,$$

. . . .

$$\frac{\Phi_n(x)}{\Phi_{n-1}(x)} \geqq k_n > 0, \text{ for } x > \max\left(c_1, X_2, c_i + \frac{\lambda_i}{k_{i-1}} + k_i\right), \quad (i = 2, 3, \cdots, n).$$

Similarly we treat the values of x for which

$$\Phi_1(x) < 0, \frac{\Phi_2(x)}{\Phi_1(x)} < 0, \cdots, \frac{\Phi_n(x)}{\Phi_{n-1}(x)} < 0.$$

We thus conclude that *the zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ of $\Phi_n(x)$ lie in the following interval:*

$$\left(\min\left[c_1, X_1, c_i + k'_i + \frac{\lambda_i}{k'_{i-1}}\right], \max\left[c_1, X_2, c_i + k_i + \frac{\lambda_i}{k_{i-1}}\right]\right)$$

$$\text{(22)} \qquad k_i > 0,\ k'_i < 0, \text{ arbitrary};\ i = 2, 3, \cdots n$$

$$X_1 = \frac{c_1 + c_2 + k'_2 - \sqrt{(c_1 + c_2 + k_2)^2 - 4(c_1 k'_2 + c_1 c_2 - \lambda_2)}}{2} \equiv X_1(c_1, c_2, k'_2);$$

$$X_2 = \frac{c_1 + c_2 + k_2 + \sqrt{\cdots}}{2} \equiv X_2(c_1, c_2, k_2).$$

Taking

$$-k'_i = k_i = \sqrt{\lambda_{i+1}}\,(i = 2, 3, \cdots, n-1), -k'_n = k_n = \sqrt{\lambda_n},$$

we conclude that

$$\begin{aligned} &\min(c_1, X_1, c_i - \sqrt{\lambda_i} - \sqrt{\lambda_{i+1}}, c_n - 2\sqrt{\lambda_n}) < x_{1,n} < x_{n,n} \\ \text{(23)} \qquad &< \max(c_1, X_2, c_i + \sqrt{\lambda_i} + \sqrt{\lambda_{i+1}}, c_n + 2\sqrt{\lambda_n}) \\ &\qquad (i = 2, 3, \cdots, n-1) \\ &X_1 = X_1(c_1, c_2, -\sqrt{\lambda_3}), X_2 = X_2(c_1, c_2, \sqrt{\lambda_3}) \text{ as in (22).} \end{aligned}$$

In particular, *in the symmetric case,*

$$\text{(24)} \quad 0 < -x_{1,n} = x_{n,n} < \max\left(\frac{\sqrt{\lambda_3} + \sqrt{\lambda_3 + 4\lambda_2}}{2}, \sqrt{\lambda_i} + \sqrt{\lambda_{i+1}}, 2\sqrt{\lambda_n}\right) \quad (i = 2, 3, \cdots, n-1).$$

We thus obtained bounds for the zeros of $\Phi_n(x)$ *applicable to any sequence of OP.* Leaving aside the discussion of the best possible choice of the constants k_i, k'_i in (22), we clearly see that (23) proves (i).[6] (ii) We have in $(0, \infty)$

$$\lambda_n = b_{2n-2} b_{2n-1},\ c_n = b_{2n-1} + b_{2n} \qquad (n \geqq 2;\ \lambda_1 = b_1;\ c_1 = b_2),$$

where the positive sequence $\{b_n\}$ is unbounded.[7] (iii) This is an immediate consequence of (i, ii). In particular, *in the symmetric case corresponding to* $(-\infty, \infty)$, *the sequence* $\{\lambda_n\}$ *is unbounded.*

Remark. The preassignment of the sequence $\{\lambda_n\}$ determines all Δ_n (see (10, 11)).

4. *Construction of the corresponding sequence of AP.* The differential equations (I, J, L) determine polynomials of degree $n = 0, 1, 2, \cdots$,

---

[6] Cf. our *Mémorial Fasc.*, p. 41, where the same method, applied to the symmetric case only, leads to

$$-x_{1,n} = x_{n,n} < \max(2\sqrt{\lambda_i}) \qquad (i = 1, 2, \cdots, n).$$

(an obvious misprint gives $x_{1,n} > \cdots$). Cf. also: O. Bottema, "Die Nullstellen gewisser durch Recursionformeln definierten Polynome," *Amsterdam Acad. Sc., Proc. Sect. Sc., V*, vol. 34 (1931), pp. 681-691, where the sufficiency of (i) (bounds for the zeros of $\Phi_n(x)$) is established by means of the theory of quadratic forms, also a proof of a part of (iii) is given, different from the one below.

[7] Stieltjes, "Recherches sur les fractions continues," *Oeuvres*, vol. 2, pp. 402-566.

$$
(25)\quad
\begin{aligned}
(J): J_n \equiv J_n(x;\alpha,\beta) &= x^n - \frac{n(n+\alpha-1)}{2n+\alpha+\beta-2}\,x^{n-1} \\
&\quad + \frac{n(n-1)(n+\alpha-1)(n+\alpha-2)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta-3)}\,x^{n-2} \mp \cdots \\
(L): l_n \equiv l_n(x;\alpha) &= x^n - \frac{n(n+\alpha-1)}{1!}\,x^{n-1} \\
&\quad + \frac{n(n-1)(n+\alpha-1)(n+\alpha-2)}{2!}\,x^{n-2} + \cdots
\end{aligned}
$$

for all α, β, unless a coefficient in $(25, J)$ has a vanishing denominator, while its numerator is different from 0 (if both vanish, we take this coefficient as $=0$). Furthermore,

$$
(26)\qquad J_n(x;\alpha,\beta) \equiv T_n(x;\alpha,\beta)\,;\; l_n(x;\alpha) \equiv L_n(x;\alpha) \qquad (\alpha,\beta > 0).
$$

$$
(27)\qquad J'_n(x;\alpha,\beta) \equiv nJ_{n-1}(x;\alpha+1,\beta+1)\,;\; l'_n(x;\alpha) \equiv nl_{n-1}(x;\alpha+1) \qquad (\text{for all } \alpha,\beta).
$$

Assuming again $\alpha, \beta > 0$, we start with a certain $J_n(x;\alpha,\beta)$ or $l_n(x;\alpha)$ and construct correspondingly infinite sequences of polynomials

$$
(28)\quad
\begin{aligned}
&(J): J_0, J_1(x;\alpha+n-1,\beta+n-1),\cdots, J_n(x;\alpha,\beta), \\
&\qquad J_{n+1}(x;\alpha-1,\beta-1),\cdots \\
&(L): l_0, l_1(x;\alpha+n-1),\cdots, l_n(x;\alpha), l_{n+1}(x;\alpha-1),\cdots
\end{aligned}
$$

of degrees $0, 1, 2, \cdots$. It is readily seen, from (25), that they are all well determined. *By virtue of* (27), *each sequence* (28) *is an AP sequence.* For Hermite polynomials we have an *AP* sequence without any additional construction [see (5)]

$$
(28^{\text{bis}})\qquad H_0, H_1(x),\cdots, H_n(x), H_{n+1}(x),\cdots.
$$

We now proceed to find the *generating functions* for (28). We recall from the theory of *AP* that the formal power series expansion

$$
(29)\qquad a(h) = \gamma_0 + \gamma_1\frac{h}{1!} + \cdots + \gamma_n\frac{h^n}{n!} + \cdots
$$

gives rise to an *AP* sequence $\{A_n(x)\}$ through the expansion

$$
(30)\qquad a(h)e^{hx} \equiv A_0 + A_1(x)\frac{h}{1!} + \cdots + A_n(x)\frac{h^n}{n!} + \cdots.
$$

Conversely, given a sequence of constants $\gamma_0(\neq 0), \gamma_1, \cdots, \gamma_n, \cdots$, constructs

$$
(31)\qquad A_n(x) = \gamma_0 x^n + \binom{n}{1}\gamma_1 x^{n-1} + \binom{n}{2}\gamma_2 x^{n-2} + \cdots + \gamma_n \qquad (n = 0, 1, 2, \cdots).
$$

The sequence $\{A_n(x)\}$ is then an AP sequence, and $a(h)$, defined by the expansion (29), (assumed to converge for $|h|$ sufficiently small), is said to be its generating function. We may write (31) symbolically (this will be designated by $\simeq$) as

$$(32) \qquad A_n(x) \simeq (x+\bar{\gamma})^n \qquad (n=1,2,\cdots;\ A_0=\gamma_0)$$

where $\bar{\gamma}$ means that we agree to replace each γ^i by γ_i, $(i=0,1,\cdots,n)$.

Regarding the sequences (28), we read at once from (25)

$$(J):\ \gamma_0=1,\ \gamma_1=-1!\cdot\frac{n+\alpha-1}{2n+\alpha+\beta-2}$$

$$\gamma_2=+2!\,\frac{n(n-1)(n+\alpha-1)(n+\alpha-2)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta-3)},\cdots$$

$$(L):\ \gamma_0=1,\ \gamma_1=-(n+\alpha-1),\ \gamma_2=+(n+\alpha-1)(n+\alpha-2),\cdots$$

whence,

$$(33) \qquad \begin{gathered}(J):\ a(h)=G(-n-\alpha+1,-2n-\alpha-\beta+2,-h)\\ \left(G(\alpha,\beta,x)\equiv 1+\frac{\alpha}{1\cdot\beta}x+\frac{\alpha(\alpha+1)}{1\cdot 2\beta(\beta+1)}x^2+\cdots\right)\\ (L):\ a(h)=(1-h)^{n+\alpha-1}.\end{gathered}$$ [8]

A direct application of (33) is the expression of x^n in terms of the polynomials (25). By the general theory of AP, if the sequence $\{B_n(x)\}$ is generated by $1/a(h)\equiv\beta_0+\beta_1h/1!+\beta_2h^2/2!+\cdots$, then

$$(34) \qquad x^n\simeq(\bar{A}+\bar{\beta})^n \quad (\simeq(AB)_n\equiv(BA)_n),\ \text{where } B_n(x)\simeq(x+\bar{\beta})^n$$

[the symbolical expressions $(\bar{A}+\bar{\beta})^n$, $(x+\bar{\beta})^n$ being interpreted as $(x+\bar{\gamma})^n$ is in (32)]. The application to Laguerre polynomials is particularly simple, since here $1/a(h)=(1-h)^{-(n+\alpha-1)}$. Thus,

$$(35) \qquad x^n=\sum_{i=0}^{n}(-1)^i\cdot i!\cdot\binom{-(n+\alpha-1)}{i}L_{n-i}(x;\alpha+i).$$

Consider (34) for the special case

$$(36) \qquad 1/a(h)\equiv a(-h).$$

A glance at (29, 31) shows that here

$$\beta_i=(-1)^i\gamma_i\ (i=0,1,\cdots);\qquad B_n(x)\simeq(x-\bar{\gamma})^n,$$

and the following very simple reciprocal relation results

$$(37) \qquad A_n(x)\simeq(x+\bar{\gamma})^n,\qquad x^n\simeq(\bar{A}-\bar{\gamma})^n \qquad (n=1,2,\cdots).$$

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[8] Cf. Appell, *loc. cit.*, where (33, L) is derived, without, however, indicating its relation to Laguerre polynomials.

This is the case of Hermite polynomials.[9]

5. *The sequences* (25) *and orthogonality.* In (25) each term, by virtue of (26), belongs to a *distinct* sequence of *OP*, as long as the parameters $\alpha \pm \nu$, $\beta \pm \nu$ remain positive. We ask now: can $(25, J)$ and or $(25, L)$, *taken as a whole*, form a sequence of *OP*? That this is conceivable follows from the fact that if we take in (7) c_i, λ_i $(i = 1, 2, \cdots, n)$ from one "permissible" sequence (i. e. all $\lambda_n > 0$), and c_{n+1}, $\lambda_{n+1}, \cdots$ from another such sequence, the first n *OP* thus derived belong to *two* sequences of *OP*.

The answer to our question is based upon Theorem I and does not utilize the properties of the zeros of the polynomials under discussion, which will be denoted by

$$A_0, A_1, \cdots, A_n, \cdots \qquad (A_n = x^n + \cdots).$$

Making use of (6), rewrite the differential equation (1) as

$$A_n = -\frac{nB}{C_n} A_{n-1} - \frac{n(n-1)A}{C_n} A_{n-2}. \tag{38}$$

$$((J): B \equiv B(\alpha, \beta), C_n \equiv C_n(\alpha, \beta); \quad (L): B \equiv B(\alpha), C_n \equiv C_n(\alpha)).$$

If $\{A_n\}$ is a sequence of *OP*, then

$$A_n = (x - c_n) A_{n-1} - \lambda_n A_{n-2} \qquad (c_n, \lambda_n \text{ are constants}),$$

whence,

$$\left(x - c_n + \frac{nB}{C_n}\right) A_{n-1} = \left(\lambda_n - \frac{n(n-1)A}{C_n}\right) A_{n-2}.$$

But two successive *OP*'s cannot have common zeros; hence, necessarily,

$$x - c_n + \frac{nB}{C_n} \equiv 0; \quad \lambda_n - \frac{n(n-1)A}{C_n} \equiv 0, \quad \text{i. e.} \quad A \equiv \text{const.},$$

which holds for Hermite polynomials only.

We indicate in passing some relations for A, B, C in (1). Differentiating (37) and using (6) once more, we have

$$A_{n-1}(C_n + B') + (n-1) A_{n-1}(A' + B) + (n-1) A A_{n-3} = 0.$$

On the other hand, by (38), changing n into $n-1$ and *increasing* α, β *by* 1, (which is indicated by writing $\bar{B}$, $\bar{C}_{n-1}$), we have

[9] We have:

$$H_n \equiv H_n(x; e^{-x^2/4}) = x^n - \frac{n(n-1)}{1!} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} x^{n-4} \mp \cdots; \quad a(h) = e^{-h^2},$$

so that

$$x^n = H_n + \frac{n(n-1)}{1!} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} x^{n-2} + \cdots.$$

$$A_{n-1}\bar{C}_{n-1} + (n-1)A_{n-2}\bar{B} + (n-1)(n-2)AA_{n-3} = 0.$$

Hence, the following relations must be satisfied for all COP:

$$C_n + B' - \bar{C}_{n-1} = 0, \qquad A' + B - \bar{B} = 0.$$

We see once more, that for Hermite polynomials, where no parameters are involved, so that $\bar{B} \equiv B$, we must have $A' = 0$, $A \equiv$ const.

6. *The sequence* $\{H_n(x)\}$ *as an AP sequence.* We close our discussion by proving a very general

THEOREM IV. *The only system of* $OP\{\Phi_n(x; d\psi)\}$ *which is at the same time an AP sequence is that with* $d\psi(x) = e^{-h^2(x-c)^2}$ *(h, c-const.), i. e. that which is reducible to Hermite polynomials by a linear transformation.*[10]

Proof. Assume that

$$\Phi'_n(x; d\psi) \equiv n\Phi_{n-1}(x; d\psi) \qquad (n = 1, 2, \cdots). \tag{39}$$

Combining (39) with the relation obtained by differentiating (7), we get, with the notations (3),

$$\frac{S_n}{n} = \frac{S_{n-1}}{n-1}\,; \qquad n\lambda_{n-1} - (n-2)\lambda_n = -\frac{2d_{n-1,n-3}}{n-1}. \tag{40}$$

Combining this with

$$c_n = S_n - S_{n-1} \qquad \text{(see (17))}$$

leads to the fundamental result:

$$c_n = \frac{S_n}{n} = \frac{S_{n-1}}{n-1} = c_{n-1}, \quad \text{i. e.} \quad c_n = c_{n-1} = \cdots = c_1 = \text{constant } c.$$

By the remark made above, *the linear substitution* $(x-c)\,|\,x$ *reduces the system* $\{\Phi_n(x; d\psi)\}$ *under discussion to the symmetric case.*

Assuming this reduction to have been made and keeping the old notations, we have now

$$c_n = S_n = 0, \qquad \alpha_{2n-1} = 0 \qquad (n = 1, 2, \cdots). \tag{41}$$

[10] Meixner, "Orthogonale Polynome mit einer besonderen Gestalt der erzeugenden Funktion," *Journal of the London Mathematical Society*, vol. 9 (1934), pp. 6-13, derives a similar theorem for polynomials satisfying a more general relation than (6), by considerations different from those developed below. A different proof of Theorem IV is to be found in the Thesis of my pupil Dr. M. Webster. Our proof is a straightforward one, applicable to similar problems. Cf. also the interesting article of I. M. Sheffer, "A differential equation for Appell polynomials," *Bulletin of the American Mathematical Society*, vol. 41 (1935), pp. 914-923.

Making use of the expression of $\Phi_n(x)$ in terms of the moments α_n, we rewrite (39) as follows:

$$(42)\qquad \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-1} & \alpha_n & \cdots & \alpha_{2n-1} \\ 0 & 1 & x \cdots & nx^{n-1} \end{vmatrix} : \Delta_n = n \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-2} & & \cdots & \alpha_{2n-3} \\ 1 & x & \cdots & x^{n-1} \end{vmatrix} : \Delta_{n-1}$$

$$(n = 2, 3, \cdots).$$

Writing (42) for $n = 3$ and making use of (41, 11), we get successively

$$(43)\qquad \alpha_4 = 3\alpha_0\gamma^2, \quad \alpha_6 = 15\alpha_0\gamma^3, \quad \text{where} \quad \alpha_2/\alpha_0 = \gamma > 0,$$

$$(44)\qquad \Delta_2 = \alpha_0^2\gamma, \quad \Delta_3 = 2\alpha_0^3\gamma^3, \quad \Delta_4 = 24\alpha_0^4\gamma^6,$$

$$(45)\qquad \lambda_2 = \gamma, \quad \lambda_3 = 2\gamma, \quad \lambda_4 = 3\gamma.$$

It remains to prove that

$$(46)\qquad \lambda_n = (n-1)\gamma \quad \textit{for all } n.$$

Here we use mathematical induction in the following manner: The recurrence relation (7) shows in the symmetric case, by comparing coefficients, that

$$d_{n,n-2} = -(\lambda_2 + \lambda_3 + \cdots + \lambda_n),$$

so that the second relation (40) gives, assuming that (46) holds up to λ_{n-1} inclusive,

$$n(n-2)\gamma - (n-2)\lambda_n = (n-2)\gamma; \qquad \lambda_n = (n-1)\gamma.$$

The induction is complete, and Theorem IV is established, for the sequence $\{c_n = 0,\ \lambda_n = (n-1)\gamma\}$ gives rise to the system of Hermite polynomials $H_n(x; e^{-x^2/\gamma})$.

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ANALYTIC FUNCTIONS STAR-LIKE IN ONE DIRECTION.

By M. S. Robertson.[1]

Introduction. Let (S) denote the class of functions $f(z)$ which have one or the other of the following sequences of properties:

Either (A): 1. $f(z) = z + \sum_2^\infty a_n z^n$ is regular for $|z| < 1$.

2. There exists a positive $\delta = \delta(f)$ so that for every r in the open interval $1 - \delta < r < 1$ $f(z)$ maps $|z| = r$ into a contour C_r which is cut by the real axis in two, and not more than two, points.

or (B): 1. $f(z) = z + \sum_2^\infty a_n z^n$ is regular for $|z| \leqq 1$.

2. $f(z)$ maps $|z| = 1$ into a contour which is cut by the real axis in two, and not more than two, points.

If $f(z)$ is a member of (S) we shall say that it is star-like in the direction of the real axis with respect to the unit circle. A function $f(z)$ which is star-like in a direction other than that of the real axis can be reduced to a function of the type considered above by taking $e^{i\alpha} f(e^{-i\alpha} z)$ with a suitable choice for the real parameter α. If $f(z)$ is a member of (S) satisfying (A) 2 there will exist two, and only two points $z_1 = re^{i\theta_1(r;f)}$ and $z_2 = re^{i\theta_2(r;f)}$ at which the imaginary part of $f(z)$, or $If(z)$, is zero. Moreover, if

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$$

then

$$\begin{aligned} v(r, \theta) &> 0 \quad \text{when} \quad \theta_1(r) < \theta < \theta_2(r) \\ v(r, \theta) &< 0 \quad \text{when} \quad \theta_2(r) < \theta < \theta_1(r) + 2\pi. \end{aligned} \tag{1.1}$$

We may define $\theta_1(r) = \theta_1(r; f)$ and $\theta_2(r) = \theta_2(r; f)$ so that

$$0 \leqq \theta_1(r) \leqq 2\pi, \qquad 0 < \theta_2(r) - \theta_1(r) < 2\pi. \tag{1.2}$$

When $f(z)$ is real on the real axis (a_n real) then $f(z)$ belongs to the sub-class of (S), consisting of typically-real functions defined by W. Rogosinski.[2] In this case,

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[1] National Research Fellow.

[2] See W. Rogosinski, "Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen," *Mathematische Zeitschrift*, Band 35 (1932), pp. 93-121.

$$\theta_1(r) \equiv 0, \qquad \theta_2(r) \equiv \pi.$$

We shall show in this paper that if $f(z)$ is a member of (S) then the coefficients a_n of $f(z)$ satisfy the inequalities

$$| a_n | \leqq n^2 \quad \text{for all } n$$

and the equality is attained for any fixed n by only one function of class (S), namely $(z+\epsilon z^2)/(1-\epsilon z)^3$, $(\epsilon = \pm i)$. If further, $f(z)$ is an odd function, then

$$| a_n | \leqq n \quad \text{for all } n.$$

and the equality sign is attained by the function $(z+z^3)/(1-z^2)^2$. If $f(z)$ is not necessarily odd but is real on the real axis then

$$| a_n | \leqq n \quad \text{for all } n,$$

and the equality sign is atttained by $z/(1-z)^2$. We shall show also that if $f(z)$ belongs to (S) $\lim_{r\to 1} f(re^{i\theta})$ exists and is finite for almost all θ.

If $g(z) = z + \sum_2^\infty c_n z^n$ is regular and univalent for $| z | < 1$ then it has been conjectured [3] that $| c_n | \leqq n$. This is known to be true in case $g(z)$ is star-like in the unit circle, or when $R\sqrt{g(z)/z} \geqq \frac{1}{2}$ for $| z | < 1$, or when there exists a complex constant α so that $R\alpha z g'(z)/g(z) \geqq 0$ for $| z | < 1$, or, finally, when $g(z)$ is real on the real axis. We shall show here that the conjecture is also true when $g(z)$ has the property that it maps $| z | = r < 1$ for every r near 1 into a contour which is convex in one direction, i. e., every straight line parallel to this direction may cut the contour in not more than two points. The equality sign is attained for any fixed n for essentially only one function of this class,

$$z/(1-e^{i\alpha}z)^2, \quad (\alpha \text{ real}).$$

2. *A representation for functions of class* (S). Suppose $f(z)$ is a member of (S) satisfying A (2) and hence (1.1) and (1.2). Let

$$\phi(r) \equiv \frac{\pi}{2} - \frac{\theta_2(r)+\theta_1(r)}{2}, \qquad 1-\delta < r < 1,$$

$$F_r(z) \equiv f(re^{-i\phi(r)}z)\left[\frac{1}{z} - z + 2i\sin\left\{\frac{\pi}{2} - \frac{\theta_2(r)-\theta_1(r)}{2}\right\}\right]$$

If $$f(re^{-i\phi(r)}z) = P(r,\theta) + iQ(r,\theta), \qquad z = e^{i\theta},$$

[3] See L. Bieberbach, *Sitzber. kgl. Akad. Berlin* 1916, pp. 940-955.

then from (1.1)

$$(2.1)\quad \begin{aligned} &Q(r,\theta) > 0 \text{ when } \frac{\pi}{2} - \frac{\{\theta_2(r) - \theta_1(r)\}}{2} < \theta < \frac{\pi}{2} + \frac{\{\theta_2(r) - \theta_1(r)\}}{2} \\ &Q(r,\theta) < 0 \text{ when } \frac{\pi}{2} + \frac{\{\theta_2(r) - \theta_1(r)\}}{2} < \theta < \frac{5\pi}{2} - \frac{\{\theta_2(r) - \theta_1(r)\}}{2} \end{aligned}$$

$$F_r(e^{i\theta}) = \{P(r,\theta) + iQ(r,\theta)\}\left[-2i\sin\theta + 2i\sin\left(\frac{\pi}{2} - \frac{\theta_2(r) - \theta_1(r)}{2}\right)\right]$$

$$\begin{aligned} \Re F_r(e^{i\theta}) &= 2Q(r,\theta)\left[\sin\theta - \sin\left(\frac{\pi}{2} - \frac{\theta_2(r) - \theta_1(r)}{2}\right)\right] \\ &\geqq 0 \text{ for all } \theta \text{ by (2.1)}. \end{aligned}$$

Hence $\Re F_r(z) \geqq 0$ for $|z| = 1$; $F_r(z)$ is regular on $|z| = 1$, $F_r(0) \neq 0$; and since the minimum of a harmonic function occurs on the boundary we have

$$(2.2)\qquad \Re F_r(z) \geqq 0 \quad \text{for} \quad |z| \leqq 1.$$

Let $\{r_i\}$ be a sequence of values of r tending to 1 so that

$$(2.3)\qquad \lim_{r_i \to 1} \theta_1(r_i) = \alpha, \qquad \lim_{r_i \to 1} \theta_2(r_i) = \beta.$$

On account of (1.2) we have

$$0 \leqq \alpha \leqq 2\pi, \qquad 0 \leqq \beta - \alpha \leqq 2\pi.$$

Let $\mu = \dfrac{\alpha + \beta}{2} - 2\kappa\pi$ with κ so chosen that $0 \leqq \mu < 2\pi$. Let $\nu = \dfrac{\beta - \alpha}{2}$.

Then $0 \leqq \nu \leqq \pi$. Let

$$F_1(z) \equiv \lim_{r_i \to 1} F_{r_i}(z) = f(-ie^{i\mu}z) \cdot \left(\frac{1 + 2iz\cos\nu - z^2}{z}\right)$$

$F_1(z)$ is regular for $|z| < 1$, and by (2.2)

$$\Re F_1(z) \geqq 0 \text{ for } |z| < 1,\ F_1(0) = \sin\mu - i\cos\mu, \text{ whence } \sin\mu \geqq 0,$$

equality holding only when $F_1(z)$ reduces to either $+i$ or $-i$. In this case $f(z)$ has the form $z(1 - 2z\cos\nu + z^2)^{-1}$. In any case we have $0 \leqq \mu \leqq \pi$, $0 \leqq \nu \leqq \pi$. If $\sin\mu \neq 0$ we let

$$F(z) \equiv (\sin\mu)^{-1} \cdot [F_1(ie^{-i\mu}z) + i\cos\mu].$$

If $\sin\mu = 0$, we may take $F(z) \equiv 1$. In both cases $F(z)$ is regular for $|z| < 1$, $F(0) = 1$, $\Re F(z) > 0$ for $|z| < 1$, and

$$(2.4)\qquad f(z) = h_\nu(e^{-i\mu}z) \cdot (\cos\mu + i\sin\mu \cdot F(z))$$

where

$$h_\nu(z) \equiv z(1 - 2z\cos\nu + z^2)^{-1}, \qquad 0 \leqq \mu \leqq \pi.$$

Since $\Re F(z) \geqq 0$ the function $\{1 + F(z)\}^{-1}$ is bounded in the unit circle, and so tends to a limit, different from zero, for almost all values of θ as $r \to 1$ $(z = re^{i\theta})$. Hence $\lim_{r\to 1} F(re^{i\theta})$ exists, and is finite for almost all values of θ. Therefore $\lim_{r\to 1} f(re^{i\theta})$ exists, and is finite for almost all values of θ.

3. *Univalent functions convex in one direction.* Let

$$g(z) = z + \sum_{2}^{\infty} c_n z^n \tag{3.1}$$

be a member of a class $(\mathfrak{J})$ of functions regular for $|z| < 1$ which are univalent and convex in one direction. Without loss of generality we may assume that this is the direction of the imaginary axis. We suppose then that there is a positive $\delta = \delta(g)$ so that for every r in the open interval $1 - \delta < r < 1$ $g(z)$ maps $|z| = r$ into a contour C_r such that every straight line parallel to the imaginary axis cuts C_r in not more than two points. If $g(z)$ is regular on $|z| = 1$ we may take $\delta = 0$ in our definition. It is readily seen that a necessary and sufficient condition that $g(z)$ map each circle $|z| = r$ on contours C_r which are of the type described above is that for every r in the interval $1 - \delta < r < 1$ there should exist two real numbers $\theta_1(r)$ and $\theta_2(r)$ satisfying (1.2) so that $\Re g(re^{i\theta})$ is a monotone decreasing function of θ in the interval (θ_1, θ_2) and a monotone increasing function of θ in the complementary interval $(\theta_2, \theta_1 + 2\pi)$. But since

$$I\{zg'(z)\} = -\frac{\partial \Re g(re^{i\theta})}{\partial \theta}$$

it follows that $g(z)$ is a member of class $(\mathfrak{J})$ if, and only if, $zg'(z)$ belongs to class (S). If $zg'(z)$ belongs to (S) then $g(z)$ is univalent. For if not, let

$$g(z_1) = g(z_2) = w_0$$

for two points z_1 and z_2 $(z_1 \neq z_2)$ lying within $|z| = r$ where r is in the interval $1 - \delta < r < 1$. Then as z describes the circle $|z| = r$ enclosing z_1 and z_2, $g(z)$ describes a continuous closed contour C_r which must consist of at least two loops [4] about the point w_0. If this were the case a straight line parallel to the imaginary axis would cut C_r in more than two points, which is a contradiction. Hence $g(z)$ is univalent for $|z| < 1$.

We may then represent $g(z)$ in the form

$$g(z) = \int_0^z \frac{1 + ie^{-i\mu} \sin \mu\{F(z) - 1\}}{1 - 2ze^{-i\mu} \cos \nu + e^{-2i\mu} z^2}\, dz \tag{3.2}$$

[4] See for example Titchmarsh, *The Theory of Functions*, 1932, p. 201.

where $\mathfrak{R}F(z) \geqq 0$ for $|z| < 1$, $0 \leqq \mu \leqq \pi$, $0 \leqq \nu \leqq \pi$. Conversely, if $F(z)$ is an arbitrary function with positive real part and regular for $|z| < 1$, and μ any parameter satisfying the inequalities $0 \leqq \mu \leqq \pi$, then the function $g(z)$ formed by the expression (3.2) is univalent for $|z| < 1$ and is the limit of functions $g_n(z)$ each of which is regular and univalent for $|z| < 1$, has at most either a simple pole or two logarithmic singularities on $|z| = 1$, and which map $|z| = 1$ on a contour convex in the direction of the imaginary axis. For let $\{r_n\}$ be a sequence of values of tending to 1 as $n \to \infty$ and define $g_n(z)$ by the equation

$$(3.3) \quad g_n(z) = \int_0^z \frac{1 + ie^{-i\mu} \sin \mu\{F(r_n z) - 1\}}{1 - 2ze^{-i\mu} \cos \nu + e^{-2i\mu} z^2} dz$$

$$(3.4) \quad g(z) = \lim_{n\to\infty} g_n(z) \text{ uniformly in any region interior to the unit circle.}$$

$F(r_n z)$ is regular for $|z| \leqq 1$, $zg'_n(z)$ is regular on $|z| = 1$ save for either one pole of multiplicity 2 ($\nu = 0$) or two simple poles. From § 2 we see that $zg'_n(z)$ maps $|z| = 1$ into a contour which is cut by the real axis in not more than two points. Hence $g_n(z)$ maps $|z| = 1$ into a contour convex in the direction of the imaginary axis and $g_n(z)$ is univalent for $|z| < 1$. It follows by the theorem of Montel [5] that $g(z)$ is also univalent for $|z| < 1$.

If in particular $g(z)$ is also real on the real axis, then $\mu = \pi/2$, $\nu = \pi/2$ and (3.2) takes the form

$$(3.5) \qquad g(z) = \int_0^z \frac{F(z)}{1 - z^2} dz.$$

We may employ here the Stieltjes formula due to Herglotz [6]

$$(3.6) \qquad F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\alpha(\theta)$$

where $\alpha(\theta)$ is non-decreasing in $(0, 2\pi)$. (3.5) can then be written in the form

$$(3.7) \qquad g(z) = \frac{1}{2\pi i} \int_0^{\pi} \log \left\{ \frac{1 - ze^{-i\theta}}{1 - ze^{i\theta}} \right\} \frac{d\alpha(\theta)}{\sin \theta}$$

when the fact is used that $g(z)$ and $F(z)$ are real on the real axis and an integration is performed.

4. *The coefficients of functions of classes* (S) *and* $(\mathfrak{F})$. If in the representation (2.4)

$$F(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

[5] See P. Montel, *Bulletin de la Société Mathématique de France*, t. 53 (1925), p. 253.

[6] See G. Herglotz, *Leipziger Berichte*, vol. 63 (1911), pp. 501-511.

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since $\mathfrak{R} F(z) \geqq 0$ for $|z| < 1$ we have $|b_n| \leqq 2$ for all n. If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

a comparison of coefficients on both sides of equation (2.4) yields

$$(4.1) \qquad a_n = \frac{\sin n\nu}{\sin \nu} e^{-(n-1)i\mu} + i \sin \mu \cdot \sum_{k=1}^{n-1} \frac{\sin k\nu}{\sin \nu} \cdot e^{-k\mu i} b_{n-k}.$$

$$(4.2) \qquad |a_n| \leqq \left| \frac{\sin n\nu}{\sin \nu} \right| + 2 \sin \mu \left[1 + \sum_{k=2}^{n-1} \left| \frac{\sin k\nu}{\sin \nu} \right| \right]$$

where $\sin \mu \geqq 0$, $\sin \nu \geqq 0$.

$$(4.3) \qquad |a_n| \leqq n + 2 \sin \mu \cdot \frac{n(n-1)}{2} = n(1 - \sin \mu) + n^2 \sin \mu$$

$$(4.4) \qquad |a_n| \leqq n^2 \text{ for all } n.$$

The equality sign $|a_n| = n^2$ is attained for a fixed n only when $\nu = 0$ or π, $\mu = \pi/2$. In this case

$$f(z) = \frac{zF(z)}{(1 - \epsilon z)^2}, \qquad (\epsilon = \pm i)$$

$$n^2 = |a_n| = |b_{n-1} + 2\epsilon b_{n-2} + \cdots + k\epsilon^{k-1} b_{n-k} + \cdots + n\epsilon^{n-1}|.$$

Equality can occur only when $b_k = 2\epsilon^k$ $(k = 1, 2, \cdots, n-1)$. In this case $f(z)$ must have the form

$$(4.5) \qquad f(z) = \sum_{k=1}^{n} k^2 \epsilon^{k-1} z^k + a_{n+1} z^{n+1} + \cdots.$$

However, since

$$g(z) = \int_0^z \frac{f(z)}{z} dz = z + c_2 z^2 + \cdots$$

is univalent and $|c_2| = 2$, $g(z)$ must be of the form

$$g(z) = z/(1 - e^{i\alpha} z)^2, \quad \alpha \text{ real}$$

as L. Bieberbach has shown.[7]
Hence (4.5) reduces to

$$(4.6) \qquad f(z) = \frac{z + \epsilon z^2}{(1 - \epsilon z)^3}, \qquad (\epsilon = \pm i).$$

Hence $|a_n| = n^2$ for a fixed n only for the function (4.6).

Again, if $f(z)$ is any odd function of class (S) then $\beta - \alpha = \pi$, $\nu = \pi/2$. Consequently, from (4.2) we have

[7] See L. Bieberbach, *loc. cit.*

$$(4.7) \qquad | a_{2n+1} | \leqq 1 + 2n \sin \mu \leqq 2n + 1.$$

If $\nu \neq 0$ or π, we also have from (4.2)

$$\overline{\lim_{n\to\infty}} \left| \frac{a_n}{n} \right| \leqq 2 \frac{\sin \mu}{\sin \nu} \cdot M(\nu)$$

where

$$M(\nu) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} | \sin k\nu |.$$

It will follow from the following lemma, due to T. Gronwall,[8] that

$$(4.8) \qquad \overline{\lim_{n\to\infty}} \left| \frac{a_n}{n} \right| \leqq \frac{4}{\pi} \frac{\sin \mu}{\sin \nu}.$$

LEMMA. *If*

$$M_n(\theta) \equiv \frac{1}{n} \sum_{k=1}^{n} | \sin k\theta |$$

then

$$M(\theta) \equiv \lim_{n\to\infty} M_n(\theta)$$

exists and

$$M(\theta) = 2/\pi$$

if θ/π is irrational;

$$M\left(\frac{k}{r}\pi\right) = \frac{\cot \pi/2r}{r} < 2/\pi, \; k \text{ and } r \; (k < r)$$

positive integers prime to each other.

If

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

is a member of class $(\mathfrak{F})$ then, as we have seen, $zg'(z)$ belongs to (S). Hence

$$(4.9) \qquad | c_n | \leqq n \text{ for all } n$$

and equality is attained for any fixed n only by the one function of class $(\mathfrak{F})$, namely

$$z(1 - \epsilon z)^{-2}, \qquad (\epsilon = \pm i).$$

For this particular class of univalent functions this constitutes a proof of the Bieberbach conjecture for the coefficients of a univalent function.

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[8] See T. Gronwall, "On a theorem of Fejér's," *Transactions of the American Mathematical Society*, 1912, pp. 445-468.

If $g(z)$ is an odd function of class $(\mathfrak{F})$ with coefficients c_{2n+1} then from (4.7) and the fact that $zg'(z)$ belongs to (S) we have

$$|c_{2n+1}| \leqq 1. \tag{4.10}$$

All the members of class $(\mathfrak{F})$ which are real on the real axis and convex in the direction of the imaginary axis (a direction perpendicular to the line on which the coefficients lie) are given by (3.7) and conversely. The coefficients c_n of $g(z)$ in this case are then given by the formula

$$c_n = \frac{1}{n\pi} \int_0^\pi \frac{\sin n\theta}{\sin \theta} \, d\alpha(\theta) \tag{4.11}$$

whence $|c_n| \leqq 1$ and equality is attained by

$$z/(1 \pm z), \qquad z/(1 - z^2).$$

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ON NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER WITH CONSTANT COEFFICIENTS.

By R. D. Carmichael.

Introduction. The methods of this paper are suggested by the stimulating memoir of A. Hurwitz [1] in which is proved the theorem of C. Guichard [2] that for every integral function $\phi(x)$ the equation $y(x+1) - y(x) = \phi(x)$ has a solution which is itself an integral function. The method also owes much to the remarkable contributions of S. Pincherle [3] to the theory of functional equations, contributions which now extend over a period of nearly fifty years.

In § 1 a formal solution of the differential equation (3) is given and a sufficient condition is set forth (Theorem I) to ensure that it shall be an actual solution. The remainder of the paper consists mainly in presenting effectively workable hypotheses under which this sufficient condition is certainly realized. The maximum simplicity in the formulation of these hypotheses is attained in Theorem VII of § 8, and this theorem is effectively supplemented by Theorem VIII. Auxiliary classifications of integral functions are indicated in §§ 3 and 4. The former is classic, but the latter seems to be new. It is a classification which appeared to be demanded by the course of the argument; it seems to be of interest for its own sake. Attention is called particularly to the invariant point property of integral functions indicated in this connection in § 4.

1. The first general theorem. Let $F(z)$ and $\phi(x)$ denote two given integral functions, neither of them being identically equal to zero, and write their power series expansions in the forms

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \tag{1}$$

$$\phi(x) = \sum_{\nu=0}^{\infty} s_\nu x^\nu/\nu!. \tag{2}$$

We consider the problem of constructing integral functions $y(x)$ satisfying the linear differential equation

$$a_0 y + a_1 y' + a_2 y'' + \cdots = \phi(x) \tag{3}$$

[1] A. Hurwitz, *Acta Mathematica*, vol. 20 (1897), pp. 285-312.

[2] C. Guichard, *Ann. Ec. Norm. Sup.* (3), vol. 4 (1887), pp. 361-380.

[3] S. Pincherle, *Acta Mathematica*, vol. 46 (1925), pp. 341-362.

of infinite order with constant coefficients. We shall say that $F(z)$ is the *characteristic function* belonging to equation (3).

Let τ be a given positive number. Let $\{\lambda_\nu\}$ and $\{\mu_\nu\}$ $(\nu = 0, 1, 2, \cdots)$ be two infinite sequences of positive numbers such that $\tau \leqq \mu_\nu \leqq \lambda_\nu$ for every ν. For each particular value of ν let C_ν be a closed contour of finite length encircling the point O and lying in the ring $\mu_\nu \leqq |z| \leqq \lambda_\nu$, and let C_ν pass through no point at which $F(z)$ vanishes. Let T_ν denote the sum of the (convergent) series in the relation

$$T_\nu = \sum_{k=0}^{\infty} |a_k| \lambda_\nu{}^k. \tag{4}$$

It is obvious that T_ν is bounded away from zero.

We introduce the functions

$$P_{\nu,n}(x) = \frac{1}{2\pi i}\int_{C_n} \frac{e^{xz}dz}{z^{\nu+1}F(z)} \tag{5}$$

where ν and n take independently the values of the set $0, 1, 2, \cdots$. For their k-th derivatives with respect to x we have

$$P^{(k)}_{\nu,n}(x) = \frac{1}{2\pi i}\int_{C_n} \frac{z^k e^{xz}dz}{z^{\nu+1}F(z)}. \tag{6}$$

We have

$$\sum_{k=0}^{\infty} a_k P^{(k)}_{\nu,n}(x) = \frac{1}{2\pi i}\int_{C_n} \sum_{k=0}^{\infty} a_k z^k \frac{e^{xz}dz}{z^{\nu+1}F(z)} = \frac{1}{2\pi i}\int_{C_n} \frac{e^{xz}dz}{z^{\nu+1}} = \frac{x^\nu}{\nu!}. \tag{7}$$

From (6) we have

$$|P^{(k)}_{\nu,n}(x)| \leqq \frac{\lambda_n{}^k e^{|x|\lambda_n}}{2\pi\mu_n{}^{\nu+1}}\int_{C_n} \frac{|dz|}{|F(z)|}. \tag{8}$$

Let us define $y(x)$ by the relation

$$y(x) = \sum_{\nu=0}^{\infty} s_\nu P_{\nu,\nu}(x) \equiv \sum_{\nu=0}^{\infty} \frac{s_\nu}{2\pi i}\int_{C_\nu} \frac{e^{xz}dz}{z^{\nu+1}F(z)}, \tag{9}$$

subject to appropriate conditions of convergence to be indicated later.

Using the notation thus described, we shall prove the following general theorem:

THEOREM I. *A sufficient condition that $y(x)$, as defined by* (9), *shall be an integral function satisfying* (3) *is that the series*

$$\sum_{\nu=0}^{\infty} e^{\rho\lambda_\nu} |s_\nu| T_\nu \mu_\nu{}^{-\nu-1}\int_{C_\nu} \frac{|dz|}{|F(z)|} \tag{10}$$

shall be convergent for every positive number ρ.

Let S be any preassigned finite closed region of the x-plane. Let ρ be the maximum value of $|x|$ for x in S. Since by hypothesis series (10) is convergent and since T_ν is bounded away from zero, it follows that the series obtained from (10) on replacing T_ν by 1 is convergent. From this fact and from relation (8) with $n=\nu$ and $k=0$ we see that for all x in S the series in (9) is dominated term by term by a convergent series of constants. Hence that series is absolutely and uniformly convergent in S and $y(x)$ is an integral function. Moreover, $y^{(k)}(x)$ may be formed from (9) by term-by-term differentiation.

If we proceed formally we have by aid of (7) the relations

$$\begin{aligned}(11)\qquad \sum_{k=0}^{\infty} a_k y^{(k)}(x) &= \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} a_k s_\nu P_{\nu,\nu}^{(k)}(x)\\ &= \sum_{\nu=0}^{\infty} s_\nu \sum_{k=0}^{\infty} a_k P_{\nu,\nu}^{(k)}(x)\\ &= \sum_{\nu=0}^{\infty} s_\nu \frac{x^\nu}{\nu!} = \phi(x).\end{aligned}$$

From (8) it follows that the series in the third member of (11) is dominated term-by-term by the series

$$(12)\qquad \sum_{\nu=0}^{\infty} |s_\nu| \sum_{k=0}^{\infty} |a_k| \lambda_\nu^k e^{|x|\lambda_\nu} \mu_\nu^{-\nu-1} \int_{C_\nu} \frac{|dz|}{|F(z)|}.$$

From the fact that $F(z)$ is an integral function it follows that the series here denoted by the second summation sign is convergent. Employing (4), we may then write the series denoted by the first summation sign in the form

$$(13)\qquad \sum_{\nu=0}^{\infty} e^{|x|\lambda_\nu} |s_\nu| T_\nu \mu_\nu^{-\nu-1} \int_{C_\nu} \frac{|dz|}{|F(z)|}.$$

For all x in S this series is dominated term-by-term by the series in (10). It follows that the repeated series in the third member of (11) is absolutely and uniformly convergent in S. Therefore the repeated series in the second member of (11) is absolutely and uniformly convergent in S and its sum is equal to that of the series in the third member. Thence it follows readily from (11) that $y(x)$ is a solution of (3).

These results imply Theorem I.

COROLLARY. *The k-th derivative of $y(x)$ is dominated as follows:*

$$(14)\qquad |y^{(k)}(x)| \leqq \sum_{\nu=0}^{\infty} e^{|x|\lambda_\nu} |s_\nu| \lambda_\nu^k \mu_\nu^{-\nu-1} \int_{C_\nu} \frac{|dz|}{|F(z)|}, \qquad (k=0,1,2,\cdots).$$

2. Consequences of Theorem I. We shall now prove the following theorem, retaining our previous notation and employing additional hypotheses:

THEOREM II. *Let $F(z)$ be a function of exponential type. Let λ_ν be further restricted by the condition $\lambda_\nu \leqq \alpha(\nu + 1)$ $(\nu = 0, 1, 2, \cdots)$, where α is a given positive constant. Let the defined elements be so related that a positive constant β exists for which the relations*

$$\int_{C_\nu} \frac{|dz|}{|F(z)|} < \frac{\mu_\nu^{\nu+1}\beta^{\nu+1}}{\nu!}, \qquad (\nu = 0, 1, 2, \cdots), \tag{15}$$

are satisfied. Then the function $y(x)$, defined in (9), is an integral function satisfying equation (3).

It is sufficient to show that the series in (10) is convergent. Since $F(z)$ is of exponential type, say of type q, we have for every positive ϵ a constant K_ϵ such that

$$|F(z)| \leqq \sum_{\nu=0}^{\infty} |a_\nu| \; |z|^\nu < K_\epsilon e^{(q+\epsilon)|z|}.$$

Then, since $\lambda_\nu < \alpha(\nu + 1)$, we have

$$T_\nu < K_\epsilon e^{\alpha(q+\epsilon)(\nu+1)}.$$

Thence, by aid of (15), we see that series (10) is dominated term-by-term by the series

$$\sum_{\nu=0}^{\infty} K_\epsilon \frac{|s_\nu|}{\nu!} e^{\rho\alpha(\nu+1)} e^{\alpha(q+\epsilon)(\nu+1)} \beta^{\nu+1}.$$

From (2) and the integral character of $\phi(x)$ it follows that the last foregoing series is convergent for every positive number ρ. Hence the same is true of series (10). Therefore Theorem II is established.

Let a, α, b, c be given constants, $a \leqq \alpha$, and take $\mu_\nu = a(\nu + 1)$, $\lambda_\nu = \alpha(\nu + 1)$. Then, if for every ν of the set $0, 1, 2, \cdots$ the length of C_ν does not exceed $b^{\nu+1}$ and if $|F(z)| > c^{-\nu-1}$ for every point z on C_ν, it is easy to show that the hypotheses of Theorem II are satisfied. It is obvious that a suitable constant b exists whenever the paths C_ν are circles about 0 as center and having radii limited by the implied conditions. This yields one of the simplest special cases of the theorem.

For the case when $F(z) = e^z - 1$ we may take the path C_ν to be a circle of radius $\pi(2\nu + 1)$. Then it is easy to show (by aid of the periodicity properties of $F(z)$) that a constant M exists such that $|F(z)| > M$ whenever z

is on a circle C_ν. By taking $\mu_\nu = \lambda_\nu = \pi(2\nu + 1)$ one may readily show that the required conditions are satisfied for the application of Theorem II and hence that the corresponding equation (3) has a solution $y(x)$ which is an integral function. This implies that the equation $y(x+1) - y(x) = \phi(x)$ has a solution $y(x)$ which is an integral function, the proof in this case being essentially that of Hurwitz (*loc. cit.*).

In a similar way it may be shown that the difference equation

$$y(x+n) + c_1 y(x+n-1) + \cdots + c_n y(x) = \phi(x)$$

with constant coefficients has a solution $y(x)$ which is an integral function provided that $\phi(x)$ is an integral function. Here the function $F(z)$ may be written

$$F(z) = e^{nz} + c_1 e^{(n-1)z} + \cdots + c_n = \prod_k (e^z - \rho_k)^{t_k},$$

where the ρ_k are constants and the t_k are positive integers. Suitable contours C_ν may be readily defined such that $F(z)$ is again bounded away from 0 for all z on all C_ν and such that Theorem II becomes applicable. In fact, the same method may be extended,[4] but with increased difficulty, to the more general case of the equation

$$c_0 y(x) + c_1 y(x + \alpha_1) + \cdots + c_m y(x + \alpha_m) = \phi(x)$$

where the c's and α's are constants and $\phi(x)$ is an integral function.

3. Integral functions of class $C(t, q)$. With a view to the extension of Theorem II it is convenient to separate certain integral functions into classes and to note some properties of the several classes.

Let t be a given positive number. Let $g(z)$ be an integral function having the power series expansion

$$g(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu.$$

We shall say that $g(z)$ is of class $C(t, q)$ if and only if

$$\limsup_{\nu=\infty} |(\nu!)^{1/t} c_\nu|^{1/\nu} = q < \infty.$$

We shall need the following known theorem[5] concerning these functions:

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[4] See R. D. Carmichael, *Transactions of the American Mathematical Society*, vol. 35 (1933), pp. 1-28.

[5] This theorem belongs to the classic theory of integral functions of order t and normal type or minimal type. For more general results of a similar character see a forthcoming paper on invariant point properties of integral functions under the joint authorship of R D. Carmichael, W. T. Martin and M. T. Bird.

Necessary and sufficient conditions in order that an integral function $g(z)$ shall be of class $C(t, q)$ are the following:

(a) numbers σ (zero or positive) shall exist for which it is true that for every positive number ϵ there exists a positive number $N_{\epsilon\sigma}$, depending on ϵ and σ but independent of z, such that for all (finite) values of z we have

$$| g(z) | < N_{\epsilon\sigma}\, e^{(\sigma+\epsilon)^t |z|^t / t};$$

(b) the least possible value of such numbers σ is q.

4. Integral functions of class $K(s)$. Let s be a given positive number not less than 1. Let $\psi(x)$ be an integral function having the power series expansion

$$\psi(x) = \sum_{\nu=0}^{\infty} \sigma_\nu x^\nu.$$

We shall say that $\psi(x)$ is of class $K(s)$ if and only if

$$\lim_{\nu=0} \sigma_\nu^{\nu^{-s}} = 0.$$

It is obvious that the class $K(s_1)$ contains the class $K(s_2)$ if $1 \leqq s_1 < s_2$.

If $\psi(x)$ is of class $K(s)$ then for every positive number ϵ there exists a positive number L_ϵ, independent of ν, such that

$$| \sigma_\nu | < L_\epsilon \epsilon^{\nu^s}.$$

Hence, from the relation

$$\psi^{(n)}(x)/n! = \sum_{\nu=0}^{\infty} \sigma_{\nu+n} \frac{(\nu+n)!}{\nu!\,n!} x^\nu$$

we have

$$| \psi^{(n)}(x)/n! | < L_\epsilon \sum_{\nu=0}^{\infty} \epsilon^{(\nu+n)^s} \frac{(\nu+n)!}{\nu!\,n!} | x |^\nu.$$

Now, since $s \geqq 1$, we have

$$(\nu+n)^s \geqq \nu^s + n^s, \quad \frac{(\nu+n)!}{\nu!\,n!} \leqq (1+\delta)^n (1+\delta^{-1})^\nu,$$

when $n + \nu \geqq 1$, where δ is any positive number whatever. Therefore if we take ϵ less than 1 (as we shall) we have

$$| \psi^{(n)}(x)/n! | < L_\epsilon \epsilon^{n^s} (1+\delta)^n \sum_{\nu=0}^{\infty} \epsilon^{\nu^s} (1+\delta^{-1})^\nu | x |^\nu,$$

whence it follows that

$$\lim_{n=\infty} | \psi^{(n)}(x)/n! |^{n^{-s}} = 0.$$

This property, for $x=0$, is the defining property for the class $K(s)$. This defining property associated with the point 0 therefore persists for all other finite points. Moreover if it holds for $x=x_0$ it clearly holds also for $x=0$. *It therefore exhibits an invariant point property of the functions of class* $K(s)$.

We shall now prove the following theorem:

A necessary and sufficient condition that the integral function $\psi(x)$ *shall be of class* $K(s)$, $s \geqq 1$, *is that the series*

$$\sum_{\nu=0}^{\infty} |\sigma_\nu| \; |x|^{\nu^s}$$

shall converge for every finite value of x.

In the proof of this theorem it is convenient to use the following lemma:

LEMMA. *Let* s *be a positive number not less than* 1. *Let*

$$u_1 + u_2 + u_3 + \cdots$$

be a series in which $u_k \geqq 0$. *Define* ρ *by the relation*

$$\limsup_{\nu=\infty} (u_\nu)^{\nu^{-s}} = \rho.$$

Then the given series is convergent if $\rho < 1$ *and divergent if* $\rho > 1$, *while for* $\rho = 1$ *there is no test.*

The usual proof for the classic case $s=1$ of the lemma holds without essential modification in the general case: it is therefore left to the reader.

From the lemma it follows at once that the series in the theorem converges for all finite x if $\psi(x)$ is of class $K(s)$. On the other hand, if the series converges for all finite x it follows from the lemma that

$$\limsup_{\nu=\infty} |\sigma_\nu|^{\nu^{-s}} |x| \leqq 1$$

for all finite x, whence we see that $\psi(x)$ is of class $K(s)$. The theorem is therefore established.

5. Generalization of Theorem II. We shall now use the notation of § 1, with the implied hypotheses, in proving the following generalization of Theorem II:

THEOREM III. *Let $F(z)$ be an integral function of class $C(t, q)$. Let $\phi(x)$ be an integral function of class $K(s)$. Let λ_ν be further restricted by the condition $\lambda_\nu \leqq \alpha(\nu + 1)^\sigma$ $(\nu = 0, 1, 2, \cdots)$, where α is a given positive constant and σ verifies the relation $1 \leqq \sigma \leqq \min(s, s/t)$. Suppose that $s \geqq t$ and $s \geqq 1$. Let the defined elements be so related that a positive constant β exists for which the relations*

$$(16) \qquad \int_{C_\nu} \frac{|\,dz\,|}{|\,F(z)\,|} < \frac{\mu_\nu^{\nu+1}\beta^{(\nu+1)^s}}{\nu!}, \qquad (\nu = 0, 1, 2, \cdots),$$

are satisfied. Then the function $y(x)$, defined in (9), *is an integral function satisfying the differential equation* (3).

In view of Theorem I it is clearly sufficient to prove that the hypotheses here involved imply the convergence of the series in (10).

From the hypothesis on $F(z)$ and from the theorem in § 3 it follows that for every positive number ϵ there exists a positive number K_ϵ such that

$$|\,F(z)\,| \leqq \sum_{\nu=0}^{\infty} |\,a_\nu\,| \; |\,z\,|^\nu < K_\epsilon e^{(q+\epsilon)^t|z|^t/t}.$$

Then, since $\lambda_\nu < \alpha(\nu+1)^\sigma$, we see from (4) that

$$T_\nu < K_\epsilon \, e^{a^t(q+\epsilon)^t(\nu+1)^s/t}.$$

Thence, by aid of (16), we see that series (10) is dominated term-by-term by the series

$$\sum_{\nu=0}^{\infty} K_\epsilon \frac{|\,s_\nu\,|}{\nu!} e^{\rho\alpha(\nu+1)^s} e^{a^t(q+\epsilon)^t(\nu+1)^s/t} \beta^{(\nu+1)^s}.$$

From (2) and from the fact that $\phi(x)$ is of class $K(s)$ and from the theorem of § 4 it follows that the foregoing series is convergent for every positive number ρ. Hence the same is true of series (10). Theorem III is therefore established.

By specialization of the hypotheses in Theorem III we obtain the following theorem which is often more convenient in use than the more general theorem:

THEOREM IV. *Let $F(z)$ and $\phi(x)$ be integral functions of classes $C(t, q)$ and $K(s)$ respectively, where s is not less than t and not less than* 1. *Let σ be such that $1 \leqq \sigma \leqq \min(s, s/t)$. Let a, α, b, c be given positive constants, $a \leqq \alpha$. If for every number ν of the set $0, 1, 2, \cdots$ there exists a contour C_ν*

encircling the point 0 *and lying in the circular ring* $a(\nu+1) \leqq |z| \leqq \alpha(\nu+1)^\sigma$ *and having a length not exceeding* $b^{(\nu+1)^s}$ *and if* $|F(z)| > c^{-(\nu+1)^s}$ *for every point* z *on* C_ν, *then the function* $y(x)$, *defined in* (9), *is an integral function satisfying equation* (3).

To prove this theorem we show that the hypotheses imply those of Theorem III. We take $\mu_\nu = a(\nu+1)$, $\lambda_\nu = \alpha(\nu+1)^\sigma$. Then, since the hypotheses in Theorem IV imply that

$$\int_{C_\nu} \frac{|dz|}{|F(z)|} < (bc)^{(\nu+1)^s},$$

it is sufficient to observe that a positive number β obviously exists such that

$$(bc)^{(\nu+1)^s} < a^{\nu+1}(\nu+1)^{\nu+1}\beta^{(\nu+1)^s}/\nu!, \qquad (\nu = 0, 1, 2, \cdots),$$

in order to recover all the hypotheses of Theorem III. Hence Theorem IV is established.

6. Another Consequence of Theorem I. We shall now prove the following theorem:

THEOREM V. *Let* $F(z)$ *and* $\phi(x)$ *be integral functions of classes* $C(t, q)$ *and* $K(s)$, *respectively, where* s *is not less than* t *and not less than* 1. *Let* σ *be such that* $1 \leqq \sigma \leqq \min(s, s/t)$. *Let* $C_0, C_1, C_2, \cdots$ *be a sequence of circles about* 0 *as center and of radii* $\sigma_0, \sigma_1, \sigma_2, \cdots$, *respectively, and let them be such that no one of them passes through a point at which* $F(z)$ *vanishes. Moreover, let us suppose that positive constants* a *and* α *exist such that* $a(\nu+1) \leqq \sigma_\nu \leqq \alpha(\nu+1)^\sigma$. *Then, if a positive constant* K *exists such that the relations*

$$(17) \qquad \int_{C_\nu} \frac{|dz|}{|F(z)|} < K^{(\nu+1)^s} \qquad (\nu = 0, 1, 2, \cdots),$$

are satisfied, the function $y(x)$, *defined in* (9) *is an integral function satisfying equation* (3).

We shall show that this proposition is a consequence of Theorem I. We take $\mu_\nu = \lambda_\nu = \sigma_\nu$. Then Theorem I implies the truth of V provided that the series

$$(18) \qquad \sum_{\nu=0}^{\infty} |s_\nu| \, e^{\rho\sigma_\nu}\sigma_\nu^{-\nu-1}\bar{T}_\nu \int_{C_\nu} \frac{|dz|}{|F(z)|}$$

converges for every positive value ρ, the symbol $\bar{T}_\nu$ being defined by the relation

$$\bar{T}_\nu = \sum_{k=0}^{\infty} | a_k | \sigma_\nu{}^k.$$

But series (18) is dominated term by term by the series

$$\sum_{\nu=0}^{\infty} \frac{| s_\nu |}{\nu !} e^{a\rho(\nu+1)^s} a^{-\nu-1} (\nu + 1)^{-\nu-1} \nu ! \bar{T}_\nu K^{(\nu+1)^s}.$$

Now take the $[(\nu + 1)^s]$-th root of the ν-th term of this series, noting that for $\bar{T}_\nu$ we have from its definition and the theorem of § 2 a relation of the form

$$\bar{T}_\nu < K_\epsilon\, e^{\alpha^t (q+\epsilon)^t (\nu+1)^s / t}$$

and employing the usual asymptotic form for $\nu !$; as ν becomes infinite this root approaches the limit 0. Hence from the lemma in § 4 it follows that series (18) converges for every positive value of ρ. Therefore the theorem is established.

We shall now show that the following theorem is a corollary of the preceding:

THEOREM VI. *Let $F(z)$ and $\phi(x)$ be integral functions of classes $C(t, q)$ and $K(s)$, respectively, where s is not less than t and not less than* 1. *Let σ be such that $1 \leqq \sigma \leqq \min(s, s/t)$. Let $m(r)$ denote the minimum value of $| F(z) |$ on the circle $| z | = r$. Let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be a sequence of positive numbers such that $a(\nu + 1) \leqq \sigma_\nu \leqq \alpha(\nu + 1)^\sigma$, where a and α are given positive constants and $a \leqq \alpha$. Let C_ν denote the circle $| z | = \sigma_\nu$. Suppose that a positive number L exists such that the relations*

$$m(\sigma_\nu) > L^{-(\nu+1)^s}, \qquad (\nu = 0, 1, 2, \cdots), \tag{19}$$

are satisfied. Then the function $y(x)$, defined by (9), *is an integral function satisfying equation* (3).

This theorem is seen to follow from the preceding by observing that conditions (19) imply conditions (18) and also imply that C_ν passes through no point at which $F(z)$ vanishes.

7. Auxiliary properties of integral functions. Let $F(z)$ be an integral function of class $C(t, q)$. In case $F(z)$ has zeros away from the point 0 we denote these by $\alpha_1, \alpha_2, \cdots$ in order of increasing moduli. If the number of zeros is infinite, then it is a classic result that the series

$$\sum_{n=1}^{\infty}\left|\frac{1}{\alpha_n}\right|^{t+\epsilon} \tag{20}$$

converges for every positive number ϵ. Moreover, we have the following well-known theorem:[6]

Let k be any positive number. For each i inclose the zero α_i by a circle with α_i as center and of radius $|\alpha_i|^{-k}$. Then in that part of the finite plane which is exterior to all these circles we have for every preassigned positive ϵ and for $|z|$ sufficiently large, the relation

$$|F(z)| > e^{-|z|^{t+\epsilon}}. \tag{21}$$

Moreover, it is known,[7] and indeed is implied by the convergence of the series in (20), that for every positive ϵ there exists a positive number M_ϵ such that $|\alpha_n|^{-t-\epsilon} < M_\epsilon/n$. If we write $k = k_1(t+\epsilon)$ we then have the existence of $M_{\epsilon k}$ such that $|\alpha_n|^{-k} < M_{\epsilon k}n^{-k_1}$. If we take $k_1 > 1$ (and this we do) and write

$$L_{\epsilon k} = 2M_{\epsilon k}\sum_{n=1}^{\infty} n^{-k_1}$$

we have

$$2\sum_{n=1}^{\infty}|\alpha_n|^{-k} < L_{\epsilon k}.$$

Therefore from the proposition associated with (21) we have the following theorem:

If $F(z)$ is of class $C(t, q)$ and ϵ is a positive number, then a number L exists ($L \geqq L_{\epsilon k}$) such that in every interval of length L sufficiently far out on the positive axis of reals there exists a value r such that

$$|F(z)| > e^{-|z|^{t+\epsilon}} \quad \textit{for} \quad |z| = r. \tag{22}$$

8. Two additional existence theorems for equation (3). We shall now employ the theorem at the end of § 7 in obtaining from Theorem VI the following more simply formulated proposition:

THEOREM VII. *Let $F(z)$ and $\phi(x)$ be integral functions of classes $C(t, q)$ and $K(s)$, respectively, and let t be less than s ($s \geqq 1$). Then the differential equation (3) has a solution $y(x)$, defined in equation (24) below, such that $y(x)$ is an integral function.*

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[6] See *Encyklopädie der Mathematische Wissenschaften*, Band II_3, p. 436.

[7] See *Encyklopädie der Mathematische Wissenschaften*, Band II_3, p. 434.

Let the positive number ϵ be such that $t+\epsilon<s$. Let a and α be two positive numbers such that $a<\alpha$. Then from the result at the end of the preceding section it follows that for sufficiently large values of ν we have on each interval $(a(\nu+1), \alpha(\nu+1))$ a number σ_ν such that

$$(23) \qquad | F(z) | > e^{-|z|^{t+\epsilon}}, \qquad | z | = \sigma_\nu.$$

We denote by C_ν the circle $| z | = \sigma_\nu$, defining σ_ν conveniently for values of ν smaller than those involved in (23). We write

$$(24) \qquad y(x) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{2\pi i} \int_{C_\nu} \frac{e^{xz} dz}{z^{\nu+1} F(z)}.$$

For σ in Theorem VI we take the value $\sigma=1$. Then the hypotheses of Theorem VI are satisfied when ν is sufficiently large, and hence for all values of ν when σ_ν is suitably defined for an appropriate finite number of values of ν. Therefore Theorem VII is established.

Corollary. *If $\phi(x)$ is an integral function and if $F(z)$ is of class $C(t,q)$ with $t<1$, then the differential equation* (3) *has a solution $y(x)$ which is an integral function.*

Whether the corollary holds when $t=1$, and more generally whether Theorem VII holds when $t=s$, I have not been able to determine. That there are cases when $t=s$ and when the conclusion holds is shown at once by the theorems for difference equations cited in the latter part of § 2. These involve cases in which $t=s=1$. But our method does not seem to yield the conclusion of Theorem VII when $t=s$ unless some additional hypotheses are placed on $F(z)$. The result associated with (21) seems not to be sufficiently restrictive for this purpose, since we seem to need an inequality more effective than (22). The nature of the additional information required is indicated by the further hypotheses on $F(z)$ introduced in the following theorem.

Theorem VIII. *Let $F(z)$ and $\phi(x)$ be integral functions of classes $C(t,q)$ and $K(t)$, respectively, where $t \geqq 1$. Let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be a sequence of positive numbers such that $a(\nu+1) \leqq \sigma_\nu \leqq \alpha(\nu+1)$, where a and α are given positive constants and $a<\alpha$. Let C_ν denote the circle $| z | = \sigma_\nu$. Suppose furthermore that $F(z)$ is such that the σ_ν may be chosen* (*subject to*

the named conditions) so that a positive constant M and a non-negative constant N exist such that

$$|F(z)| > M\, e^{-N|z|^t} \tag{25}$$

for all z on the circles C_ν. Then the function $y(x)$, defined by (9), *is an integral function satisfying the differential equation* (3).

This theorem is an almost immediate corollary of Theorem VI. In the latter theorem we take $s = t$, whence we must have $\sigma = 1$. Then in the two theorems the constants σ_ν belong to the same intervals. Using the symbol $m(r)$ of Theorem VI, we see from (25) that

$$m(\sigma_\nu) > M\, e^{-N\sigma_\nu^t} \geqq M\, e^{-Na^t(\nu+1)^t} > L^{-(\nu+1)^t},$$

where L is a sufficiently large positive number. Hence we have recovered all the hypotheses of Theorem VI. Therefore Theorem VIII is established.

9. Further properties of the solution of Equation (3). In all the cases treated we have shown that the solution $y(x)$ is an integral function. The corollary to Theorem I gives upper bounds to the values of $|y^{(k)}(x)|$, $(k = 0, 1, 2, \cdots)$, valid in the general case of § 1; and this may obviously be specialized so as to yield more precise inequalities under the additional hypotheses involved in the less general theorems. We shall now show that there is a class of cases, depending on the nature of the characteristic function $F(z)$, in which additional information concerning $y(x)$ may be obtained directly from the convergence of the series in the left member of (3).

Let us suppose that $F(z)$ is of class $C(t, q)$ with q greater than 0 and let the coefficients $a_0, a_1, a_2, \cdots$ have a certain character of regularity implied by the required condition that

$$\lim_{\nu=\infty} |(\nu!)^{1/t} a_\nu|^{1/\nu} = q, \tag{26}$$

the superior limit in the defining condition of the class $C(t, q)$ being thus replaced by an actual limit. Then the condition

$$\limsup_{\nu=\infty} |a_\nu y^{(\nu)}(x)| \leqq 1,$$

necessary for the convergence of the series in the left member of (3), may be written in the form

$$\limsup_{\nu=\infty} |(\nu!)^{1/t} a_\nu \cdot (\nu!)^{-1/t} y^{(\nu)}(x)| \leqq 1.$$

Then from (26) we see that for any solution $y(x)$ of (3) we must have

$$\limsup_{\nu=\infty} |(\nu!)^{-1/t} y^{(\nu)}(x)| \leqq q^{-1}.$$

When $t \leqq 1$ this does not yield any information beyond that implied by the fact that the given solution $y(x)$ is an integral function. But when $t > 1$ it does give such additional information; in fact, it implies that the solution must then be of class $C(1 - t^{-1}, q_1)$ where $0 \leqq q_1 \leqq q^{-1}$. Under the named conditions every solution of (3) must be of the class indicated.

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ON GENERALIZATIONS OF SUM FORMULAS OF THE EULER-MACLAURIN TYPE.

By Marion T. Bird.

Introduction. We seek solutions $F(x)$ of the difference equation

$$(0.1) \qquad \underset{\omega}{\Delta} F(x) \equiv [F(x+\omega) - F(x)]/\omega = \phi(x)$$

where $\phi(x)$ is a known function. By the method of symbolic operators introduced by Lagrange [1] (see also Pincherle)[2] we see that a formal solution is afforded by the expression

$$F(x+\omega y) = \omega e^{\omega y D}(e^{\omega D} - 1)^{-1}\phi(x).$$

Carmichael [3] has emphasized the fact that by this procedure classes of solutions of the equation (0.1) may be made to depend upon suitable expansions of the expression

$$g(D) = \omega e^{\omega y D}(e^{\omega D} - 1)^{-1}.$$

Such considerations lead to the introduction of the function $A_{\nu,r}(y)$ by means of the Laurent expansion (see Von Koch)[4]

$$(0.2) \qquad \omega e^{\omega y z}(e^{\omega z} - 1)^{-1} = \sum_{\nu=-\infty}^{+\infty} \omega^{\nu+1} A_{\nu+1,r}(y) z^{\nu}, \quad 2r\pi < |\omega z| < 2(r+1)\pi, \quad (r = 0, 1, 2, \cdots).$$

It is convenient to define $\phi^{(-\nu)}(x)$ by means of the equation

$$\phi^{(-\nu)}(x) = \int_a^x \phi^{(-\nu+1)}(t)\,dt \qquad (\nu = 1, 2, 3, \cdots),$$

where a is in the region of continuity of $\phi(x)$. We may transform $\phi^{(-\nu)}(x)$ into the alternative form

$$\phi^{(-\nu)}(x) = \int_a^x \frac{(x-t)^{\nu-1}}{(\nu-1)!}\phi(t)\,dt, \qquad (\nu = 1, 2, 3, \cdots).$$

[1] J. L. de Lagrange, "Sur une nouvelle espèce de calcul, rélatif à la différentiation et à l'intégration des quantités variables," *Nouveaux Mémoires de L'Académie Royale des Sciences et Belles-Lettres a Berlin* (1772), pp. 185-221.

[2] S. Pincherle, "Funktionaloperationen und -gleichungen," *Encyklopädie der Mathematischen Wissenschaften,* II A 11, pp. 761-817.

[3] R. D. Carmichael, "The present state of the difference calculus and the prospect for the future," *The American Mathematical Monthly,* vol. 31 (1924), pp. 169-183.

[4] Helge Von Koch, "On a class of equations connected with Euler-Maclaurin's sumformula," *Arkiv för Matematik, Astronomi Och Fysik,* vol. 15 (1921), N:o 26.

With the foregoing definitions in mind one is led to contemplate

$$(0.3) \qquad F(x+\omega y)=\sum_{\nu=-\infty}^{+\infty}\omega^{\nu+1}A_{\nu+1,r}(y)\phi^{(\nu)}(x) \qquad (r=0,1,2,\cdots),$$

as a possible solution of equation (0. 1). In the following sections this will be exhibited as an actual solution. However, we shall need to know more properties of the function $A_{\nu,r}(y)$ and the expansions associated with it.

From the generating equation (0. 2) it follows that the function $A_{\nu,r}(y)$ satisfies the following relationships:

$$(0.4) \qquad \Delta A_{\nu+1,r}(y)=\begin{Bmatrix} 0, & \nu=-1,-2,-3,\cdots \\ 1, & \nu=0 \\ y^{\nu}/\nu!, & \nu=1,2,3,\cdots \end{Bmatrix} \qquad (r=0,1,2,\cdots);$$

$$(0.5) \qquad D_yA_{\nu+1,r}(y)=A_{\nu,r}(y) \qquad (\nu=0,\pm1,\pm2,\cdots,r=0,1,2,\cdots);$$

$$(0.6) \qquad A_{\nu,r}(1-y)=(-1)^{\nu}A_{\nu,r}(y) \quad (\nu=0,\pm1,\pm2,\cdots,r=0,1,2,\cdots).$$

Furthermore, if we introduce the symbol $[y]$ to represent the largest integer that does not exceed y we have

$$(0.7) \qquad \sum_{s=0}^{p-1}A_{\nu+1,r}(y+sp^{-1})=p^{-\nu}A_{\nu+1,[rp^{-1}]}(py) \qquad (\nu=0,\pm1,\pm2,\cdots,r=0,1,2,\cdots).$$

If C_r denotes a circle with center at the origin and of radius $2\pi r+\delta$, $0<\delta<2\pi$, then it follows from (0. 2) that $A_{\nu,r}(y)$ has the alternative definition

$$(0.8) \qquad A_{\nu,r}(y)=(2\pi i)^{-1}\int_{C_r}e^{yt}(e^t-1)^{-1}t^{-\nu}dt \qquad (\nu=0,\pm1,\pm2,\cdots,r=0,1,2,\cdots).$$

This latter equation enables us to relate the function $A_{\nu,r}(y)$ with the Bernoulli-Hurwitz [5] function and enables us to deduce the relation

$$(0.9) \qquad A_{\nu,r}(y)=\begin{Bmatrix} \displaystyle\sum_{k=-r}^{+r}(2k\pi i)^{-\nu}e^{2k\pi yi}=2\sum_{k=1}^{r}(2k\pi)^{-\nu}\cos(2k\pi y-\nu\pi/2), \\ (\nu=-1,-2,-3,\cdots), \\ \displaystyle\sum_{k=-r}^{+r}e^{2k\pi yi}=1+2\sum_{k=1}^{r}\cos 2k\pi y=\frac{\sin(2r+1)\pi y}{\sin\pi y} \\ (\nu=0), \\ \displaystyle A_{\nu,0}(y)+2\sum_{k=1}^{r}(2k\pi)^{-\nu}\cos(2k\pi y-\nu\pi/2), \\ (\nu=1,2,3,\cdots), \end{Bmatrix} \quad (r=0,1,2,\cdots).$$

[5] A. Hurwitz, "Sur l'intégrale finie d'une fonction entière," *Acta Mathematica*, vol. 20 (1897), pp. 285-312; vol. 22 (1899), pp. 179-180.

The equation (0.8) also permits us to deduce the following lemma in the manner of Carmichael [6] or Lindelöf: [7]

LEMMA. *For every positive δ less than 2π there exists a constant K_δ, depending on δ alone, such that*

$$| A_{\nu+1,r}(y) | < K_\delta(2r\pi + \delta)^{-\nu} e^{|y|(2r\pi+\delta)}, \quad (\nu = 0, \pm 1, \pm 2, \cdots, r = 0, 1, 2, \cdots),$$

for all finite y.

In the study of the given difference equation for real variables it will be convenient to associate with $A_{\nu,r}(y)$ for real values of y the periodic function $\bar{A}_{\nu,r}(y)$. It is defined by the equation

$$\bar{A}_{\nu,r}(y) = A_{\nu,r}(y - [y]), \qquad (\nu = 1, 2, 3, \cdots, r = 0, 1, 2, \cdots).$$

In order to proceed with rigor we shall have to discuss separately the situations in the real and the complex domains.

I. Real Variables.

1.1. **The modified Euler-Maclaurin expansion formula.** Let us assume that x, y, z are real variables and that ω is positive. Furthermore, let us use m, $n+1$ to represent positive integers. Let us consider a function $\phi(x)$ which together with its first m derivatives is assumed to be continuous for $b \leqq x \leqq c$, $b + \omega \leqq c$.

Let us contemplate the function

$$P_{\nu,r} = \omega^\nu \int_z^y A_{\nu,r}(y - t + z)\phi^{(\nu)}(x + \omega t)\,dt, \quad (v = -n, -n+1, \cdots, m),$$

where r is a fixed non-negative integer and z, x, y are such that $0 \leqq z \leqq 1$, $b \leqq x \leqq c - \omega$, $b \leqq x + \omega y \leqq c$. Integrating by parts we have at once from (0.5) the relationship

$$P_{\nu,r} = P_{\nu-1,r} + \omega^{\nu-1}\{A_{\nu,r}(z)\phi^{(\nu-1)}(x + \omega y) - A_{\nu,r}(y)\phi^{(\nu-1)}(x + \omega z)\},$$
$$(\nu = -n+1, -n+2, \cdots, m).$$

If we replace ν by $-n+1, -n+2, \cdots, m-1, m$ successively in this last equation and add the resulting equations we have

[6] R. D. Carmichael, "Summation of functions of a complex variable," *Annals of Mathematics* (2), vol. 34 (1933), pp. 349-378.

[7] Ernst Lindelöf, *Le Calcul des Résidus, et ses Applications a la Théorie des Fonctions*, Paris 1905.

$$(1.1)\quad P_{m,r} = P_{-n,r} + \sum_{\nu=-n}^{m-1} \omega^{\nu}\{A_{\nu+1,r}(z)\phi^{(\nu)}(x+\omega y) - A_{\nu+1,r}(y)\phi^{(\nu)}(x+\omega z)\}.$$

If we difference both members of this equation with respect to z and set z equal to zero in the result we have as a consequence of (0.4) an expansion which we incorporate in the following theorem:

THEOREM 1.1. *Let $\phi(x)$ together with its first m, $m \geqq 1$, derivatives be continuous for x in the interval $b \leqq x \leqq c$, $b + \omega \leqq c$. Then $\phi(x)$ has the expansion*

$$(1.2)\qquad \phi(x+\omega y) = \sum_{\nu=-n}^{m-1} \omega^{\nu+1} A_{\nu+1,r}(y) \underset{\omega}{\Delta} \phi^{(\nu)}(x) - R_m + R_{-n}$$

for ω positive, r and n non-negative integers, and for x and y such that $b \leqq x \leqq c - \omega$ and $b \leqq x + \omega y \leqq c$ where

$$R_m = \omega^m \int_0^1 \bar{A}_{m,r}(-t)\phi^{(m)}(x+\omega y+\omega t)\,dt$$
$$- \omega^m \int_0^y A_{m,r}(y-t)[\phi^{(m)}(x+\omega+\omega t) - \phi^{(m)}(x+\omega t)]\,dt,$$
$$R_{-n} = \omega^{-n-1}\int_x^{x+\omega} A_{-n,r}(\{x-t+\omega y\}/\omega)\phi^{(-n)}(t)\,dt.$$

We shall designate the expansion (1.2) the *modified Euler-Maclaurin expansion formula* (see Euler,[8] Maclaurin[9]). This expansion is true independently of the value of n and so n may be taken equal to zero without loss of generality.

Derivation of expansion formulas by integration by parts was studied by Darboux[10] but the ideas presented by Nörlund[11] suggested the foregoing derivation.

If z is positive and m is a positive integer all the hypotheses of Theorem 1.1 are met if $\phi(x) = e^{zx}$, $-\infty \leqq x \leqq c - \omega$. In that case we have

$$\phi^{(\nu)}(x) = z^{\nu}e^{xz}, \qquad (\nu = 0, 1, 2, \cdots).$$

Furthermore, if a is taken to be negative infinity we have

$$\phi^{(-\mu)}(x) = z^{-\mu}e^{xz}, \qquad (\mu = 1, 2, 3, \cdots).$$

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[8] Leonhard Euler, "Methodus generalis summandi progressiones," *Commentarii Academiae Scientiarum Imperialis Petropolitanea*, vol. 6 (1732-33), pp. 68-97.

[9] Maclaurin, *A Treatise of Fluxions*, Edinburgh 1742.

[10] G. Darboux, "Sur les développements en série des fonctions d'une seule variable," *Journal de Mathématiques Pures et Appliquées, Paris* (3), vol. 2 (1876), pp. 291-312.

[11] N. E. Nörlund, "Mémoire sur le calcul aux différences finies," *Acta Mathematica*, vol. 44 (1923), pp. 71-211.

Consequently, we have as a special case of Theorem 1.1 the expansion

$$(1.3)\qquad \omega e^{\omega yz}(e^{\omega z}-1)^{-1}=\sum_{\nu=-n}^{m-1}\omega^{\nu+1}A_{\nu+1,r}(y)z^{\nu}+B_m+B_{-n}$$

for ω and z positive, for $r, n, m-1$ non-negative integers and for real finite y where

$$B_m=\omega^{m+1}z^m\int_0^y A_{m,r}(y-t)e^{\omega tz}dt-\omega^{m+1}z^m(e^{\omega z}-1)^{-1}\int_y^{y+1}\bar{A}_{m,r}(y-t)e^{\omega tz}dt,$$

$$B_{-n}=\omega^{-n+1}z^{-n}(e^{\omega z}-1)^{-1}\int_0^1 A_{-n,r}(y-t)e^{\omega tz}dt=\omega\sum_{k=-r}^{+r}\left(\frac{2k\pi i}{\omega z}\right)^n\frac{e^{2k\pi yi}}{\omega z-2k\pi i}.$$

1.2. **The modified Euler-Maclaurin sum formula.** The formula (1.3) might be used in place of (0.2) to suggest a solution of the difference equation (0.1). Using the suggested expansion we are led to contemplate the function

$$\begin{aligned}(1.4)\quad F_r(x+\omega y|\omega)=&\sum_{\nu=-n}^{m-1}\omega^{\nu+1}A_{\nu+1,r}(y)\phi^{(\nu)}(x)\\&+\omega^{m+1}\int_0^y A_{m,r}(y-t)\phi^{(m)}(x+\omega t)dt\\&+\omega^{m+1}\int_0^\infty \bar{A}_{m,r}(-t)\phi^{(m)}(x+\omega y+\omega t)dt\\&+\omega^{-n}\int_a^x A_{-n,r}(\{x-t+\omega y\}/\omega)\phi^{(-n)}(t)dt\end{aligned}$$

as a consequence of the usual interpretation of the symbolic operators involved. As a result of equation (1.1) it can be shown that $F_r(x+\omega y|\omega)$ is actually a function of $x+\omega y$ and, furthermore, that it is independent of the value of n so long as n is a non-negative integer. Moreover, we can prove the theorem:

THEOREM 1.2. *Let $\phi(x)$ together with its first m, $m\geqq 1$, derivatives be continuous for x such that $x\geqq b$ and, moreover, such that for a particular non-negative integral value of r the integral*

$$\int_0^\infty \bar{A}_{m,r}(-t)\phi^{(m)}(x+\omega t)dt$$

converges for x in the interval $b\leqq x\leqq b+\omega$. Then the function $F_r(x+\omega y|\omega)$ exists and affords a solution of the difference equation (0.1) for positive ω and non-negative n and for x and y such that $x\geqq b$ and $x+\omega y\geqq b$.

It is readily seen that the hypotheses imply the existence of $F_r(x+\omega y|\omega)$. That $F_r(x+\omega y|\omega)$ satisfies the given difference equation is a consequence of Theorem 1.1 all of whose hypotheses are satisfied.

We shall designate the solution (1.4) the *modified Euler-Maclaurin sum formula* (Clearly some distinction must be made between formula (1.2) and formula (1.4)). This formula is true independently of the value of n and without loss of generality n may be taken to be zero. For extensive references to the literature of the Euler-Maclaurin formulas one might see Burkhardt,[12] Runge-Willers,[13] Nörlund,[14] and Walther.[15]

If we introduce the function

$$Q_r(x+\omega y) = \omega^{-n}\int_a^x A_{-n,r}(\{x-t+\omega y\}/\omega)\,\phi^{(-n)}(t)\,dt$$
$$+\sum_{\nu=-n}^{m-1}\omega^{\nu+1}A_{\nu+1,r}(y)\phi^{(\nu)}(x)+\omega^{m+1}\int_0^y A_{m,r}(y-t)\phi^{(m)}(x+\omega t)\,dt$$

it is clear that we may write the function $F_r(x+\omega y|\omega)$ in the form

$$(1.5)\quad F_r(x+\omega y|\omega) = Q_r(x+\omega y)$$
$$-\omega\sum_{s=0}^{\infty}\{\phi(x+\omega y+s\omega)-\underset{\omega}{\Delta}\,Q(x+\omega y+s\omega)\}$$

so that the modified Euler-Maclaurin sum appears as a particualr modified progressive sum of $\phi(x)$. It is the only solution of (0.1) having the property

$$\lim_{h\to 0}\{F_r(x+\omega y+h\omega|\omega)-Q_r(x+\omega y+h\omega)\}=0.$$

Let $\phi(x)$ together with its derivatives of all orders be continuous for x such that $x\geqq b$ and, moreover, such that the integrals

$$\int_0^\infty \bar{A}_{\nu,r}(-t)\phi^{(\nu)}(x+\omega t)\,dt, \qquad (\nu=N+1, N+2,\cdots),$$

converge for x in the interval $b\leqq x\leqq b+\omega$ and possess finite limits as ω approaches zero. Then, considered as a function of ω, $F_r(x+\omega y|\omega)$ is such that

$$F_r(x+\omega y|\omega)\sim\omega^{-n}\int_a^x A_{-n,r}(\{x-t+\omega y\}/\omega)\phi^{(-n)}(t)\,dt$$
$$+\sum_{\nu=-n}^{m-1}\omega^{\nu+1}A_{\nu+1,r}(y)\phi^{(\nu)}(x),\qquad \omega>0.$$

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[12] H. Burkhardt, "Restglied der Euler-Maclaurinschen Summenformel," *Encyklopädie der Mathematischen Wissenschaften*, II A 12, Nr. 105, pp. 1324-1337.

[13] C. Runge-Fr. A. Willers, "Die Eulersche Formel," *Encyklopädie der Mathematischen Wissenschaften*, II C 2, Nr. 9, pp. 91-96.

[14] N. E. Nörlund, "Einfache Summen," *Encyklopädie der Mathematischen Wissenschaften*, II C 7, Nr. 10, pp. 711-716.

[15] Alwin Walther, "Numerische Integration," *Pascals Repertorium der Höhern Mathematik*, 2 Auflage 1929, I 3 XXIII, 4, pp. 1200-1210.

As a consequence of the property (0. 7) of the Bernoulli-Hurwitz function the modified Euler-Maclaurin sum enjoys the relation

$$\sum_{s=0}^{p-1} F_r(x + \omega s p^{-1}|\omega) = pF_{[rp^{-1}]}(x|\omega p^{-1}) \tag{1. 6}$$

whenever the left member exists.

Let $\phi(x)$ together with its first m, $m \geqq 1$, derivatives be continuous for x such that $x \geqq b$ and, moreover, such that for a particular non-negative value of r the integral

$$\int_0^\infty \bar{A}_{m,r}(-t)\phi^{(m)}(x + \omega t)\,dt$$

converges uniformly for x such that $b \leqq x \leqq b + \omega$. Then we have

$$\omega^{-1}\int_x^{x+\omega} F_r(t|\omega)\,dt = \int_a^x \phi(t)\,dt. \tag{1. 7}$$

Let $\phi(x)$ together with its first m, $m \geqq 1$, derivatives be continuous for x such that $x \geqq b$ and, moreover, such that the series

$$\sum_{s=0}^{\infty} \phi^{(m)}(x + s\omega)$$

converges uniformly in the interval $b \leqq x \leqq b + \omega$. Then, if $F_r{}^{\mu}(x|\omega)$ denotes the modified Euler-Maclaurin sum of $\phi^{(\mu)}(x)$, the first m derivatives of the modified Euler-Maclaurin sum of $\phi(x)$ exist and are such that

$$F_r{}^{(\mu)}(x|\omega) = F_r{}^{\mu}(x|\omega) + \sum_{\nu=0}^{\mu-1} \omega^{-\nu} A_{-\nu,r}(\{x + \omega y - a\}/\omega)\phi^{(\mu-\nu-1)}(a), \tag{1. 8}$$
$$(\mu = 0, 1, \cdots, m).$$

Under the conditions named in the foregoing paragraph together with the condition that $\phi(x)$ have a zero of order m at some point a, $a \geqq b$, the modified Euler-Maclaurin sum of $\phi^{(\mu)}(x)$ equals the μ-th derivative of the modified Euler-Maclaurin sum of $\phi(x)$.

The proofs of the theorems listed above follow from Theorem 1. 2 and the properties of the Bernoulli-Hurwitz function by allowing y to be the variable of integration, summation, and differentiation.

As an example of the modified Euler-Maclaurin sum let us take $\phi(x)$ equal to $x^{\mu}/\mu!$, $\mu \geqq 0$. It is clear that all the conditions of Theorem 1. 2 are satisfied if m is taken to be $\mu + 1$. Then if a is taken to be zero we have

$$F_r(x|\omega) = \omega^{\mu+1} A_{\mu+1,r}(x/\omega)$$

and the Bernoulli-Hurwitz function itself is seen to be the sum of x^{μ} afforded by the modified Euler-Maclaurin sum formula. This leads us to expect the Euler-Maclaurin sum is intimately related to a modification of Nörlund's [11] principal sum. We propose to indicate this relationship more precisely in the next section.

1.3. **Modified principal solutions.** Under suitable hypotheses $\phi(x)$ has the expansion

$$\phi(x+s\omega)=\omega^{-1}\int_{s\omega}^{s\omega+\omega}\phi(t)\,dt+2\omega^{-1}\sum_{k=1}^{\infty}\int_{s\omega}^{s\omega+\omega}\cos[2k\pi(x-t)/\omega]\phi(t)\,dt$$

and consequently $\phi(x+s\omega)$ is approximated by the quantity

$$\omega^{-1}\int_{s\omega}^{s\omega+\omega}\phi(t)\,dt+2\omega^{-1}\sum_{k=1}^{r}\int_{s\omega}^{s\omega+\omega}\cos[2k\pi(x-t)/\omega]\phi(t)\,dt$$
$$=\omega^{-1}\int_{s\omega}^{s\omega+\omega}A_{0,r}(\{x-t\}/\omega)\phi(t)\,dt.$$

Now a solution of (0.1) is afforded by the series

$$-\omega\sum_{s=0}^{\infty}\phi(x+s\omega) \tag{1.9}$$

if it converges; if it diverges one might naturally contemplate the difference

$$\int_{a}^{\infty}A_{0,r}(\{x-t\}/\omega)\phi(t)\,dt-\omega\sum_{s=0}^{\infty}\phi(x+s\omega)$$

in order to find a valid solution of (0.1) (see Kronecker).[16] Nörlund [11] has contemplated the difference for the special case where r is zero.

If we follow the procedure which Nörlund has given for the special case we would define the modified principal solution $F_r(x|\omega)$ to be the limit of $F_r(x|\omega;\eta)$ as η approaches zero where

$$F_r(x|\omega;\eta)=\int_{a}^{\infty}A_{0,r}(\{x-t\}/\omega)e^{-\eta t}\phi(t)\,dt-\omega\sum_{s=0}^{\infty}\phi(x+s\omega)e^{-\eta(x+s\omega)},\quad \eta>0.$$

The relations (1.6) and (1.7) which held for the modified Euler-Maclaurin sum are seen to hold for the modified principal solution as a consequence of this definition.

Furthermore, the procedure of Nörlund leads to the theorem:

[16] Leopold Kronecker, "Ueber eine bei Anwendung der partiellen Integration nützliche Formel," *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin* (1885), pp. 841-862; *Leopold Kronecker's Werke*, vol. 5 (1930), pp. 267-294.

THEOREM 1.3. *Let $\phi(x)$ together with its first m, $m \geqq 1$, derivatives be continuous for $x \geqq b$ and such that for a fixed value of r the integral*

$$\int_0^\infty \bar{A}_{m,r}(-t)\phi^{(m)}(x+\omega t)\,dt$$

converges uniformly for x such that $b \leqq x \leqq b+\omega$. Then the modified principal solution exists as a continuous function of x and is afforded by the modified Euler-Maclaurin sum formula

$$F_r(x+\omega y|\omega) = \int_a^x A_{0,r}(\{x-t+\omega y\}/\omega)\phi(t)\,dt + \sum_{\nu=1}^m A_{\nu,r}(y)\omega^\nu\phi^{(\nu-1)}(x) + \omega^{m+1}\int_0^\infty \bar{A}_{m,r}(y-t)\phi^{(m)}(x+\omega t)\,dt$$

for $x \geqq b$, $0 \leqq y \leqq 1$ provided $a \geqq b$.

Solutions valid for $x \leqq b$ would be obtained by proceeding from the series

$$\omega\sum_{s=1}^\infty \phi(x-s\omega)$$

rather than from the series (1.9).

II. Complex Variables.

We shall confine attention to integral functions. A classification of integral functions developed in an unpublished paper by Carmichael, Martin, and Bird is used.

2.1. **A classification of integral functions.** If the sequence of positive numbers $t_0, t_1, t_2, \cdots$ has, in addition to the property that

$$\lim_{n\to\infty}(t_n)^{1/n} = \infty, \tag{2.1}$$

the property that for every analytic function $F(x)$ we have

$$\limsup_{n\to\infty} |\, t_n F^{(n)}(x_0)/n!\,|^{1/n} = \limsup_{n\to\infty} |\, t_n F^{(n)}(x_1)/n!\,|^{1/n}$$

for any two regular points x_0 and x_1 of $F(x)$, then we shall say that $\{t_n\}$ is an I-*sequence.* If $\{t_n\}$ is an I-sequence and $F(x)$ is an analytic function such that at the regular point x_0 we have

$$\limsup_{n\to\infty} |\, t_n F^{(n)}(x_0)/n!\,|^{1/n} = q, \qquad 0 \leqq q < \infty,$$

then $F(x)$ is an integral function and we shall say it is a function of *sort* $\{t_n\}$, *type* q.

It is convenient to introduce the particular function $E(x)$ of sort $\{t_n\}$, type 1, defined by the expansion

$$E(x) = \sum_{\nu=0}^{\infty} x^{\nu}/t_{\nu}.$$

It is also useful to introduce the sequences $\{\lambda_n\}$ and $\{\rho_n\}$ such that

$$\lambda_0 = t_0,\ \lambda_1 = t_1,\ \lambda_n = t_n/t_{n-1}, \qquad (n = 2, 3, 4, \cdots),$$
$$\rho_0 = t_0,\ \rho_n = (t_n)^{1/n}, \qquad (n = 1, 2, 3, \cdots).$$

We have the following theorems:

THEOREM 2. 1. *A necessary and sufficient condition that a sequence* $\{t_n\}$ *shall be an* I-*sequence is that it shall satisfy condition* (2. 1) *and the condition*

$$\lim_{n\to\infty} (t_n E^{(n)}(a)/n!)^{1/n} = 1.$$

COROLLARY. *A necessary condition that* $\{t_n\}$ *be an* I-*sequence is that*

$$\lim_{n\to\infty} \rho_n = \infty.$$

A sufficient condition that $\{t_n\}$ *be an* I-*sequence is that*

$$\lim_{n\to\infty} \lambda_n = \infty.$$

THEOREM 2. 2. *Let* $F(x)$ *be any integral function. If* $F(x)$ *is a polynomial then* $F(x)$ *is of sort* $\{t_n\}$, *type* 0, *for every* I-*sequence* $\{t_n\}$. *If* $F(x)$ *is not a polynomial then there is associated with* $F(x)$ *an* I-*sequence* $\{t_n\}$ *such that* $F(x)$ *is of sort* $\{t_n\}$, *type* 1.

COROLLARY. *In order to classify all integral functions it is sufficient to consider only the* I-*sequences* $\{t_n\}$ *for which* ρ_n *is monotonic.*

THEOREM 2. 3. *Let* $\{t_n\}$ *be an* I-*sequence such that* ρ_n *is monotonic. Then if* $F(x)$ *is of sort* $\{t_n\}$, *type* q, *we have for every positive* ϵ *an* M_ϵ *such that*

$$| t_n F^{(n)}(x)/n! | < M_\epsilon (q+\epsilon)^n (1 + 1/\delta)^n E([q+\epsilon][1+\delta] \, | x |)$$

for every finite x *and positive* δ.

THEOREM 2. 4. *Let* $F(x)$ *be a function of sort* $\{t_n\}$, *type* q. *It is necessary and sufficient in order that the derivative of* $F(x)$ *be a function of sort* $\{t_n\}$, *type* q, *that*

$$\lim_{n\to\infty} (\lambda_n)^{1/n} = 1.$$

2.2. **The generalized Euler-Maclaurin expansion formula.** We consider x, y, z and ω as complex variables. We shall assume the line integrals are taken along straight line paths. Let us assume that m and μ are non-negative integers such that μ equals or exceeds m.

As in Section 1.1 let us consider the function

$$P_{\nu,r}=\omega^{\nu}\int_{z}^{y}A_{\nu,r}(y-t+z)\phi^{(\nu)}(x+\omega t)\,dt,\ (\nu=0,1,2,\cdots,r=0,1,2,\cdots).$$

If we integrate by parts and then replace ν by $m+1, m+2,\cdots,\mu+1$ in the result we have upon addition

$$P_{\mu+1,r}=P_{m,r}+\sum_{\nu=m}^{\mu}\omega^{\nu}\{A_{\nu+1,r}(z)\phi^{(\nu)}(x+\omega y)-A_{\nu+1,r}(y)\phi^{(\nu)}(x+\omega z)\},\quad (r=0,1,2,\cdots).$$

Let us difference both members of this equation with respect to z and then set z equal to zero. For abbreviation we use

$$R_{\nu,r}=[\Delta P_{\nu,r}]_{z=0},\qquad (\nu=0,1,2,\cdots,r=0,1,2,\cdots).$$

Then we have the formula

$$\sum_{\nu=m}^{\mu}\omega^{\nu+1}A_{\nu+1,r}(y)\,\underset{\omega}{\Delta}\,\phi^{(\nu)}(x)+R_{\mu+1,r}-R_{m,r}=\begin{cases}0, & m>0,\\ \phi(x+\omega y), & m=0\end{cases}\quad (r=0,1,2,\cdots).$$

Let the integer r depend upon ν for its value and let us designate this relationship by the notation r_ν, $\nu=0,1,2,\cdots$. If we have

$$r_\nu=r_m,\qquad (\nu=m,m+1,\cdots,\mu),$$

the preceding formula assumes the form

$$\sum_{\nu=m}^{\mu}\omega^{\nu+1}A_{\nu+1,r_\nu}(y)\,\underset{\omega}{\Delta}\,\phi^{(\nu)}(x)+R_{\mu+1,r_\mu}-R_{m,r_m}=\begin{cases}0, & m>0,\\ \phi(x+\omega y), & m=0.\end{cases}$$

Let us arrange successive values of r_ν which are equal in groups. Let us designate the number in successive groups by ν_1, $\nu_2-\nu_1$, $\nu_3-\nu_2,\cdots$. It is convenient to adopt the convention that

$$\sum_{k=r+1}^{\rho}f(k)=\sum_{k=0}^{\rho}f(k)-\sum_{k=0}^{r}f(k),\qquad r\geqq\rho\geqq 0.$$

Now let us take m equal successively to $0, \nu_1,\cdots,\nu_j$ where

$$\nu_j\leqq n<\nu_{j+1}$$

and let us take μ equal successively to $\nu_1 - 1, \nu_2 - 1, \cdots, n$. If we add the resulting expansions we obtain as a consequence of the properties of the Bernoulli-Hurwitz functions the expansion

$$(2.2) \qquad \phi(x + \omega y) = I_{n+1,r_n} + \sum_{\nu=0}^{n} \omega^{\nu+1} A_{\nu+1,r_\nu}(y) \underset{\omega}{\Delta} \phi^{(\nu)}(x) + R_{n+1,r_n}$$

where

$$(2.3) \qquad R_{n+1,r_n} = \omega^{n+1}\{\int_1^y A_{n+1,r_n}(y - t + 1)\phi^{(n+1)}(x + \omega t)\,dt - \int_0^y A_{n+1,r_n}(y - t)\phi^{(n+1)}(x + \omega t)\,dt\},$$

$$(2.4) \qquad I_{n+1,r_n} = \int_0^1 \{\phi(x + \omega t) + 2 \sum_{\nu} \sum_{k=r_{\nu-1}+1}^{r_\nu} \omega^\nu (2k\pi)^{-\nu} \cos[2k\pi(y - t) - \nu\pi/2]\phi^{(\nu)}(x + \omega t)\}\,dt.$$

In this latter integral ν runs over the values $0, \nu_1, \nu_2, \cdots, \nu_j$ with r_{-1} equal to zero.

We shall refer to (2.2) as the *generalized Euler-Maclaurin expansion.*

Let us assume that $\phi(x)$ is of sort $\{t_n\}$, type q, where $\{t_n\}$ is an I-sequence such that ρ_n is monotonic. Let us assume that ω is held fixed while x and y are confined to the finite regions X and Y of the complex plane. Then the lengths of the paths of the integrals in (2.2) as well as the absolute values of each of the quantities $y - t + 1$, $y - t$, $x + \omega t$ are dominated by the constant a. Furthermore, as a consequence of Theorem 2.3 and the Lemma in the Introduction we have

$$\limsup_{n\to\infty} (R_{n+1,r_n})^{1/n} < 1$$

if r_n is suitably restricted. Indeed it is sufficient that we have

$$q\,|\,\omega\,|\,(1 + \gamma)/(2\pi e \rho_{n+1}) \leqq (r_n + 1)/(n + 1) \leqq [\log \rho_{n+1}]^\alpha$$

where α is any positive constant less than 1 and γ is any positive constant. The proof is facilitated if a division of cases is made at

$$r_n + 1 = (n + 1)\xi$$

where ξ is any positive constant which satisfies the inequality

$$2\pi a\xi - \log(1 + \gamma) < 0.$$

If we restrict the sequence $\{t_n\}$ to be such that

$$\lim_{n\to\infty} (\lambda_n)^{1/n} = 1,$$

under the conditions named above we also have

$$\limsup_{n\to\infty} | \omega^{n+1} A_{n+1,r_n}(y) \underset{\omega}{\Delta} \phi^{(n)}(x) |^{1/(n+1)} < 1.$$

Consequently, we have the theorem:

THEOREM 2. 5. *Let $\phi(x)$ be a function of sort $\{t_n\}$, type q, where $\{t_n\}$ is an* I-*sequence such that ρ_n is monotonic and*

$$\lim_{n\to\infty} (\lambda_n)^{1/n} = 1.$$

Then for x, y confined to finite regions X, Y and for a fixed value of ω, $\phi(x)$ has the expansion

$$\phi(x + \omega y) = \lim_{n\to\infty} I_{n+1,r_n} + \sum_{\nu=0}^{\infty} \omega^{\nu+1} A_{\nu+1,r_\nu}(y) \underset{\omega}{\Delta} \phi^{(\nu)}(x) \tag{2.5}$$

where for all n greater than some fixed value we have

$$q \mid \omega \mid (1+\gamma)/(2\pi e \rho_{n+1}) \leqq (r_n + 1)/(n+1) \leqq [\log \rho_{n+1}]^{\alpha},$$
$$0 < \alpha < 1, \qquad 0 < \gamma,$$

and where the integral I_{n+1,r_n} is defined in the preceding remarks. This expansion converges uniformly in the regions indicated. Furthermore, this expansion is unique for any particular sequence $\{r_n\}$.

In particular, one may choose r_n to be n. This is the choice Hurwitz [5] made in his study of the equation (0. 1) for integral functions.

For functions of exponential type q (see Carmichael),[17] i. e., $\lambda_n = n$, it is sufficient according to Theorem 2. 5 to define r_n by the inequalities

$$(q \mid \omega \mid /2\pi) - 1 < r_n \leqq q \mid \omega \mid /2\pi.$$

In such a case Theorem 2. 5 recovers the theorem of Carmichael [6] for expansions of functions of exponential type q in series of Bernoulli-Hurwitz functions. For the case where $\lambda_n = n^{1/t}$, $1 \leqq t < \infty$, it is sufficient that

$$(q \mid \omega \mid /2\pi) n^{1-1/t} < r_n \leqq (q \mid \omega \mid /2\pi) n^{1-1/t}, \qquad (n = 1, 2, 3, \cdots).$$

When q is different from zero this gives expansions of functions of order t, normal type q (see Bieberbach).[18]

[17] R. D. Carmichael, "Functions of exponential type," *Bulletin of the American Mathematical Society*, vol. 41 (1934), pp. 241-261.

[18] Ludwig Bieberbach, "Grundbegriffe," *Encyklopädie der Mathematischen Wissenschaften*, II C 4, Nr. 29, pp. 429-431.

2.3. **The generalized Euler-Maclaurin sum formula.** In this section we propose to modify the results of Guichard,[19] Appell,[20] Hurwitz,[5] Weber,[21] Barnes,[22] Carmichael [23] so that instead of finding an integral solution of the equation (0.1) when the known function is integral we shall find an integral solution of the same character as the known function.

We state the result in two theorems:

THEOREM 2.6. *Let $\phi(x)$ be an integral function of sort $\{t_n\}$, type q, $0 < q < \infty$, where $\{t_n\}$ is an* I-*sequence such that*

$$\lim_{n\to\infty} \lambda_n = \infty, \qquad \limsup_{n\to\infty} (\lambda_{n+1}/\lambda_n)^n = \theta e < e.$$

Then there exists a solution of the difference equation (0.1) *which is of sort $\{t_n\}$, type q, and indeed one such solution is afforded by the function*

$$F(x|\omega) = \sum_{\nu=0}^{\infty} \phi^{(\nu)}(0) A_{\nu+1,r_\nu}(x/\omega)\omega^{\nu+1}, \tag{2.6}$$

where r_n is the integer defined by the relation

$$r_n = [q \mid \omega \mid n/2\pi\lambda_n], \qquad (n = 0, 1, 2, \cdots).$$

THEOREM 2.7. *Let $\phi(x)$ be an integral function of sort $\{t_n\}$, type q, $0 < q < \infty$, where $\{t_n\}$ is an* I-*sequence such that*

$$\lim_{n\to\infty} n/\lambda_n = \tau < \infty.$$

Then there exists a solution of the difference equation (0.1) *which is of sort $\{t_n\}$, type q, and indeed one such solution is afforded by*

$$F(x|\omega) = \sum_{\nu=0}^{\infty} \phi^{(\nu)}(0) A_{\nu+1,r}(x/\omega)\omega^{\nu+1} \tag{2.7}$$

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[19] G. Guichard, "Sur la résolution de l'équation aux différences finies $G(x+1) - G(x) = H(x)$," *Annales Scientifiques de l'École Normale Supérieure* (3), vol. 4 (1887), pp. 361-380.

[20] P. Appell, "Sur les fonctions périodiques de deux variables," *Journal de Mathématiques Pures et Appliquées, Paris* (4), vol. 7 (1891), pp. 157-219.

[21] Heinrich Weber, "Über Abel's Summation endlicher Differenzenreihen," *Acta Mathematica,* vol. 27 (1903), pp. 225-233.

[22] E. W. Barnes, "The linear difference equation of the first order," *Proceedings of the London Mathematical Society* (2), vol. 2 (1904), pp. 439-469.

[23] R. D. Carmichael, "On the theory of linear difference equations," *American Journal of Mathematics,* vol. 35 (1913), pp. 163-182.

where r is the integer defined by the relation

$$r = [q \mid \omega \mid \tau/2\pi].$$

It is clear that Theorem 2. 6 for the special case $\lambda_n = n^{1/t}$, $1 \leqq t < \infty$ refines the result which Whittaker[24] has given for functions of order t.

Obviously, if the series (2. 6) converges it defines a solution of (0. 1). It follows from the hypotheses of Theorem 2. 6 that we have

$$\lim_{n \to \infty} n/\lambda_n = \infty.$$

Consequently, r_n becomes infinite with n (although not necessarily monotonically) and for any positive numbers δ, δ' we have the inequalities

$$\begin{aligned} 2\pi r_n + \delta &< (q \mid \omega \mid n/\lambda_n) + 2\delta < (q + 2\delta) \mid \omega \mid n/\lambda_n \\ 2\pi(r_n + 1) - \delta' &> (q \mid \omega \mid n/\lambda_n) - \delta' > (q - \delta') \mid \omega \mid n/\lambda_n \end{aligned}$$

for all n after some fixed value. This together with the Lemma of the Introduction and Theorem 2. 3 enables us to prove that the series (2. 6) converges absolutely and uniformly for x confined to a finite region X.

Consideration of the successive derivatives of the integral function defined by the series (2. 6) proves that the function is of sort $\{t_n\}$, type not exceeding q. We are led to a contradiction of our hypotheses if $F(x|\omega)$ is of sort $\{t_n\}$, type less than q. This establishes Theorem 2. 6.

The proof of Theorem 2. 7 follows the same method.

Corollary. *The solutions $F(x|\omega)$ defined in Theorem* 2. 6 *and Theorem* 2. 7 *are such that we have*

$$F(x - \omega | - \omega) = F(x|\omega).$$

This is an immediate consequence of the relation (0. 6).

2. 4. **Modified principal solutions.** In a manner analogous to that found in Section 1. 3 we may modify Nörlund's definition of principal solution in the complex domain. With this modification many of Nörlund's results may be extended. In particular, we have the theorem:

Theorem 2. 8. *Let $\phi(x)$ be an integral function of exponential type q, $0 \leqq q < \infty$. Then the modified principal solution $F_r(x|\omega)$ exists as an analytic function of x for all finite x and of ω for all ω in the interior of the circle*

[24] J. M. Whittaker, "On the asymptotic periods of integral functions," *Proceedings of the Edinburgh Mathematical Society* (2), vol. 3 (1932-33), pp. 241-258.

with center at the origin and of radius $2(r+1)\pi/q$ *with a neighborhood about the origin deleted if* $r > 0$ (*If* $q = 0$, *the region of validity is the* ω*-plane with a neighborhood about the origin deleted*). *Furthermore, if* ω *is confined to the annular ring*

$$2r\pi \leqq |\,\omega\,|\, q < 2(r+1)\pi,$$

the modified principal solution is of exponential type q.

COROLLARY. *Under the conditions named in the preceding theorem the modified principal solution may be exhibited in the forms:*

$$\begin{aligned} F_r(x|\omega) &= \int_a^{x-a\omega} A_{0,r}(\{x-t\}/\omega)\phi(t)\,dt \\ &\quad + \omega\int_{-\alpha}^{-\alpha+\infty i} \phi(x+\omega t)e^{2r\pi t i}(1-e^{-2\pi t i})^{-1}\,dt \\ &\quad + \omega\int_{-\alpha}^{-\alpha-\infty i} \phi(x+\omega t)e^{-2r\pi t i}(1-e^{2\pi t i})^{-1}\,dt, \end{aligned}$$

$$\begin{aligned} F_r(x|\omega) &= (2r+1)(2\pi i)^{-1}\int_{C_2}\int_a^{x+\omega t} A_{0,r}(\{x-t\}/\omega)\phi(u)\,du \\ &\quad \times \{\pi \csc(2r+1)\pi t\}^2\,dt, \end{aligned}$$

$$\begin{aligned} F_r(x|\omega) &= \int_a^{x} A_{0,r}(\{x-t\}/\omega)\phi(t)\,dt - \omega\phi(x)2^{-1} \\ &\quad + i\omega\int_0^{\infty}\{\phi(x+i\omega t)-\phi(x-i\omega t)\}e^{-2r\pi t}(1-e^{2\pi t})^{-1}\,dt, \end{aligned} \tag{2.8}$$

$$\begin{aligned} F_r(x|\omega) &= \int_a^{x} A_{0,r}(\{x-t\}/\omega)\phi(t)\,dt \\ &\quad - \omega\phi(x)2^{-1} + \sum_{\nu=1}^{\infty} A_{\nu+1,r}(0)\omega^{\nu+1}\phi^{(\nu)}(x), \end{aligned}$$

$$F_r(x|\omega) = \sum_{\nu=1}^{\infty} \omega^{\nu}\phi^{(\nu-1)}(0)A_{\nu+1,r}(x/\omega), \tag{2.9}$$

where a *is an arbitrary constant,* α *is a positive constant less than one, and* C_2 *is the contour through the point* $-\alpha$ *along a line parallel to the axis of imaginaries and traversing it in the negative sense. In the last formula we have taken the arbitrary constant equal to zero.*

We might refer to (2.8) as the *modified Abel sum formula* (see Abel,[25] also Plana).[26]

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[25] Niels Henrik Abel, "Solution de quelques problèmes à l'aide d'intégrales définies," *Magazin for Naturvidenskaberne, Christiana* (1), vol. 2 (1823), pp. 55-63; *Oeuvres Complètes de Niels Henrik Abel*, Ed. 2, vol. 1, Christiana (1881), pp. 11-27.

[26] Plana, "Note sur une nouvelle expression analytique des nombres Bernoulliens, propre à exprimer en termes finis la formule générale pour la sommation des suites," *Memorie della Reale Accademia delle Scienze di Torino*, vol. 25 (1820), pp. 403-418.

The formula (2. 9) was exhibited as a solution of the difference equation (0. 1) in Theorem 2. 7. In this connection one is led to contemplate as a possible solution of equation (0. 1) the function defined by the series

$$F_r(x+\omega y|\omega)=\int_a^x A_{0,r}(\{x-t+\omega y\}/\omega)\phi(t)\,dt+\sum_{\nu=1}^{\infty}A_{\nu,r}(y)\omega^{\nu}\phi^{(\nu-1)}(x),$$
$$(r=0,1,2,\cdots).$$

If $\phi(x)$ is of exponential type q this function exists as an analytic function of x, y, ω. If we restrict ω to the annular ring

$$2r\pi \leqq |\,\omega\,|\,q < 2(r+1)\pi$$

it is seen that $F_r(x+\omega y|\omega)$ affords a solution of exponential type q considered as a function of x or of y. This can be established independently of the definition of principal solution.

It is well to note that the modified principal solution extends the range of validity of solutions in the ω-plane. This is rather nicely exhibited by consideration of the particular function $\phi(x)=e^{tx}$.

The extension of our results to functions of two or more variables in the way in which Nörlund,[27] Baten,[28] and others have led is direct but the results become very involved.

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[27] N. E. Nörlund, "Mémoire sur les polynomes de Bernoulli," *Acta Mathematica*, vol. 43 (1922), pp. 121-196.

[28] W. D. Baten, "A remainder for the Euler-Maclaurin summation formula in two independent variables," *American Journal of Mathematics*, vol. 54 (1932), pp. 265-275.

ON A FUNDAMENTAL THEOREM IN MATRIC THEORY.[1]

By C. C. MacDuffee.

1. The following theorem was first proved by Muth[2] when A is a bilinear form in $2n$ contragradient variables. It was given for a general matrix by Kreis,[3] and by a number of later writers.[4] It is believed, however, that the present proof is the shortest.

Let A be a matrix of order n with elements in a field $\mathfrak{F}^*$ having the single elementary divisor $(\lambda - a)^n$ where a is in $\mathfrak{F}^*$. Let ϕ be any polynomial with coefficients in $\mathfrak{F}^*$.

Theorem 1. *If the first number of the sequence*

$$\phi'(a), \phi''(a), \cdots, \phi^{(n-1)}(a), 1$$

which is not 0 *is the i-th, define q and r by the relations*

$$n = qi + r \qquad 0 \leqq r < i.$$

Then $\phi(A)$ *has the elementary divisors* $[\lambda - \phi(a)]^{q+1}$ *taken r times and* $[\lambda - \phi(a)]^q$ *taken* $i - r$ *times.*

Let A be taken in the Jordan normal form. Then

$$\phi(A) = \left\| \begin{matrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_0 \end{matrix} \right\| \tag{1}$$

where $a_0 = \phi(a)$, $a_i = (1/i!)\phi^{(i)}(a)$. Using the notation of Turnbull and Aitken,[5] write

$$\phi(A) = a_0 I + a_1 U + a_2 U^2 + \cdots + a_{n-1} U^{n-1}$$

where

$$U = (\delta_{r+1,s}), \qquad U^i U^j = U^{i+j}, \qquad U^n = 0.$$

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[1] Presented to the American Mathematical Society, April 10, 1936.

[2] P. Muth, *Journal für reine und angewandte Mathematik*, vol. 125 (1903), p. 291.

[3] H. Kreis, *Contribution a la théorié des systémes linéaires*, Zurich, 1906, p. 47.

[4] A number of other references are given by N. H. McCoy, *American Journal of Mathematics*, vol. 57 (1935), pp. 491-502.

[5] Turnbull and Aitken, *Canonical Matrices*, Blackie and Son, 1932, p. 62.

If $a_1 = a_2 = \cdots = a_{i-1} = 0$, $a_i \neq 0$, then

$$\phi(A) - a_0 I = a_i U^i + \cdots, \qquad [\phi(A) - a_0 I]^k = a_i^k U^{ik} + \cdots.$$

The nullity (order minus rank) of U^h is h, so the nullities of the powers of $\phi(A) - a_0 I$ are

$$i, 2i, 3i, \cdots, qi, n,$$

and the Weyr characteristic[6] of $\phi(A)$ relative to the root $a_0 = \phi(a)$ is $(i, i, \cdots, i, r)$. The Segre characteristic of $\phi(A)$ is the conjugate partition of n, namely $(q+1, \cdots, q+1, q, \cdots, q)$ where the number of $q+1$'s is r. This proves the theorem.

2. McCoy[7] has recently extended this theorem to matrices having elements in a general field. The following proof, using results from algebraic number theory, is more direct.

Suppose that $\mathfrak{F}$ is any field, and that the matrix A of order n has elements in $\mathfrak{F}$ and that the polynomial ϕ has coefficients in $\mathfrak{F}$. In the polynomial ring $\mathfrak{F}[\lambda]$, let A have the single elementary divisor $[p(\lambda)]^k$ where $p(\lambda)$ is irreducible of degree h, $n = hk$.

Let $\Phi(\lambda) = 0$ be the equation of degree h obtained by applying ϕ as a Tschirnhausen transformation to the roots of $p(\lambda) = 0$. That is, the roots of $\Phi = 0$ are the functions $\phi(\lambda)$ of the roots of $p(\lambda) = 0$. Then

$$\Phi(\lambda) = [\psi(\lambda)]^m, \tag{2}$$

where $\psi(\lambda)$ has coefficients in $\mathfrak{F}$, is irreducible in $\mathfrak{F}$, and $h = mt$.

THEOREM 2. *If the first function of the sequence*

$$\phi'(\lambda), \phi''(\lambda), \cdots, \phi^{(k-1)}(\lambda), 1$$

which is not divisible by $p(\lambda)$ is the i-th, define q and r by the relations

$$k = qi + r \qquad 0 \leqq r < i.$$

Then $\phi(A)$ has the elementary divisors $[\psi(\lambda)]^{q+1}$ taken mr times and $[\psi(\lambda)]^q$ taken $m(i-r)$ times, where $\psi(\lambda)$ is defined by (2).

Consider $\mathfrak{F}$ extended to a field $\mathfrak{F}^*$ in which

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_h).$$

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[6] Turnbull and Aitken, *loc. cit.*, p. 80.

[7] N. H. McCoy, *loc. cit.* McCoy's treatment is more explicit in actually exhibiting the transforming matrix.

Then A has the elementary divisors $(\lambda-\lambda_j)^k$. Since $p(\lambda)$ is irreducible, $\phi^{(g)}(\lambda_j)=0$ if and only if $p(\lambda)$ divides $\phi^{(g)}(\lambda)$. Thus for each elementary divisor $(\lambda-\lambda_j)^k$, the same i, q and r are obtained. Then by Theorem 1 the elementary divisors of $\phi(A)$ relative to $\mathfrak{F}^*$ are $[\lambda-\phi(\lambda_j)]^{q+1}$ repeated r times and $[\lambda-\phi(\lambda_j)]^q$ repeated $i-r$ times, $(j=1,2,\cdots,h)$.

It may not be true that the $\phi(\lambda_i)$ are distinct, but if they are not, they fall into t sets of $m=h/t$ equal values each, and the product

$$\psi(\lambda)=\Pi[\lambda-\phi(\lambda_j)],$$

where $\phi(\lambda_j)$ ranges over t distinct values, has coefficients in $\mathfrak{F}$ and is irreducible in $\mathfrak{F}$.[8]

The invariant factors of $\phi(A)$ are the same for $\mathfrak{F}$ as for $\mathfrak{F}^*$, and by the usual rule for forming them from the elementary divisors there are mr of them of the form $[\psi(\lambda)]^{q+1}$ and $m(i-r)$ of them of the form $[\psi(\lambda)]^q$. Since $\psi(\lambda)$ is irreducible in $\mathfrak{F}$, these are the elementary divisors of $\phi(A)$ relative to $\mathfrak{F}$.

The function $\psi(\lambda)$ is readily calculated as follows. Let B be the companion matrix of $p(\lambda)=0$. Then $\Phi(\lambda)=|\phi(B)-\lambda I|$.[9] Then $\psi(\lambda)$ is $\Phi(\lambda)$ divided by the g. c. d. of $\Phi(\lambda)$ and $\Phi'(\lambda)$.

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[8] See any standard work on algebraic number theory, as E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale,* Teubner 1918, p. 11, Theorem 33. Or L. W. Reid, *Elements of the Theory of Algebraic Numbers,* Macmillan 1910, p. 273.

[9] C. C. MacDuffee, "The theory of matrices," *Ergeb. Math.*, II, vol. 5, Springer 1933, p. 24.

A GENERALIZED LAMBERT SERIES.[1]

By Mary Cleophas Garvin.

I. *Introduction.* In 1771 Lambert[2] introduced the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{nz^n}{1 - z^n}.$$

It seems probable that he made a complete study of the more general series now known as the Lambert series:

$$L(z) = \sum_{n=1}^{\infty} a_n \frac{z^n}{1 - z^n}.$$

About a century later Weierstrass[3] treated the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}.$$

He showed that its region of convergence consists of two distinct parts, the interior and the exterior of the unit circle, in each of which the series represents a function which cannot be continued over the boundary $|z| = 1$.

In 1907 Hansen[4] set up the series

$$\sum_{n=1}^{\infty} \frac{z^{rn+t}}{1 - z^{rn+t}}.$$

It reduces to the Lambert series with coefficient unity when $r = 1$ and $t = 0$, while the difference of two series of this type gives that of Weierstrass.

Knopp, in 1913, published an extensive treatment of the Lambert series, in which he discusses its region of convergence and shows that under certain restrictions placed on the coefficients a_n, the function represented by the series cannot be continued beyond the unit circle.[5]

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[1] This paper was presented to the American Mathematical Society on June 20, 1934 under the title, "On the convergence of a generalized series and the relation of its coefficients to those of the corresponding power series."

[2] J. H. Lambert, *Anlage zur Architektonik*, vol. 2, Riga, 1771.

[3] L. Weierstrass, "Zur Functionentheorie," *Monatsberichte der Berliner Akademie* (1880), pp. 719-743.

[4] C. Hansen, "Démonstration de l'impossibilité du prolongement analytique de la série de Lambert," *Oversigt over det kgl. danske videnskabernes selskabs fordhandlinger* (1907), pp. 1-19.

[5] K. Knopp, "Ueber Lambertsche Reihen," *Journal für Mathematik*, vol. 142 (1913), pp. 283-315.

The purpose of this paper is to present a series which includes the three above mentioned as particular cases and to show how some remarkable properties of these series can be extended to a much larger class.

II. *Definition of the series; its convergence.* The series which we shall study is:

$$\sum_{n=1}^{\infty} a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} \tag{2.1}$$

where λ and μ are any positive integers. We shall refer to it briefly as the F-series. It reduces to the Lambert series $L(z)$ when $\lambda = \mu = 1$, or to $L(z^k)$ when $\lambda = \mu = k$. The series of Weierstrass is the difference of two series of type (2.1):

$$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}} = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^{4n}} - \sum_{n=1}^{\infty} \frac{z^{3n}}{1 - z^{4n}}. \tag{2.2}$$

Finally, if a_n is any constant c, we obtain

$$\sum_{n=1}^{\infty} \frac{cz^{\lambda n}}{1 - z^{\mu n}} = \sum_{n=1}^{\infty} \frac{cz^{\mu n + (\lambda - \mu)}}{1 - z^{\mu n + (\lambda - \mu)}}. \tag{2.3}$$

This is Hansen's series if $c = 1$.

It is important at the outset to determine the exact region of convergence of the F-series, and for this we have the following

THEOREM 1.[6] (A) *If Σa_n diverges, and* (i) $\mu > \lambda$, *the F-series and the power series $\Sigma a_n z^{\lambda n}$ converge and diverge at the same points within the unit circle, while for values of z beyond the unit circle the F-series and the power series $\Sigma a_n z^{(\lambda - \mu) n}$ converge and diverge together.*

If Σa_n diverges and (ii) $\mu \leq \lambda$, *the F-series and $\Sigma a_n z^{\lambda n}$ converge and diverge at the same points when $|z| < 1$, but the F-series diverges for every $|z| > 1$.*

(B) *If Σa_n converges and* (i) $\mu \geq \lambda$, *the F-series converges for every z whose modulus is not equal to unity.*

If Σa_n converges and (ii) $\mu < \lambda$, *the F-series converges for all $|z| < 1$, but converges and diverges at the same points as $\Sigma a_n z^{(\lambda - \mu) n}$ when $|z| > 1$.*[7]

The proof of this theorem is left to the reader. It is easily shown that its results can be extended to series in which $\lambda \leq 0$ or $\mu < 0$ or both.[8]

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[6] Throughout the remainder of this paper all summations, unless otherwise specified, are to be taken from $n = 1$ to $n = \infty$.

[7] This theorem does not exclude the possibility of convergence at some points on the unit circle, but such cases will not be considered.

[8] In establishing further properties of the series it will not be necessary to take into

By means of Weierstrass's M-test the uniform convergence of the series is established, so that we have

THEOREM 2. *The F-series converges uniformly in every closed sub-region lying completely within one of its regions of convergence and including no point of modulus unity.*

Whenever the F-series converges for values of z such that $|z| > 1$, a relation can be set up between the sum of the series for a point z outside the unit circle and the sum of some typical F-series at a point $1/z$ inside it. If $\lambda < \mu$ we have:

$$\Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} = - \Sigma a_n \frac{(1/z)^{(\mu - \lambda) n}}{1 - (1/z)^{\mu n}}, \tag{2.4}$$

while for $\mu = \lambda$ this becomes:

$$\Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} = - \Sigma a_n - \Sigma a_n \frac{(1/z)^{\mu n}}{1 - (1/z)^{\mu n}}. \tag{2.5}$$

When $\lambda > \mu$ we can use a lemma similar to that developed by Ananda-Rau [9] for the Lambert series, and obtain the relation:

$$\Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} = - \Sigma a_n \frac{(1/z)^{(q\mu - k) n}}{1 - (1/z)^{\mu n}} - \sum_{p=1}^{q} f[(1/z)^{(p-1)\mu - k}], \tag{2.6}$$

where $k = \lambda - \mu$, $f(z^s) = \Sigma a_n z^{sn}$, and q is the smallest positive integer for which $q\mu > k$.

Hence we may limit our considerations to the region of convergence of the F-series which lies within the unit circle. The radius of this region is $r \leq 1$.

III. *Expansion as a power series; inversion of a power series.* Every term of the F-series is analytic in $|z| < r$ and the series is uniformly convergent in $|z| \leq \rho < r$. Therefore the theorem of Weierstrass [10] on double series may be applied to obtain the expansion in power series of the function represented by the F-series in $|z| < r$. The result is:

$$\Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} = \Sigma A_n z^n, \tag{3.1}$$

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consideration any but positive values of λ and μ since a series in which $\lambda \leq 0$, or $\mu < 0$ can be transformed in such a way that the theorems developed for positive values will be applicable.

[9] K. Ananda-Rau, "On Lambert's series," *Proceedings of the London Mathematical Society*, ser. 2, vol. 18 (1920), p. 3.

[10] K. Knopp, *Theory and Application of Infinite Series*, Blackie and Son, Ltd., London and Glasgow, 1928, p. 430.

where A_n is the sum of all the coefficients a_k whose subscript k is such a divisor of n that $n \equiv \lambda k \pmod{\mu k}$, or, to borrow a notation from Knopp, $A_n = \sum_{k|n} a_k$.

In order that a power series may be expressed as an F-series it is necessary and sufficient that the coefficients a_n be known in terms of the A's appearing in the power series. If we consider the case in which λ and μ are relatively prime or are multiples of relatively prime integers, we shall find by expanding the series that the a_n's of the F-series are not in general uniquely expressible in terms of A's. Similar results are obtained when λ is a multiple of μ. But if $\mu = m\lambda$ $(m = 1, 2, 3, \cdots)$ every a_n is a unique sum of certain A's in the power series $\Sigma A_{\lambda n} z^{\lambda n}$. Thus we have

THEOREM 3. *The necessary and sufficient condition that a power series* $\Sigma A_{\lambda n} z^{\lambda n}$ *be expressible as an F-series is that* μ *be a multiple of* λ.

To determine the form of the relation which expresses any given a_n in terms of A's, we shall introduce an inversion function denoted by $R(n)$. Let $\{n\}$ be a set of positive integers defined by $n \equiv 1 \pmod{m}$, where $m = 1, 2, 3, \cdots$, each value of m determining a particular set $\{n\}$. We shall call the k-divisors of any integer n those divisors which satisfy the congruence $n/k \equiv 1 \pmod{m}$. For the elements of the set $\{n\}$, $R(n)$ is defined to be an integer, positive or negative, such that

$$(3.2) \qquad \sum_{k|n} R(k) = R(1) + R(k_1) + R(k_2) + \cdots + R(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$$

where $1, k_1, k_2, \cdots, n$ are the k-divisors of n.

When $m = 1$, $R(n)$ reduces to the Möbius function $\mu(n)$,[11] for then $\{n\}$ is the complete set of positive integers and the k-divisors are all the factors of n. When $m = 2$, $R(n)$ may be defined similarly to the Möbius function, $\{n\}$ being the set of all the odd positive integers whose k-divisors are the odd factors of n.

We can now show that if $A_{\lambda n} = \sum_{k|n} a_k$, the inverse relation is given by

$$a_n = \sum_{k|n} R(n/k) A_{\lambda k}.$$

Since $A_{\lambda k} = \sum_{\kappa|k} a_\kappa$, it follows that

$$\sum_{k|n} R(n/k) A_{\lambda k} = \sum_{k|n} R(n/k) \sum_{\kappa|k} a_\kappa.$$

[11] A. F. Möbius, "Ueber eine besondere Art von Umkehrung der Reihen," *Journal für Mathematik*, vol. 9 (1832), p. 105.

For a fixed κ, the coefficient of a_κ is

$$\sum_{K|n/\kappa} R\left(\frac{n/\kappa}{K}\right) = \sum_{K|n/\kappa} R(K),$$

since a given a_κ will have as coefficient only those terms of the sum $\sum_{k|n} R(n/k)$ for which k has the factor κ, that is, $k = \kappa K$. But by the definition of $R(n)$, the sum $\sum_{K|n/\kappa} R(K)$ vanishes unless $n/\kappa = 1$, that is, $n = \kappa$, in which case the coefficient of a_n is unity, since $\sum_{K|1} R(K) = R(1) = 1$. Hence

$$\sum_{k|n} R(n/k)\ A_{\lambda k} = a_n. \tag{3.3}$$

Thus any power series $\Sigma A_{\lambda n} z^{\lambda n}$ can be expressed as an F-series in which μ is a multiple of λ and a_n is given by (3.3).

IV. *Existence of natural boundaries.* Knopp has shown that when $a_n = 1$ in the Lambert series the function represented by the series cannot be continued beyond the unit circle. This property can be extended to series having $\mu = \lambda = k$ and also to those in which $\mu \gtrless \lambda$, as given in the following theorem, the method of proof being that of Landau as presented by Knopp.[12]

THEOREM 4. *If in the F-series $a_n = 1$ and* (i) $\mu \leq \lambda$, *the series represents a function $F(z)$ which cannot be continued beyond the unit circle*; (ii) $\mu > \lambda$, *it represents two functions in $|z| < 1$ and $|z| > 1$ respectively, both of which have the unit circle as a natural boundary.*

The proof of this theorem consists in showing that $F(z)$ becomes infinite as $z \to z_0$, z_0 being one of a set of points everywhere dense on the unit circle. If we take z_0 to be a rational point, namely, $z_0 = e^{2\pi i l'/l}$, where l is any positive integer except one of the finite number of factors of μ, and l' is relatively prime to l, Landau's method can be applied.

For values of a_n other than unity, Knopp[13] has demonstrated that under certain restrictions placed upon a_n, Lambert's series represents a function for which $|z| = 1$ is a natural boundary, and a parallel to his theorem can be developed for the F-series.

THEOREM 5. *If the coefficients a_n of the F-series are such that the radius of convergence of $\Sigma a_n z^{\lambda n}$ is unity, and also such that for an integer l (except the finite number of factors of μ) the series*

[12] K. Knopp, *Ueber Lambertsche Reihen*, p. 291.

[13] *Ibid.*, p. 292.

$$\sum_{\nu=1}^{\infty} \frac{a_{l\nu+q}}{\lambda\nu + q} \qquad [q = 0, 1, 2, \cdots, (l-1)]$$

converges, and if for such an l and an l' relatively prime to l, $z_0 = e^{2\pi i l'/l}$, then for all positive values of λ and μ

$$\lim_{z \to z_0} \left\{ \left(1 - \frac{z}{z_0}\right) \Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} \right\} = \frac{1}{\mu} \sum_{\nu=1}^{\infty} \frac{a_{l\nu}}{l\nu}.$$

If this be true for an infinite number of l's and if $\sum_{\nu=1}^{\nu=\infty} a_{l\nu}/l\nu \neq 0$, then the function represented by the F-series cannot be continued beyond the unit circle.

Proof. We wish to determine

$$\lim_{z \to z_0} \left(1 - \frac{z}{z_0}\right) \Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}} \quad \text{or} \quad \lim_{\rho \to 1} (1 - \rho) \Sigma a_n \frac{z^{\lambda n}}{1 - z^{\mu n}}.$$

Following Knopp's procedure at this point, let us break up the series into Σ_1 and Σ_2 according as $n \equiv 0$ or $\not\equiv 0 \pmod{l}$. Then if $n = l\nu$,

$$\begin{aligned}
\lim_{\rho \to 1} (1 - \rho) \Sigma_1 &= \lim_{\rho \to 1} (1 - \rho) \sum_{\nu=1}^{\infty} a_{l\nu} \frac{\rho^{\lambda l \nu}}{1 - \rho^{\mu l \nu}} \\
&= \lim_{\rho \to 1} \frac{1 - \rho}{1 - \rho^{\mu l}} (1 - \rho^{\mu l}) \sum_{\nu=1}^{\infty} a_{l\nu} \frac{\rho^{\lambda l \nu}}{1 - \rho^{\mu l \nu}} \\
&= \frac{1}{\mu l} \lim_{y \to 1} (1 - y) \sum_{\nu=1}^{\infty} a_{l\nu} \frac{y^{\lambda \nu/\mu}}{1 - y^{\nu}} \qquad (y = \rho^{\mu l}) \\
&= \frac{1}{\mu} \lim_{y \to 1} \sum_{\nu=1}^{\infty} \frac{a_{l\nu}}{l\nu} \frac{\nu y^{l\nu/\mu}}{1 + y + y^2 + \cdots + y^{\nu-1}}.
\end{aligned}$$

Since $\sum_{\nu=1}^{\nu=\infty} a_{l\nu}/l\nu$ is convergent, it follows that the entire series on the right is uniformly convergent in $0 \leq y \leq 1$ if $\nu y^{\lambda\nu/\mu}/(1 + y + y^2 + \cdots + y^{\nu-1})$ is uniformly bounded for every $0 \leq y \leq 1$ and for every $\nu \geq 1$.[14] Now for $0 < y < 1$ and for $\nu \geq 1$

$$\frac{\nu y^{\lambda\nu/\mu}}{1 + y + y^2 + \cdots + y^{\nu-1}} = \frac{\nu y^{\lambda\nu/\mu}(1 - y)}{1 - y^{\nu}} \leq \frac{\nu y^{\lambda\nu/\mu}(1 - y)}{1 - y} = \nu y^{\lambda\nu/\mu} < k,$$

since $\nu(y^{\lambda/\mu})^{\nu}$ is a null sequence and therefore bounded. For $y = 0$,

$$\frac{\nu y^{\lambda\nu/\mu}}{1 + y + y^2 + \cdots + y^{\nu-1}} = 0$$

[14] K. Knopp, *Theory and Application of Infinite Series*, p. 346.

and for $y=1$ it is equal to unity. Thus for every $0 \leq y \leq 1$ and for every $\nu \geq 1$

$$\frac{\nu y^{\lambda\nu/\mu}}{1+y+y^2+\cdots+y^{\nu-1}} < K,$$

where K is the larger of the two quantities 1 and k. Hence the series is uniformly convergent in $0 \leq y \leq 1$ and

$$\lim_{\rho\to 1}(1-\rho)\Sigma_1 = \frac{1}{\mu}\lim_{y\to 1}\sum_{\nu=1}^{\infty}\frac{a_{l\nu}}{l\nu}\,\frac{\nu y^{\lambda\nu/\mu}}{1+y+y^2+\cdots+y^{\nu-1}}$$
$$= \frac{1}{\mu}\sum_{\nu=1}^{\infty}\lim_{y\to 1}\frac{a_{l\nu}}{l\nu}\,\frac{\nu y^{\lambda\nu/\mu}}{1+y+y^2+\cdots+y^{\nu-1}} = \frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{a_{l\nu}}{l\nu}.$$

Knopp's method may be applied to show that

$$\lim_{z\to z_0}\left\{\left(1-\frac{z}{z_0}\right)\Sigma_2\right\} = 0,$$

or that for every $q = 1, 2, \cdots, (l-1)$

$$\lim_{z\to z_0}\left\{\left(1-\frac{z}{z_0}\right)\sum_{\nu=0}^{\infty}a_{l\nu+q}\,\frac{z^{\lambda(l\nu+q)}}{1-z^{\mu(l\nu+q)}}\right\} = 0.$$

Thus we have

$$\lim_{z\to z_0}\left\{\left(1-\frac{z}{z_0}\right)\Sigma a_n\,\frac{z^{\lambda n}}{1-z^{\mu n}}\right\} = \frac{1}{\mu}\sum_{\nu=1}^{\infty}\frac{a_{l\nu}}{l\nu}.$$

If this be true for an infinite number of l's and $\sum_{\nu=1}^{\nu=\infty} a_{l\nu}/l\nu \neq 0$, then all the rational points defined by z_0 are singular points and the function represented by the F-series cannot be continued beyond the unit circle.

We have taken into consideration the radial approach of z to z_0 from within the unit circle, and have therefore shown that when the F-series converges for $|z| < 1$ the function which it represents has the unit circle as a natural boundary. However, whenever the F-series converges for $|z| > 1$ it can be expressed in terms of some other F-series convergent for $|z| < 1$, to which the theorem applies. Thus, if the series converges for $|z| < 1$ and also for values of z such that $|z| > 1$, it will in general represent two functions, neither of which can be continued across the unit circle.

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GEODESIC CONTINUA IN ABSTRACT METRIC SPACE.

By Orrin Frink, Jr.

Menger has treated the subject of geodesics in abstract metric space. (*Mathematische Annalen*, vol. 103 (1930), pp. 466-501.) In Menger's work, a geodesic is an arc whose length, properly defined, is less than or equal to the lengths of comparison arcs. Existence theorems for geodesics, with more general end conditions than those of Menger, can be proved by showing that the length or linear measure function is lower semicontinuous.

This method of approach to the subject of geodesic arcs by means of lower semicontinuity is subject to the difficulty that the limit set of a convergent sequence of arcs is not necessarily an arc. Such a limit set of arcs is, however, always a continuum, which suggests generalizing the notion of a geodesic arc to that of geodesic continuum. A geodesic continuum may be defined as one whose length, or linear measure, is less than or equal to the linear measures of comparison continua.

Menger has shown (*loc. cit.*, p. 477) that one of his definitions of length, called "Längeninhalt," which is defined for continua as well as for arcs, is lower semicontinuous. However, this particular definition of length, while satisfactory for arcs, has disadvantages when applied to continua in general. For, as can be seen from an example given by Menger (*loc. cit.*, p. 476), the Längeninhalt of a set consisting of the two diagonals of a unit square is 3, while each diagonal by itself has a Längeninhalt equal to $\sqrt{2}$. Menger does not use his theorem on lower semicontinuity in proving his existence theorems.

In the present paper a form of the Caratheodory definition of linear measure, applicable to metric sets in general, is used, rather than any special definition of length suitable only for arcs. It is shown that for continua, the Caratheodory linear measure is lower semicontinuous. This, taken together with some results on the compactness of collections of continua, leads to existence theorems for geodesic continua with rather general end conditions. (See Theorems 4, 5, and 7 below.) Aside from the gain in generality due to the end conditions, it is interesting to note that many of the functionals of the calculus of variations, defined in terms of an integral J, become cases of the Caratheodory linear measure function when the metric of the space is redefined in terms of J. (Morse, *Calculus of Variations in the Large*, p. 208; Menger, *Fundamenta Mathematica*, vol. 25, p. 441.)

Definitions.

Two sets A and B are said to be *adjacent* if they have a point in common, or if either contains a limit point of the other. Note that a set is connected if it is not the sum of two non-empty sets which are not adjacent.

The *distance* $\overline{AB}$ between two closed sets A and B is the greatest lower bound of numbers ϵ such that $A \subset U(B,\epsilon)$ and $B \subset U(A,\epsilon)$, where $U(A,\epsilon)$ means the ϵ-neighborhood of A. (Hausdorff, *Mengenlehre,* 2nd ed., p. 146.)

An *ϵ-partition* of a set A is a collection of a finite number of subsets $A_1, A_2, \cdots, A_n$ of A, whose logical sum is A, the diameter $d(A_r)$ of each subset being less than ϵ. The subsets $A_1, \cdots, A_n$ are called the *subdivisions* of the partition, and their number, n, is called the *degree* of the partition. The sum of the diameters of all the subdivisions is called the *diameter sum* of the partition, and is less than $n\epsilon$.

The (Caratheodory) *linear measure* $m(A)$ of the set A is the greatest lower bound of numbers l such that for every $\epsilon > 0$ there exists an ϵ-partition of A whose diameter sum is $< l$. If there is no such number l, then $m(A)$ is said to be infinite. (Caratheodory, *Göttinger Nachrichten* 1914, pp. 404-426.) A more general form of the Caratheodory definition, where the number of subdivisions is allowed to be countably infinite, is not needed for continua, with which this paper is chiefly concerned.

Lemmas.

The first five lemmas are stated without proof.

Lemma 1. *Every subdivision of a partition of a connected set is adjacent to at least one other subdivision of the partition.*

Lemma 2. *If A and B are adjacent, then $d(A+B) \leqq d(A) + d(B)$.*

Lemma 3. *The diameter sum of a partition is not increased if the partition is altered by replacing two adjacent subdivisions by their logical sum.*

Lemma 4. $d[U(A,\epsilon)] \leqq d(A) + 2\epsilon$.

Lemma 5. *If the sets $B_1, B_2, \cdots, B_n$ are each adjacent to A, and are each of diameter less than ϵ, then*

$$d(A + B_1 + B_2 + \cdots + B_n) < d(A) + 2\epsilon.$$

The next lemma states a property of partitions of connected sets that does not hold for disconnected sets. This accounts for the fact that the linear measure function is not a lower semicontinuous function of sets in general, but only of connected sets.

LEMMA 6. *If A is connected, and $m(A) < l$, then for every $\epsilon > 0$ there exists an ϵ-partition of A of diameter sum $< l$, every subdivision of which is of diameter $\geqq \epsilon/4$.*

Proof. Since $m(A) < l$, by definition of linear measure there exists an $\epsilon/2$-partition of A of diameter sum $< l$. The idea of the proof is to combine adjacent subdivisions of this partition, using Lemma 3, until all subdivisions are of diameter $\geqq \epsilon/4$. Accordingly, let two adjacent subdivisions of diameter $< \epsilon/4$ be replaced by their logical sum, and let this step be repeated until no two adjacent subdivisions are each of diameter $< \epsilon/4$. It follows from Lemmas 2 and 3 that the new partition is still an $\epsilon/2$-partition of diameter sum $< l$.

The new partition may still have subdivisions of diameter $< \epsilon/4$, but by Lemma 1 each such subdivision must be adjacent to one of diameter $\geqq \epsilon/4$. The next step is to replace such a subdivision of diameter $\geqq \epsilon/4$, together with all subdivisions of diameter $< \epsilon/4$ adjacent to it, by the logical sum of all these subdivisions. This step is repeated until no subdivisions of diameter $< \epsilon/4$ remain, which is one of the things to be proved. The resulting partition is an ϵ-partition by Lemma 5, although it may no longer be an $\epsilon/2$-partition. By Lemma 3, its diameter sum is still $< l$. This completes the proof of Lemma 6.

LEMMA 7. *If A is a connected set, and $m(A) < l$, then for every $\epsilon > 0$ there exists an ϵ-partition of A whose diameter sum is $< l$, and whose degree is $< 4l/\epsilon$.*

This follows from Lemma 6. For if the partition provided by Lemma 6 were of degree $\geqq 4l/\epsilon$, then since the diameter of each subdivision is $\geqq \epsilon/4$, the diameter sum of the partition would be $\geqq 4l/\epsilon \cdot \epsilon/4 = l$, contrary to Lemma 6.

THEOREMS ON LOWER SEMICONTINUITY.

THEOREM 1. *In a metric space, if A is any set of linear measure $\geqq l$, then for every $\theta > 0$ there exists a $\delta > 0$ such that if C is any connected set for which $A \subset U(C, \delta)$, then $m(C) > l - \theta$.*

This theorem states that connected sets sufficiently close to a fixed set have a linear measure not much less than that of the fixed set.

Proof. Since $m(A) \geqq l$, it follows from the definition of linear measure that for $\theta > 0$ there exists an $\epsilon > 0$ such that no ϵ-partition of A has a diameter sum $< l - \theta/3$. For otherwise $m(A)$ would be $\leqq l - \theta/3$.

Now choose the δ required by the theorem such that $\delta < \epsilon/4$ and $\delta < (\epsilon \cdot \theta)/48l$. For this choice of δ it must be shown that if C is any connected set such that $A \subset U(C, \delta)$, then $m(C) > l - \theta$. Suppose on the contrary that $m(C) \leqq l - \theta$. Then by Lemma 7 there exists an $(\epsilon/2)$-partition of C whose diameter sum is $< l - 2\theta/3$, and whose degree is $< 8l/\epsilon$. Let $\{C_k\}$ be the subdivisions of this partition, and n its degree.

By hypothesis $A \subset U(C, \delta)$, hence the sets $\{A \cdot U(C_k, \delta)\}$ form the subdivisions of a partition of A, also of degree n. To estimate the diameter sum of this ϵ-partition of A, note that from Lemma 4 it follows for any particular subdivision $A \cdot U(C_k, \delta)$ of this partition that $d[A \cdot U(C_k, \delta)] \leqq d(C_k) + 2\delta$. Hence the diameter sum of this partition of A exceeds the diameter sum of the partition of C by at most $2n\delta$, there being n subdivisions. Since $n < 8l/\epsilon$, and $\delta < (\epsilon \cdot \theta)/48l$, it follows that $2n\delta < \theta/3$.

That is, the diameter sum of the partition of A exceeds the diameter sum of the partition of C by less than $\theta/3$. But the diameter sum of the partition of C was $< l - 2\theta/3$. Hence the diameter sum of the partition of A is $< l - \theta/3$, contrary to the assumption above that no such partition exists. This contradiction proves Theorem 1.

THEOREM 2. *In a metric space whose elements are continua of some other metric space, the Caratheodory linear measure function is lower semicontinuous at every element of the space.*

Proof. To prove lower semicontinuity at an element A it must be shown that if $m(A) \geqq l$, then for every $\theta > 0$ there exists a $\delta > 0$ such that if C is another element (continuum) of the space, for which the distance $\overline{AC} < \delta$, then $m(C) > l - \theta$. Theorem 1 supplies just such a value of δ, since if $\overline{AC} < \delta$, by definition of the distance $\overline{AC}$ it follows that $A \subset U(C, \delta)$, which is the condition of Theorem 1. Since the sets A and C are continua, C is connected, and both A and C are closed, so that the definition of distance between closed sets applies.

APPLICATIONS OF LOWER SEMICONTINUITY.

To derive an existence theorem for geodesic continua from Theorem 2, it is sufficient to show that the collection of admissible continua, among which one of minimum linear measure is to be found, is closed and compact, in which case the theorem applies that a function lower semicontinuous on a closed and compact set attains its minimum value for some element of the set.

Hausdorff has shown (*Grundzüge der Mengenlehre*, 1st ed., p. 302; *Mengenlehre*, 2nd ed., p. 150) that the space of all continua of a compact

metric space is itself a compact metric space. Hence the collection of all continua of a compact metric space which satisfy given end conditions is closed and compact provided the limit element of any convergent sequence of the collection also satisfies the end conditions.

THEOREM 3. *In a metric space, if the closed set A is the limit of the sequence of closed sets $\{C_r\}$, and every set C_r of the sequence has at least one point in common with the closed and compact set B, then A has at least one point in common with B.*

Proof. Suppose A and B have no common point. Then, since A and B are closed, and B is compact, there exists a δ-neighborhood $U(A, \delta)$ of A which contains no point of B. Since A is the limit of the sequence $\{C_r\}$, there exists a C_r such that $C_r \subset U(A, \delta)$. But this is impossible, since C_r contains a point of B, and $U(A, \delta)$ does not. This proves Theorem 3.

COROLLARY. *All continua of a compact metric space which have at least one point in common with each set of a collection of closed sets, constitute a compact metric space.*

This follows from Theorem 3. For, the collection of all continua having at least one point in common with one fixed closed set C_r of the collection of closed sets $\{C_r\}$, is closed by Theorem 3. Hence the logical product of all such collections is closed. But this logical product is just the collection of continua mentioned in the corollary. Being a closed subset of the compact metric space of all continua, its elements constitute a compact metric space. Similar results may be found in R. L. Moore's *Foundations of Point Set Theory*, Chapter 5.

As a consequence of the corollary, together with Theorem 2 and the property that a function which is lower semicontinuous over a compact space assumes its minimum, we have

THEOREM 4. *In a compact metric space, if there exists a continuum of finite linear measure which has at least one point in common with each set of a collection of closed sets, then there exists a continuum of minimum linear measure having this property.*

Each closed set of the collection mentioned in Theorem 4 corresponds to an end condition on the geodesic continuum. If the closed set consists of a single point, it is a fixed end point condition; otherwise a variable end point condition. In particular, if each closed set of the collection consists of a single point, the end conditions amount to requiring that the continua being con-

sidered all contain the set A (consisting of all these single points). The set A is arbitrary. This proves

THEOREM 5. *In a compact metric space, if there exists a continuum of finite linear measure containing the set A, then there exists a continuum of minimum linear measure containing A.*

A more general type of variable end condition, corresponding to the end conditions of Morse, *The Calculus of Variations in the Large*, pp. 19 and 65, is obtained if in addition to the condition that the continuum have a point, say c_r, in common with each set C_r of a collection $\{C_r\}$ of closed sets, it is required that certain relations hold between the endpoints $\{c_r\}$. Such relations between the endpoints are most conveniently described in terms of the notion of the product space P of the closed end sets $\{C_r\}$.

Suppose in the compact metric space M there are given n such end sets $C_1, \cdots, C_r, \cdots, C_n$, no two having a point in common. To every selection $(c_1, \cdots, c_r, \cdots, c_n)$ of points, one from each of the end sets C_r, corresponds a point π of the product space P. The points $(c_1, \cdots, c_n)$ corresponding to π will be called the *coördinates* of π. The notions of neighborhood and distance in the product space are defined in the usual way.

Now let Q be any closed set of the product space P. A continuum K of the original compact metric space M is said to satisfy the *end condition* Q if K contains all the coördinates $(c_1, \cdots, c_n)$ of at least one point π of Q. This type of end condition is stronger than that of Theorem 4, since K is still required to have at least one point in common with each end set, and in addition the end points must satisfy the condition of being the coördinates of some point π of Q. What is now needed is the result that the collection of all continua satisfying an end condition Q is closed and compact.

THEOREM 6. *If Q is a closed set of the product space P of n mutually exclusive closed sets $C_1, \cdots, C_n$ of the compact metric space M, then the space of all continua of M which contain all the coördinates $(c_1, \cdots, c_n)$ of at least one point π of Q, is a compact metric space.*

Proof. Suppose the continuum K is the limit of the convergent sequence of continua $\{K_m\}$, and suppose each continuum K_m of the sequence satisfies the end condition Q. It must be shown that K also satisfies the end condition Q. For each m, let π_m be the point of Q corresponding to K_m. Then K_m contains all the coördinates of π_m. Since Q is closed and compact, the sequence $\{\pi_m\}$ contains a subsequence $\{\pi_{mj}\}$ converging to a point π of Q. It will now be shown that K contains all the coördinates of π. Let c_p be the

p-th coördinate of π, and c_{pj} be the p-th coördinate of π_{mj}. Since the sequence $\{\pi_{mj}\}$ converges to π, it follows that for every p the sequence $\{c_{pj}\}$ converges to c_p. Now since K is the limit of the convergent sequence of continua $\{K_{mj}\}$, every δ-neighborhood of K contains all except a finite number of the continua $\{K_{mj}\}$, and hence every δ-neighborhood of K contains all except a finite number of the points $\{c_{pj}\}$. Since K is closed, it follows that K also contains the limit c_p of the sequence $\{c_{pj}\}$. This is true for every p, hence K contains all the coördinates $\{c_p\}$ of π, and therefore satisfies the end condition Q. This proves that the collection of all continua of M which satisfy the end condition Q, is closed, since any limit continuum of the collection also satisfies the end condition Q, and is therefore a member of the collection. Since the collection is a closed subset of the compact metric space of all continua of M, it is itself a compact metric space.

THEOREM 7. *If Q is a closed set of the product space P of n mutually exclusive closed sets $(C_1, \cdots, C_n)$ of the compact metric space M, and if there exists a continuum of M of finite linear measure which contains all the coördinates of at least one point of Q, then there exists a continuum of minimum linear measure having this property.*

This follows from Theorems 2 and 6.

In the existence Theorems 4 and 7, if there are more than two end sets C_r, it cannot be expected in general that the geodesic continuum whose existence is asserted will be an arc. However, it is interesting to note that in the important case where there are just two end sets C_1 and C_2, the geodesic continuum must be an arc. To show this, it is sufficient to show that the geodesic continuum is locally connected and irreducible between its endpoints (Hausdorff, *Mengenlehre*, 2nd ed., p. 222). It follows from a theorem of R. L. Moore (*Foundations of Point Set Theory*, p. 95, Theorem 8) that a continuum of finite linear measure is locally connected. For Moore's theorem asserts that if the continuum K is not locally connected, then K contains infinitely many mutually exclusive subcontinua, each of diameter greater than a fixed constant. From this it follows that the linear measure of K is infinite. That a geodesic continuum with two end conditions is irreducible between its endpoints follows from the fact that it is geodesic. For, any proper subcontinuum of K is necessarily of smaller linear measure than K. This proves

THEOREM 8. *A geodesic continuum with two end conditions is an arc.*

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PROOF OF THE IDEAL WARING THEOREM FOR EXPONENTS 7-180.

By L. E. Dickson.

1. If q denotes the greatest integer $< (3/2)^n$, then $2^n q - 1$ is a sum of $I = 2^n + q - 2$, but not fewer, n-th powers (Lemma 1). The ideal Waring theorem states that every positive integer is a sum of I n-th powers; for example, 4 squares, 9 cubes, 19 fourth powers. Proofs for squares and cubes are classic. For $n \leqq 20$, proof has been found, but not yet published, by use of a "constant" far exceeding our new constant N in Theorem 1, and the use of more or less extensive tables.

Using the new N, we here prove without any tables the ideal Waring theorem for $9 \leqq n \leqq 180$, and by use of tables also for $n = 7$ and 8. For $n = 6$ we cannot attain the ideal 73, but reach 115 (the best earlier result being 160).

Part I. Asymptotic Theory.[1]

2. If $(x]$ denotes the distance from x to the nearest integer, let

$$\Omega' \leqq \sum_{y=1}^{Y} \sum_{y_1=1}^{Y} \frac{1}{2(\alpha t]}, \qquad t = y^n - y_1{}^n, \quad t \neq 0.$$

This sum is double of that obtained when $t > 0$. When t and y_1 are fixed there is a single real positive y. Thus $y^n - y_1{}^n$ has at most Y sets of integral solutions y, y_1, each chosen from $1, \cdots, Y$. Hence

$$\Omega' \leqq Y \Sigma 1/(\alpha t],$$

summed for certain integers $t \geqq 1$, $t \leqq Y^n$. Let l be the absolutely least residue of at modulo q. By A, § 13, $1/(\alpha t] < 2q/i$, where $i = |\, l \,|$ and the values of i are in $(1, 1)$ correspondence with those of $t > 0$. In ascending order of magnitude, the values of i are $\geqq 1, \geqq 2, \cdots$, terminating at or before Y^n. Hence

$$\Omega' \leqq 2Yq \sum_{j=1}^{Y^n} (1/j) \leqq 2Yq(1 + \log Y^n),$$

as is shown by comparing the sum (for $j \geqq 2$) of the rectangles under the curve $y = 1/x$ with the area $=$ integral from 1 to Y^n.

[1] We avoid the divisor function in § 13 of our exposition and generalization of Vinogradow's theory, *Annals of Mathematics*, vol. 36 (1935), pp. 395-405, cited as A.

Since we are dealing with intervals of the second class, $q < \tau = 2mnP^{n-\frac{1}{2}}$. Hence (92) of A becomes

$$|\sum_Y V_y S_y|^2 < X_1 RY(a2^n R^n + 4mnP^{n-\frac{1}{2}}(1+\log Y^n)),$$

in which we have dropped the subscript $k+k_1$ of a. By (9), A,

(1) $$Y^n \leqq CP^{(n-1)f}, \qquad C = m/(a2^{n-1}),$$

(2) $$1+\log Y^n \leqq 1+\log C + (n-1)f\log P$$

will be later verified to be $\leqq P^z$. By (7), A, $R \leqq P^{1-f}$. Hence

$$\begin{aligned} |\Sigma V_y S_y|^2 &< X_1 RY\{a2^n P^{n(1-f)} + 4mnP^{z+n-\frac{1}{2}}\} \\ &< X_1 RY \cdot 8mnP^{z+n-\frac{1}{2}}, \end{aligned}$$

since the latter exponent of P exceeds $n(1-f)$ by more than z in view of $f \geqq \frac{1}{2}\nu$, $\nu = 1/n$. Hence by A, (103), and the line below it, we see that (88) gives

(3) $$|H_2| < DP^{s-2}PX(X_1 RYP^{z+n-\frac{1}{2}})^{\frac{1}{2}}, \qquad D = 3(8mn)^{\frac{1}{2}},$$

By (A), (7), (9), (19), (24),

(4) $$R > \sqrt{1/2}\,P^{1-f}, \quad Y > \sqrt{1/2}\,C^\nu P^{(1-\nu)f}, \quad X \geqq \tfrac{1}{2}\kappa n^\mu P^{n-1-\sigma},$$

while X_1 exceeds the last product with P replaced by R, and κ by κ_1, since we assume that $k_1 = k$. Note that

(5) $$\mu = \sigma - n + 1, \qquad \sigma = n(1-\nu)^k.$$

Multiply and divide (3) by $X(X_1RY)^{\frac{1}{2}}$ and apply (4). Thus

(6) $$|H_2| < C_1 P^J X^2 X_1 RYP^{s-n}, \qquad C_1 = \frac{4D2^{(n-1-\sigma)/4}}{\kappa n^\mu(\kappa_1 n^\mu C^\nu)^{1/2}},$$

(7) $$2J = \sigma(3-f) - g, \; g = \tfrac{1}{2} - z - nf + (1-\nu)f.$$

The Waring theorem is true for every integer $\geqq N_0$, if the integral $I(N_0) > 0$. By the above and A, (89), the condition is

(8) $$P^{-J} > C_1/c_9.$$

For P large, this holds if $J < 0$. This is true by (5_2) if

(9) $$r(1-\nu)^k < 1, \qquad r = (3-f)n/g.$$

But r increases with f since

$$dr/df = (n/g^2)(3n - 7/2 + 3\nu + z) > 0.$$

Since (9_1) is equivalent to

$$k > \log r / \{-\log(1-\nu)\}, \tag{10}$$

k will be a minimum if r and hence f is. But $f \geqq \frac{1}{2}\nu$. Hence we employ the minimum f, viz.,

$$f = \tfrac{1}{2}\nu. \tag{11}$$

The resulting value of r in (9) is

$$r = n^2(6n-1)/(n-d), \qquad d = 1 + 2n^2 z. \tag{12}$$

For a small z, say $\nu^3/24$, let k_0 be the least integer k satisfying (10). Then all sufficiently large integers are sums of $4n - 2 + 3k_0$ n-th powers.

3. Results which are better for universal theorems are obtained by taking [2] $k = 2k_0$, whence k exceeds the double of the second member of (10). By the definition (5) of σ and (10), $\log \sigma$ is just $< \log n - 2 \log r$. Hence

$$\sigma < \frac{(n-d)^2}{n^3(6n-1)^2}, \qquad 3 - f = \frac{6n-1}{2n}, \qquad \tfrac{1}{2}\sigma(3-f) < \frac{(n-d)^2}{4n^4(6n-1)}.$$

Take $z = \nu^3/24$. Then $1 < d < 2$. Thus

$$(12d-1)n > 11n > 24 > 6d^2, \qquad (n-d)^2/(6n-1) < n/6,$$

$$-J = -\frac{1}{2}\sigma(3-f) - \frac{1}{2}z + \frac{1}{4}(\nu - \nu^2) > \frac{-1}{24n^3} - \frac{1}{48n^3} + \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n^2}\right) \geqq \frac{2}{9n},$$

if $(2n-9)^2 \geqq 90$ and hence if $n \geqq 10$. Thus (8) holds if

$$\log_{10} P \geqq 2n \log_e C_1/c_9, \qquad n \geqq 9, \tag{13}$$

since it is readily verified if $n = 9$.

4. *Evaluation of* c_9. Let S denote the singular series and

$$n \geqq 4, \qquad s \geqq 4n, \qquad b = (1 + n^s)^{2/(s-5)}.$$

Then [3] $S > b_4$, where, if the products range over the primes p indicated,

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[2] We discard (119) of A, whose object was to make σ very small, a property first used in § 18, which is to be omitted now. We may omit the condition $s \geqq 70$.

[3] Landau, *Vorlesungungen über Zahlentheorie*, vol. 1, p. 303; James, *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 408, 434-435.

$$b_4 = \prod_{p \leqq b} b(p) \cdot \prod_{p > b} (1 - p^{-3/2}), \qquad b(p) > \begin{cases} p^{-n(2n-1)}, & p > 2, \\ 2^{-n(4n-1)}, & p = 2. \end{cases}$$

The second factor exceeds the like product over all primes. Hence its reciprocal is

$$< \sum_{x=1}^{\infty} x^{-3/2} \leqq 1 + \int_1^{\infty} x^{-3/2} dx = 3.$$

The reciprocal R of the first factor is

$$R < 2^{n(4n-1)} \prod_{p \leqq b} p^{n(2n-1)} \cdot 2^{-n(2n-1)}.$$

$$\log R < 2n^2 \log 2 + n(2n - 1)\vartheta(b), \quad \vartheta(b) < (6/5)ab + 3\log^2 b + 8 \log b + 5,$$

where[4] $\vartheta(b)$ is the sum of the natural logarithms of all primes $\leqq b$, and $a = .92129 \cdots$. Let $n \geqq 9$, $s = 36$. We increase $2n - 1$ to $2n$, which more than compensates for dropping term 1 in b. Then

$$\log n \leqq n/9M, \qquad \log b \leqq 72/31 \cdot n/9M < .6n, \qquad M = .4343.$$

Thus $\log R$ will be $< n^5 - 2n^2$ if, after cancelling n^2,

$$1.3863 + 2\{(6/5)an^{72/31} + 1.08\, n^2 + 4.8\, n + 5\} < n^3 - 2.$$

Multiply the combined constant term by $1 \leqq n/9$ and likewise for the resulting terms in n and n^2. It suffices to verify

$$2.4\, an^{72/31} < .62312\, n^3, \qquad \log 2.4\, a = .3446 < .4410 = \log(.62312 \cdot 9^{21/31}).$$

When s increases, the exponent of n in b decreases, whence $\log R$ decreases. Hence, if $n \geqq 9$, $s \geqq 4n$, then $\log R < n^5 - 2n^2$. By A, $c_9 = \nu(1/3)^n b_4$. Hence

$$-\log c_9 < A + n^5 - 2n^2, \qquad A = \log n + n \log 3 + \log 3 < .18184\, n^2.$$

5. *Evaluation of C_1.* Restrict attention to the classic Waring problem. Then $\kappa = \kappa_1 = 1$, $m = 3^{n-1}$, $a = 1$. We increase C_1 by suppressing terms $-\sigma$. Thus

$$C_1 < 12(8n \cdot 3^{n-1})^{1/2} 2^{(n-1)/4} n^{3(n-1)/2} / (3/2)^{(1-\nu)/2}.$$

We increase C_1 by suppressing the denominator. Thus

$$\log_{10} C_1 < 1.53073 + (1/4)(n - 1)1.25527 + ((3/2)n - 1)\log n.$$

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[4] Landau, *Handbuch . . . Verteilung der Primzahlen*, vol. 1, p. 91.

Take $n \geqq 9$. Then $\log n \leqq n/9$. Multiply the combined constant term by $1 \leqq n/9$ and likewise for the combined term in n. Hence

$$\log_{10} C_1 < .20420\, n^2, \qquad \log_e C_1 < (1/2)n^2, \qquad n \geqq 9.$$

6. *The constant N.* Hence $\log_e C_1/c_9 < n^5$ if $n \geqq 9$. Thus (13) holds if $\log_{10} P \geqq 2n^6$. To verify that (2) is $\leqq P^z$, it suffices to use our least P. Then (2) is $< 3n^6 < P^z = 10^{n^8/12}$. Finally, P is the greatest integer $\leqq (1/3)N_0^{\nu}$. Thus (13) holds if $\log_{10} N_0 \geqq 2n^7$.

In A, (120), $\log_{10} c_{15} < (1/2)n^3$, $c_1 = (1/2)c_{15}{}^s$, $\log c_1 < 2n^4$. Hence in (85) the term $c_1 P^{-1}$ is insignificant. Likewise for $c_3 P^{-g}$ in (87).

THEOREM 1. *Let k_0 be the least integer exceeding* (10) *with r as in* (12). *Take $k = 2k_0$, $\log_{10} N = 2n^7$. If $n \geqq 9$, every integer $\geqq N$ is a sum of $4n - 2 + 3k$ integral n-th powers $\geqq 0$.*

PART II. THE IDEAL WARING THEOREM.

7. Write $3^n = 2^n q + r$, $0 \leqq r < 2^n$. Then $q = [(3/2)^n]$.

LEMMA 1. *All positive integers $\leqq 2^n q$ are sums of $I = 2^n + q - 2$ integral n-th powers $\geqq 0$. But $2^n q - 1$ is not a sum of $I - 1$ n-th powers.*

Let $B \leqq 2^n q - 1$. Then $B = 2^n x + y$, $0 \leqq y \leqq 2^n - 1$, and $x < q$. Thus B is a sum of $x + y \leqq 2^n - 1 + q - 1 = I$ powers.

LEMMA 2. *All integers in the interval $(2^n q, 2^n q + 2^n)$ are sums of E n-th powers, where $E = max(q + r - 1, 2^n - r)$.*

First, $2^n q, \cdots, 2^n q + r - 1$ are sums of $q + r - 1$ powers. Second, the integers $2^n q + r = 3^n, \cdots, 2^n q + 2^n - 1 = 3^n - r + 2^n - 1$ are sums of $2^n - r$ powers. Third, $2^n q + 2^n$ is a sum of $q + 1$ powers. But if $r = 1$, n would be divisible by 2^{n-2} since 3 belongs to the exponent 2^{n-2} modulo 2^n, whence $n \leqq 4$.

LEMMA 3. *Let L, p, n, z be positive integers. If all integers in the interval $(L, L + p^n z)$ are sums of m integral n-th powers $\geqq 0$, then all in the interval $(L, L + p^n z + p^n)$ are sums of $m + 1$ powers.*

Proof is needed only for integers g satisfying

$$L + p^n z < g \leqq L + p^n z + p^n, \qquad L + p^n(z - 1) < g - p^n \leqq L + p^n z.$$

Since $g - p^n$ is in the first given interval, it is a sum of m powers. Hence g is a sum of $m + 1$ n-th powers.

By induction on v, Lemma 3 yields.

LEMMA 4. *Let L, p, n, z, v be positive integers. If all integers in $(L, L+p^n z)$ are sums of m integral n-th powers $\geqq 0$, then all in $(L, L+p^n(z+v))$ are sums of $m+v$ n-th powers.*

Taking $p=2$, $z=1$, and applying Lemma 2, we get

LEMMA 5. *All integers in $(2^n q, 2^n(q+y))$ are sums of $E+y-1$ n-th powers.*

Take $y=1+q$. Then $y>(3/2)^n$, $2^n y>3^n$. This proves

LEMMA 6. *All integers in $(2^n q, 2^n q+3^n)$ are sums of $E+q$ n-th powers.*

This and Lemma 4 with $p=3$, $z=1$, yield

LEMMA 7. *All in $(2^n q, 2^n q+3^n(1+v))$ are sums of $E+q+v$.*

Take $v=[(4/3)^n]$. Then $1+v>(4/3)^n$. This proves

LEMMA 8. *All in $(2^n q, 2^n q+4^n)$ are sums of $E+q+[(4/3)^n]$.*

By induction, we get

THEOREM 2. *All integers in the interval $(2^n q, 2^n q+(n+1)^n)$ are sums of*

$$E+q+\left[\left(\frac{4}{3}\right)^n\right]+\left[\left(\frac{5}{4}\right)^n\right]+\cdots+\left[\left(\frac{n+1}{n}\right)^n\right] \tag{14}$$

n-th powers, and hence are sums of $E+q+(n-2)[(4/3)^n]$ powers.

8. *Ascent.* Let $v=(1-l/L_0)/n$, $v^n L_0 \geqq 1$. If all integers between l and L_0 inclusive are sums of Q n-th powers $\geqq 0$, then [5] all between l and L_t inclusive are sums of $Q+t$ n-th powers, where

$$\log L_t=\left(\frac{n}{n-1}\right)^t(\log L_0+n\log v)-n\log v. \tag{15}$$

The interval in Theorem 2 includes (l, L_0), where $l=3^n$ and $L_0=(n+1)^n$. The above inequality reduces to $v(n+1)\geqq 1$ and hence to

$$\left(\frac{n+1}{3}\right)^{n-1}\geqq 3,$$

which holds if $n\geqq 4$. Let $n\geqq 11$. Then $v=1/n$ to seven decimal places. By the series for $\log(1+x)$, $\log L_0-n\log n$ is the product of n by

[5] Dickson, *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 711.

$$\log(1+1/n) > M(1/n - 1/2n^2), \qquad M = .4343,$$

where (as henceforth) we use logarithms to base 10.

We desire that $L_t > N$, for N in Theorem 1. This holds if

$$\left(\frac{n}{n-1}\right)^t M\left(1-\frac{1}{2n}\right) > 2n^7, \qquad n \geqq 11. \tag{16}$$

Let $n \geqq 14$. Then (16) follows from the like inequality with the denominator $2n$ replaced by 28, and hence if

$$t \log \frac{n}{n-1} > .67904 + 7 \log n.$$

The coefficient of t is the product of the modulus M by

$$\log_e\left(1+\frac{1}{n-1}\right) > \frac{1}{n-1} - \frac{1}{2(n-1)^2} > \frac{1}{n} \qquad (n > 2).$$

Hence $L_t > N$ if

$$t > M^{-1}n(.67904 + 7\log n), \qquad n \geqq 14. \tag{17}$$

9. First, let $E = 2^n - r$. By Theorems 1 and 2, all integers $\geqq 2^n q$ are sums of

$$t + 2^n - r + q + (n-2)[(4/3)^n]$$

n-th powers, where t is the least integer satisfying (17). This number is $\leqq I = 2^n + q - 2$, if

$$r/2^n \geqq F(n), \qquad F(n) = t/2^n + (n-2)/q. \tag{18}$$

But if $E = q + r - 1$, the like conclusion holds if

$$r/2^n \leqq 1 - F(n) - q/2^n. \tag{19}$$

For $n = 14$, $t = 281$ and these limits are $F(14) = .058388$ and $.923851$. Since $F(n)$ decreases when n increases, the ideal Waring theorems holds for any $n \geqq 14$ for which $r/2^n$ lies between these decimals. Now q is the integral part and $r/2^n$ the decimal part of $(3/2)^n$. By adding the latter to one half of itself we get $(3/2)^{n+1}$. In this way we form a table of values of $(3/2)^n$ for successive n's. The above condition is seen to hold for $n = 15\text{-}21$, but not for $n = 14$.

When $n = 14$, $q = 291$, $r = 15225$, $2^n = 16384$, $t = 281$,

$$\left[\left(\frac{4}{3}\right)^{14}\right] = 56, \qquad \left[\left(\frac{5}{4}\right)^{14}\right] = 22, \qquad \left[\left(\frac{6}{5}\right)^{14}\right] < 22, \cdots.$$

Hence (14) is < 16104. Adding t, we get $16385 < I = 16673$.

Hence the ideal Waring Theorem holds for $n = 14$-21.

For $n \geqq 16$, the decimal in (17) reduces to .677034. We get $F(21) = .00403868$, $q/2^{21} = .00237799$. Hence the ideal Waring theorem holds for any $n \geqq 21$ for which the decimal part of $(3/2)^n$ lies between .00403868 and .99358333. This is true if $21 \leqq n \leqq 162$. But for $n = 163$, the decimal part is .9955.

For $n = 163$, $F(n)$ begins with 28 zeros, while $q/2^n$ begins with 19 zeros, and hence (19) with 19 nines. Our condition holds on to $n = 180$.

Theorem 3. *The ideal Waring theorem holds for exponents* 14-180.

10. When $n = 11$, the least integer t satisfying (16) is 184. Since $2^{11} = 2048$, $q = 86$, $r = 1019$, we have $E = 1104$. The final sum in Theorem 2 is 1397. This plus t is $1581 < I = 2132$.

11. When $n = 12$, $t = 228$. The limits (18) and (19) are .13318 and .83533. For $n = 12$ or 13, the decimal part of $(3/2)^n$ lies between these limits.

12. When $n = 10$, $\log v = \bar{2}.9999990$ and (15) gives

$$.0457575\, t \geqq 7.684117, \qquad t = 168.$$

Since $q = 57$, $r = 681$, we get $E = 737$. Then (14) is < 842. This plus t is $1010 < I = 1079$.

13. When $n = 9$, $q = 38$, $r = 227$, $E = 285$,

$$\log v = \bar{1}.045673, \qquad .0511525\, t \geqq 7.366816, \qquad t = 145,$$

$$[(4/3)^9] = 13, \qquad [(5/4)^9] = 7, \qquad (14) < 323 + 13 + 6 \times 7 = 378,$$
$$t + 378 = 523 < I = 548.$$

14. For $n = 7$ and 8, it is not possible to prove that $I = 143$ or 279 powers suffice by the preceding method, but is possible by use of results from tables.

When $n = 7$, $s = 28$, we perform the calculations in § 4 (without approximations) and get

$$\log_e 1/b_4 < 21589.453, \qquad \log_{10} c_9 = -9380.362.$$

Choose $z = .00001$. By (10) and (12), $k \geqq 38$. We take $k = 39$, which gives $4n - 2 + 3k = 143 = I$. By (5) and (6), $\sigma = .01714655$, $\log C_1 = 11.34357$. Then (7) and (8) give

$$-J = .0055, \qquad \log\log P > 6.2324, \qquad \log\log N = 7.0775.$$

But[6] 44 seventh powers suffice from $l = 14\,782\,969$ to L_0, where

$$\log L_0 = 13.8435104.$$

Here $v = 1/n$ to 7 decimal places. By (15) with $t = 99$, $\log\log L_t = 7.52688$. Thus $L_t > N$. Hence every integer $\geqq l$ is a sum of 143 seventh powers. The same is true for integers $\leqq l$ (*Bulletin, loc. cit.*, p. 713).

15. If $n \geqq 6$, $s \geqq 24$, we find as in §§ 4, 5 that

$$\log 1/b_4 < 2n^5 - 2n^2 + \log 3, \qquad \log C_1 < .6n^2, \qquad \log C_1/c_9 < 2n^5,$$

where the logarithms are natural.

Let $n = 8$ and take $z = .00002$. By (10) and (12), $k \geqq 46$. We take $k = 83$, for which $4n - 2 + 3k = 279 = I$. Then

$$\log \sigma = \bar{4}.089754, \qquad -J = .0271531, \qquad \log\log N = 6.92353.$$

Every integer[7] between $l = 2\,460\,866$ and $2\,851\,491$ is a sum of 81 eighth powers. By ascent, all from l to $L_0 = 2\,235\,617 \times 10^9$ are sums of 102. By (15), $L_t \geqq N$ if

$$.057992\, t + .909806 \geqq 6.92353, \qquad t \geqq 104.$$

Hence all integers $\geqq l$ are sums of I eighth powers. But (*Bulletin, loc. cit.*, p. 713), I powers suffice from 1 to far beyond l.

16. For $n = 6$, take $z = .00002$. Then $k \geqq 31$. For $k = 31$,

$$\sigma = .0210635, \qquad -J = .0039946, \qquad \log\log P > 6.2280972.$$

Adding $\log 6$ to the last, we get $\log\log N$. In his Chicago Doctor's thesis, R. C. Shook proved that every integer between $l = 2120044$ and $L_0 = 516798 \times 10^8$ is a sum of 33 sixth powers. Apply (15), where $v = 1/n$ to 7 decimal places. Thus $L_t \geqq N$, if $t = 77$. Hence 110 powers suffice from l to N. But $4n - 2 + 3k = 115$ suffice after N, while (Shook) 86 suffice before l.

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[6] Dickson, *Researches on Waring's Problem*, Carnegie Institution of Washington, 1935, p. 81.

[7] A. Sugar, *Chicago Master's dissertation*, 1934.

SOLUTION OF WARING'S PROBLEM.

By L. E. Dickson.

I shall prove the Waring theorem in its original sense, in contrast to an asymptotic result. For every $n > 6$ I shall evaluate $g(n)$ such that every positive integer is a sum of $g(n)$ integral n-th powers $\geqq 0$, while not all are sums of $g - 1$. Since this paper is a sequel to my preceding one, I shall continue the numbering of formulas, sections, theorems and lemmas.

17. If $[x]$ denotes the greatest integer $\leqq x$, write

$$q = [(3/2)^n], \quad f = [(4/3)^n], \quad g = [(5/4)^n], \quad s = f + 2g,$$
$$I = 2^n + q - 2.$$

Lemma 9. *If all integers in the interval $(L, L + 2^n)$ are sums of m n-th powers, then all in $(L, L + (n+1)^n)$ are sums of*

$$(20) \qquad m + q + f + g + [(6/5)^n] + \cdots + [(n+1)^n/n^n].$$

This follows from Lemma 4 as in the proof of Theorem 2.

Lemma 10. *If $n \geqq 35$ and all integers in the interval $(L, L + 2^n)$ are sums of m n-th powers, then every integer $\geqq L$ is a sum of $m + q + s - 2$ n-th powers.*

Apply[1] § 8 with $n \geqq 35$. We get (17) with the decimal replaced by .66950. Let t be the least integer satisfying (17). We shall prove that the sum of t and (20) is $< m + q + s - 1$. Cancel m, q, f, g and multiply by $(4/5)^n$. In the resulting inequality every term decreases when n increases since this is true of $p^n, np^n, n \log n \cdot p^n$, if $0 < p < 4/5$ and $n \geqq 6$. Hence it suffices to verify the initial inequality when $n = 35$, whence $t = 912$. S. S. Pillai[2] stated Lemma 10 with an undetermined limit for n, which exceeds 300 for his constant β, and is about 180 for our smaller constant N. We have

$$(21) \qquad 3^n = 2^n q + r, \qquad 1 \leqq r < 2^n.$$

[1] With $l = 3^n$ replaced by $l = 4^n$, whence in the inequality displayed below (15), 3 is replaced by 4. The new inequality holds if $n \geqq 5$. To explain these replacements, note that, in the proofs of Lemmas 12 and 14, $L < U3^n$, $U < 1 + s/4 \leqq h$, whence $L < h3^n = (3/4)4^n$. For Lemma 11, $L < 3^n$. Hence for all three lemmas, the second interval in Lemma 9 includes $(4^n, (n+1)^n)$.

[2] *Annamalai University Journal*, vol. 5 (1936), pp. 145-166. His auxiliary Lemma 13 must be altered as to the definitions of the r_i.

LEMMA 11. *If $n \geqq 35$ and $s \leqq r \leqq 2^n - q - s$, every positive integer is a sum of I n-th powers.*

The integers $2^n q, \cdots, 2^n q + r$ are sums of $q + r \leqq 2^n - s$ n-th powers. The integers $\geqq 2^n q + r = 3^n$ and $< 2^n(q+1)$ and hence $\leqq 3^n + 2^n - r - 1$ are sums of $2^n - r \leqq 2^n - s$ powers. Finally $2^n(q+1)$ is a sum of $q+1$ powers. Applying Lemma 10 with $L = 2^n q$, $m = 2^n s$, we see that every integer $\geqq 2^n q$ is a sum of I n-th powers. Apply also Lemma 1.

LEMMA 12. *If $n \geqq 35$ and $r < s$, every positive integer is a sum of I n-th powers.*

Since 3 belongs to the exponent 2^{n-2} modulo 2^n, $r = 1$ in (21) would show that n is divisible by 2^{n-2}, whence $n \leqq 4$. If $r = 3$, then $3^{n-1} = \frac{1}{3}q2^n + 1$, which was seen to be impossible unless $n - 1 \leqq 4$. Evidently r is not even. Hence $r \geqq 5$.

For $x = 0, 1, \cdots, u(3^n - 2^n q) - 1 = ur - 1$, $2^n qu + x$ is a sum of $qu + x \leqq qu + ur - 1$ n-th powers. If $2^n > ur$, there exist integers $\geqq 3^n u$ and $\leqq 2^n qu + 2^n - 1 = 3^n u + 2^n - 1 - ur$, and each is a sum of $u + 2^n - 1 - ur$ n-th powers. The latter and $qu + ur - 1$ will both be $\leqq 2^n - s$ if

$$s/(r-1) \leqq u \leqq (2^n - s)/(q+r). \tag{22}$$

We decrease r to 5 on the left and increase r to s on the right and obtain the subinterval

$$s/4 \leqq u \leqq (2^n - s)/(q+s). \tag{23}$$

Thus any u satisfying (23) will satisfy (22). The last fraction in (23) is $< 2^n/r$ since $q + s > s > r$, whence our former condition $2^n > ur$ is satisfied.

18. There will exist an integer u satisfying (23), if the difference between the limits is $\geqq 1$ and hence, if

$$2^n \geqq P, \qquad P = 2s + s^2/4 + (1 + \tfrac{1}{4}s)q.$$

But $q < (3/2)^n$, $s < 3(4/3)^n$. Hence $2^n \geqq P$, if

$$2^n > 24(4/3)^n + 9(16/9)^n + 4(3/2)^n.$$

Multiplication by $(9/16)^n$ yields the equivalent inequality

$$(9/8)^n > 9 + 24(3/4)^n + 4(27/32)^n.$$

This holds if $n = 20$ and hence if $n \geqq 20$.

Thus all integers in $(2^n qu, 2^n qu + 2^n)$ are sums of $2^n - s$ n-th powers. By Lemma 10, every integer $\geqq 2^n qu$ is a sum of I n-th powers.

19. To complete the proof of Lemma 12, it remains to prove that every positive integer $< 2^n qu$ is a sum of I n-th powers. By Lemmas 1 and 2, this is true for integers $< 3^n$. It remains to prove it for integers between 3^n and $3^n U$, where U is the least integer u satisfying (23).

LEMMA 13. *All integers in $(3^n w, 3^n w + 2^n)$ are sums of M n-th powers, where M is the maximum of $A = 2^n - wr + w - 1$ and $B = w(q + r)$.*

For, those $\leqq w3^n + 2^n - wr - 1$ are sums of A powers. The next integer is equal to $2^n(qw + 1)$. To this we have to add $1, \cdots, wr - 1$, and obtain $3^n w + 2^n - 1$ as the final sum.

When $w \leqq U - 1$, we shall prove that $B \leqq A$, viz.,

$$2^n \geqq wq + 2wr - w + 1. \tag{24}$$

By definition of U, $U - 1 < s/4$. Employ $W = [h]$, $h = \frac{3}{4}(4/3)^n$. Then $W > U - 1$ if

$$W + 1 > h \geqq 1 + s/4,$$

which holds since $n \geqq 35$. In (24) we replace w by h, increase r to $3(4/3)^n$ and q to $(3/2)^n$, and suppress $1 - w$. Thus (24) holds if $(9/8)^n \geqq 18$, which is true for $n \geqq 25$.

We may therefore take $M = A$ in Lemma 13. Hence by Lemma 4, all in $(3^n w, 3^n w + 2^n(v + 1))$ are sums of $A + v$ n-th powers. But $A + v \leqq I$ if $v \leqq wr - w + q - 1$. For the largest such v,

$$3^n w + 2^n(v + 1) = 3^n(w + 1) + 2^n(wr - w) - r > 3^n(w + 1).$$

But (24) implies the like inequality for a smaller w. Hence all integers in $(3^n w, 3^n(w + 1))$ are sums of I n-th powers for $w = 1, 2, \cdots, U - 1$.

LEMMA 14. *If $n \geqq 35$ and $2^n - q - s < r \leqq 2^n - q - 5$, every positive integer is a sum of I n-th powers.*

For $x = 0, \cdots, 3^n u - 2^n(uq + u - 1) - 1 = 2^n - u(2^n - r) - 1$, $2^n(uq + u - 1) + x$ is a sum of $C = 2^n - u(2^n - r - q - 1) - 2$ n-th powers. The integers $\geqq 3^n u$ and $\leqq 2^n(uq + u) - 1 = 3^n u + u(2^n - r) - 1$ are sums of $D = u + u(2^n - r) - 1$ n-th powers. Then C and D are both $\leqq 2^n - s$ if

$$s/(2^n - r - q - 1) \leqq u \leqq (2^n - s)/(2^n - r + 1). \tag{25}$$

We increase r to $2^n - q - 5$ on the left and decrease r to $2^n - q - s + 1$ on the right and obtain (23) as a subinterval.

Note that the greatest x is positive if $u \leqq (2^n - 1)/(2^n - r)$, which follows from (25). The discussion in § 18 applies also here and shows that every integer $\geqq 3^n u$ is a sum of I n-th powers.

20. We shall prove that the latter is true also of the integers in $(3^n w, 3^n(w+1))$ for $w = 1, \cdots, U-1$, where U is the least integer u satisfying (23). We shall later restrict n so that

$$2^n \geqq wq + (w+1)s. \tag{26}$$

Then $2^n > wq + ws$, whence

$$wr > w(2^n - q - s) > (w-1)2^n, \qquad w(2^n - r) < 2^n.$$

As explained in various recent papers, my method to obtain the minimum decompositions of integers in $J = (3^n w, 3^n w + 2^n)$ is here based on the equations

		Diff.	Wt.
$j\{-3^n + (q+1)2^n\}$	$= j(2^n - r)$	$2^n - r$	$jq \quad (j = 1, \cdots, w-1)$.
$w\{-3^n + (q+1)2^n\}$	$= w(2^n - r)$	$2^n - w(2^n - r)$	wq

To Diff. + Wt. — 1 we add w (since $w3^n$ must be added to get decompositions). For $j \leqq w - 1$, the largest sum is $F = 2^n - r + (w-1)(q+1)$. For our final equation, the sum is $G = 2^n - w(2^n - r) + wq + w - 1$. Then[3] $G \geqq F$ if $(w+1)r \geqq 2^n w - q$. By an hypothesis in Lemma 14, this follows from (26). This proves that every integer in the interval J is a sum of G n-th powers. Hence all in $(3^n w, 3^n w + (y+1)2^n)$ are sums of $G + y$ powers. Use the largest y for which $G + y \leqq I$. The interval now ends with

$$3^n w + 2^n\{w(2^n - r) + (1-w)q - w\} \geqq (w+1)3^n$$

if the number in brackets is $\geqq 1 + q$ and hence is $\geqq (3/2)^n$. The condition is $w(2^n - r - q - 1) > 0$ and is satisfied.

For W below (24), $W > U - 1 \geqq w$. Hence (26) follows from the like inequality with w replaced by W, and q increased to $(3/2)^n$, and s increased to $(4/3)^n + 2(5/4)^n$. Division by $2^n/4$ yields

$$1 \geqq 4(2/3)^n + 8(5/8)^n + 3(8/9)^n + 6(5/6)^n.$$

The second member evidently decreases when n increases. But the inequality holds if $n = 23$ and hence if $n \geqq 23$.

This proof of Lemma 14 is readily extended to $r = 2^n - q - i$, $i = 3$ or 4, by employing (25) instead of (23), and using $s < (4/3)^n + 2(5/4)^n$ in § 18.

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[3] Expressed untechnically, the minimum decompositions are free of terms 3^n.

Lemmas 11-14 yield

THEOREM 4. *If* $n \geqq 35$ *and* $r \leqq 2^n - q - 3$, *every positive integer is a sum of I n-th powers.*

This inequality has been verified for $4 \leqq n \leqq 180$.

21. Let $r \geqq 2^n - q$. If $r = 2^n - 1$, (21) gives $3^{2n} \equiv 1 \pmod{2^n}$. Since 3 belongs to the exponent 2^{n-2}, $2n$ is divisible by 2^{n-2}, but is less than it if $n \geqq 6$. Next, if $r = 2^n - 3$, (21) gives $3^{n-1} = \frac{1}{3}(q+1)2^n - 1$, whence $n - 1 \leqq 5$. Since r is odd, we conclude that $r \leqq 2^n - 5$. Thus

$$(27) \qquad 5 \leqq R \leqq q, \qquad R = 2^n - r.$$

Since many integers $< 4^n$ require more than I summands (§ 22), we shall make our ascents from an interval containing 4^n. Let

$$(28) \qquad 4^n = 3^n f + 2^n h + j, \qquad 0 \leqq 2^n h + j < 3^n, \qquad 0 \leqq j < 2^n.$$

Then $f = [(4/3)^n]$. Fortunately we can determine h and j in terms of q, r, f. In (28) replace 3^n by its value (21). Then $j + fr \equiv 0$, $j - fR \equiv 0 \pmod{2^n}$. But $fR \leqq fq < (4/3 \cdot 3/2)^n$. Hence gR and j are numbers x for which $0 \leqq x < 2^n$. Thus $|j - fR| < 2^n$. The multiple $j - fR$ of 2^n is therefore zero. Eliminating 3^n, j, r from (28) by use of (21) and (27), we get $4^n = 2^n(fq + h + f)$. Hence

$$(29) \qquad j = fR, \qquad h = 2^n - fq - f.$$

For these values, the first inequality (28) is satisfied.

If $E = \max(A = 2^n - f(q - R) - 1, 2^n - fR)$, every integer in the interval $(3^n f + 2^n h, 3^n f + 2^n(h+1))$ is a sum of E n-th powers, as shown by

$$\begin{aligned} &3^n f + 2^n h + x, \qquad && (x = 0, 1, \cdots, fR - 1), \\ &4^n + y, && (y = 0, 1, \cdots, 2^n - fR - 1). \end{aligned}$$

By Theorem 2, all in $(4^n, (n+1)^n)$ are sums of (14) n-th powers.

Let $n \geqq 35$. Let t be the least integer satisfying (17) with the decimal replaced by .66950.

First, let $E = 2^n - fR$. Then all integers $\geqq 4^n$ are sums of

$$t + 2^n - fR + q + f + K, \qquad K = [(5/4)^n] + \cdots + [(n+1)^n/n^n].$$

Since $K < (n-3)(5/4)^n$, this sum is $< I$ if

$$R - 1 \geqq (t+2)(3/4)^n + (n-3)(15/16)^n.$$

The latter decreases when n increases. For $n = 35$ it becomes $R \geqq 4.3816$ since $t = 912$. Hence it holds when $n \geqq 35$.

Second, let $E = A$. Then all integers $\geqq 4^n$ are sums of $t + A + q + f + K$ powers. This will be $\leqq I$, if $f(q - R - 1) \geqq t + 1 + K$. If $n = 35$, this holds if $q - R - 1 \geqq .21$. Hence if $n \geqq 35$, it holds if $R \leqq q - 2$.

THEOREM 5. *If $n \geqq 35$ and $r \geqq 2^n - q + 2$, every integer $\geqq 4^n$ is a sum of I n-th powers.*

22. *Minimum decomposition of integers between 3^n and 4^n when $r \geqq 2^n - q$.* My general theory (§ 20) here has the peculiarity that there is no effective equation. For, such an equation is a multiple of $(q + 1)2^n = 3^n + 2^n - r$. The weight $q + 1$ of the left member is $\geqq$ the weight $1 + 2^n - r$ of the right. Expressed otherwise every minimum decomposition of an integer $< 4^n$ is a normal decomposition $3^n x + 2^n y + z$ in which x, y, z are integers $\geqq 0$ and x is a maximum, so that $2^n y + z < 3^n$, $z < 2^n$. Comparison with (28) and (21) gives $x \leqq f$, $y \leqq q$. Write $S = x + y + z$.

If $y = q$, then $z < r$. But $r \leqq 2^n - 5$ (§ 20). Hence $S < I + f - 3$.

Let $x = f$. Then $y \leqq h$. But $h \leqq q$. If $y = h$, then $z < fR < 2^n$, and $S \leqq I + f$, with equality only when $h = q$, $fR = 2^n - 1$. Next, if $y \leqq h - 1$, $S \leqq f + h + 2^n - 2 \leqq I + f$, and $S = I + f$ only when $y = h - 1$, $h = q$.

Finally, if $x \leqq f - 1$ and $y \leqq q - 1$, then $S \leqq I + f - 1$.

Hence if $r \geqq 2^n - q$, every integer $\leqq 4^n$ is a sum of $I + f$ n-th powers. The integer $3^n(f - 1) + 2^n(q - 1) + 2^n - 1$ is a sum of $I + f - 1$, but not fewer, n-th powers. Only in the special case $h = q$ does there exist an integer requiring $I + f$ powers.

These results and Theorem 5 yield

THEOREM 6. *If $n \geqq 35$ and $r \geqq 2^n - q + 2$, then $g(n) = I + f$ or $I + f - 1$ according as $2^n = fq + f + q$ or $2^n < fq + f + q$.*

23. To prove Theorem 6 also when $r = 2^n - q + 1$, whence $R = q - 1$, we permit $I + f - 1$ summands instead of I in the second case $E = A$ of § 21. The condition is now $f(q - R) \geqq t + 2 + K$, which holds if $q - R \geqq .21$ and hence if $R \leqq q - 1$.

For $r = 2^n - q$ (or $R = q$) a like proof applies if we permit $I + f(1 + d)$ summands, where $d = .21$ if $n \geqq 35$, $d = .001614$ if $n \geqq 180$. But if we permit only $I + f - 1$ summands, we must start our ascents from an interval beyond 4^n.

24. There remain only the cases $r = 2^n - q - i$, $i = 1, 2$.

If $r = 2^n - q - 1$, (21) gives $3^n = (q + 1)(2^n - 1)$. Hence $2^n - 1 = 3^t$, $t \leqq n$, $3^{2t} \equiv 1 \pmod{2^n}$. Since 3 belongs to the exponent 2^{n-2}, the latter divides $2t$. But $2t \leqq 2n < 2^{n-2}$ if $n \geqq 6$. Hence $t = 0$, $n = 1$.

THE UNIVERSITY OF CHICAGO.

ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OF $2h$ SQUARES.[1]

By R. D. James.

1. *Introduction.* Let $N(2^\alpha m, 2h)$ denote the number of representations of an integer $2^\alpha m$, $\alpha \geqq 0$, m an odd[2] integer > 0, as a sum of $2h$ squares. Both the arrangement of the squares and the signs of their square roots are relevant in counting the representations. Let $M(2^\alpha m, 2h, b)$ denote the number of representations of $2^\alpha m$ as a sum of $2h$ squares, exactly b of which are odd and occupy the first b places in the representations. Further, let

$$\xi_r(m) = \sum_{d|m} (-1|d) d^r, \qquad \sigma_r(m) = \sum_{d|m} d^r,$$

where $(-1|d) = (-1)^{(d-1)/2}$ is the Jacobi symbol.

In this paper we prove the following results.

THEOREM 1. *For $\alpha \geqq 3$, $h \geqq 2$, we have*

$$(1.11) \quad M(2^\alpha m, 2h, 4s) = 2^{2s} \sum_{s/2 \leqq v \leqq (h-s)/2} \binom{h-2s}{2v-s} M(2^{\alpha-1} m, 2h, 4v).$$

For $h \geqq 2$ we have

$$(1.12) \quad M(4m, 2h, 4s) = 2^{2s} \sum_{(s-1)/2 \leqq v \leqq (h-s-1)/2} \binom{h-2s}{2v-s+1} M(2m, 2h, 4v+2).$$

For $h \geqq 2$ and $m \equiv 1 \pmod 4$ we have

$$(1.13) \quad M(2m, 2h, 4s+2) = 2^{2s+1} \sum_{s/2 \leqq v \leqq (h-s-1)/2} \binom{h-2s-1}{2v-s} M(m, 2h, 4v+1)$$

For $h \geqq 2$ and $m \equiv 3 \pmod 4$ we have

$$(1.14) \quad M(2m, 2h, 4s+2) = 2^{2s+1} \sum_{(s-1)/2 \leqq v \leqq (h-s-2)/2} \binom{h-2s-1}{2v-s+1} M(m, 2h, 4v+$$

[1] Presented to the American Mathematical Society, November 30, 1935 and December 31, 1935 under the titles "The number of representations of an integer as a sum of six, ten, or fourteen squares" and "The number of representations of an integer as a sum of twelve, sixteen, or twenty squares."

[2] Throughout this paper m will always denote an odd integer.

THEOREM 2. *Let α and β be integers such that $\alpha \geqq \beta + 2 \geqq 2$. Then, given the integers $A_s(h)$, $1 \leqq s \leqq (h-1)/2$, there exist integers $A_v(h,\beta)$, $1 \leqq v \leqq (h-1)/2$, $0 \leqq \beta \leqq \alpha - 2$ such that*

$$\sum_{1 \leqq s \leqq (h-1)/2} A_s(h) M(2^\alpha m, 2h, 4s) = \sum_{1 \leqq v \leqq (h-1)/2} A_v(h,\beta) M(2^{\alpha-\beta} m, 2h, 4v). \tag{1.21}$$

The integers $A_v(h,\beta)$ are defined by the formulas

$$A_v(h,0) = A_v(h),$$

$$A_v(h,\beta) = \sum_{1 \leqq s \leqq \min(2v, h-2v)} 2^{2s} \binom{h-2s}{2v-s} A_s(h, \beta - 1). \tag{1.22}$$

Moreover they satisfy the equation

$$\sum_{1 \leqq v \leqq (h-1)/2} A_v(h,\beta) = 2^{(h-1)\beta} \sum_{1 \leqq s \leqq (h-1)/2} A_s(h). \tag{1.23}$$

THEOREM 3. *For $\alpha \geqq 0$, $m > 0$, $t \geqq 1$ there exist integers $\beta(t)$, $\gamma_r(t)$, $0 \leqq r \leqq t-1$, such that*

$$\begin{aligned} &\beta(t) N(2^\alpha m, 4t+2) - 4[(-1)^t(-1|m) 2^{2t(\alpha+1)} + 1]\xi_{2t}(m) \\ &\qquad = \sum_{r=0}^{t-1} \gamma_r(t) M(2^{\alpha+2} m, 4t+2, 4t-4r), \end{aligned} \tag{1.31}$$

with

$$\sum_{r=0}^{t-1} \gamma_r(t) = 0. \tag{1.32}$$

THEOREM 4. *For $\alpha \geqq 0$, $m > 0$, $t \geqq 1$ there exist integers $\lambda(t)$, $\mu_r(t)$, $0 \leqq r \leqq t-1$, such that*

$$\begin{aligned} &\lambda(t) N(2^\alpha m, 4t+4) \\ &\qquad - 2^{2t+3}\left[2^{(2t+1)\alpha} - (-1)^t \frac{2^{(2t+1)\alpha} - 2^{2t+2} + 1}{2^{2t+1} - 1}\right] \sigma_{2t+1}(m) \\ &\qquad = \sum_{r=0}^{t-1} \mu_r(t) M(2^{\alpha+2} m, 4t+4, 4r+4), \end{aligned} \tag{1.41}$$

with

$$\sum_{r=0}^{t-1} \mu_r(t) = 0. \tag{1.42}$$

Theorem 1 is proved by a straightforward examination of the relations between $M(2^\alpha m, 2h, 4s)$ and $M(2^{\alpha-1} m, 2h, 4v)$. Theorem 2 is a consequence of Theorem 1. In applications the condition (1.23) is the important part of this theorem. Theorems 3 and 4 depend on results from the theory of theta-functions. They are obtained in the usual way by equating the coefficients of two power series representing the same function. The conditions (1.32) and (1.42) are interesting results.

As examples of the application of the above general theorems to particular problems we quote the following results.

THEOREM 5. *For no* $\alpha \geqq 1$ *is an equation of the form*

$$N(2^\alpha m, 14) = b_6(\alpha)\xi_6(m)$$

valid for all odd integers m *with* $b_6(\alpha)$ *independent of* m.

THEOREM 6. *For no* $\alpha \geqq 1$ *is an equation of the form*

$$N(2^\alpha m, 4t+4) = c_{2t+1}(\alpha)\sigma_{2t+1}(m), \qquad (t = 3, 4, \text{ or } 5),$$

valid for all odd integers m *with* $c_{2t+1}(\alpha)$ *independent of* m.

Theorems 5 and 6 are extensions of results proved by E. T. Bell.[3] He showed that no equations of the form $N(m, 4t+2) = b_{2t}\xi_{2t}(m)$, $t \geqq 3$ or $N(2m, 4t+4) = c_{2t+1}\sigma_{2t+1}(m)$, $t \geqq 3$ are possible for all odd integers m with b_{2t} and c_{2t+1} independent of m. J. W. L. Glaisher[4] gave results of the type of Theorems 5 and 6 but without proof.

2. *Proof of Theorem* 1. For a, b, h, and s integers such that $0 \leqq 2s \leqq h$, $0 \leqq a \leqq 2s$, $0 \leqq b \leqq h - 2s$ let $M_{ab}(2^\alpha m, 2h, 4s)$ denote the number of solutions of the equation

$$2^\alpha m = \sum_{i=1}^{4s} x_i^2 + \sum_{j=1}^{2h-4s} y_j^2 \tag{2.11}$$

with the added restrictions

$$x_i \text{ odd}, \qquad y_j \text{ even}; \tag{2.12}$$

$$\begin{aligned} &x_{2\mu-1} + x_{2\mu} \equiv 2 \pmod 4, && (\mu = 1, \cdots, a); \\ &x_{2\mu-1} + x_{2\mu} \equiv 0 \pmod 4, && (\mu = a+1, \cdots, 2s); \end{aligned} \tag{2.13}$$

$$\begin{aligned} &y_{2\nu-1} + y_{2\nu} \equiv 2 \pmod 4, && (\nu = 1, \cdots, b); \\ &y_{2\nu-1} + y_{2\nu} \equiv 0 \pmod 4, && (\nu = b+1, \cdots, h-2s). \end{aligned} \tag{2.14}$$

We have the following result.

LEMMA 1. *For* $\alpha \geqq 1$ *we have*

$$M_{ab}(2^\alpha m, 2h, 4s) = M(2^{\alpha-1}m, 2h, 2s+2b).$$

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[3] *Journal of the London Mathematical Society*, vol. 4 (1929), pp. 279-285; *Journal für die reine und angewandte Mathematik*, vol. 163 (1930), pp. 65-70.

[4] *Quarterly Journal of Mathematics*, vol. 38 (1906-7), pp. 178-237; *Proceedings of the London Mathematical Society*, Series 2, vol. 5 (1907), pp. 479-490.

Proof. Write

$$(2.21)\quad \begin{aligned}
X_\mu &= \tfrac{1}{2}(x_{2\mu-1} + x_{2\mu}), && (\mu = 1, \cdots, a);\\
X_\mu &= \tfrac{1}{2}(x_{2\mu-1} - x_{2\mu}), && (\mu = a+1, \cdots, 2s);\\
X_{2s+2\nu-1} &= \tfrac{1}{2}(y_{2\nu-1} + y_{2\nu}), && (\nu = 1, \cdots, b);\\
X_{2s+2\nu} &= \tfrac{1}{2}(y_{2\nu-1} - y_{2\nu}), && (\nu = 1, \cdots, b);\\
Y_\mu &= \tfrac{1}{2}(x_{2\mu-1} - x_{2\mu}), && (\mu = 1, \cdots, a);\\
Y_\mu &= \tfrac{1}{2}(x_{2\mu-1} + x_{2\mu}), && (\mu = a+1, \cdots, 2s);\\
Y_{2s+2\nu-2b-1} &= \tfrac{1}{2}(y_{2\nu-1} + y_{2\nu}), && (\nu = b+1, \cdots, h-2s);\\
Y_{2s+2\nu-2b} &= \tfrac{1}{2}(y_{2\nu-1} - y_{2\nu}), && (\nu = b+1, \cdots, h-2s).
\end{aligned}$$

Using (2.12), (2.13), and (2.14) we find that the X's are odd integers and the Y's are even integers. It then follows from (2.11) and (2.21) that

$$(2.22)\quad 2^{a-1}m = \sum_{i=1}^{2s+2b} X_i^2 + \sum_{j=1}^{2h-2s-2b} Y_j^2, \qquad \begin{matrix} X_i \text{ odd}, \\ Y_j \text{ even}. \end{matrix}$$

Conversely write

$$(2.23)\quad \begin{aligned}
x_{2\mu-1} &= X_\mu + Y_\mu, && (\mu = 1, \cdots, a);\\
x_{2\mu} &= X_\mu - Y_\mu, && (\mu = 1, \cdots, a);\\
x_{2\mu-1} &= X_\mu + Y_\mu, && (\mu = a+1, \cdots, 2s);\\
x_{2\mu} &= Y_\mu - X_\mu, && (\mu = a+1, \cdots, 2s);\\
y_{2\nu-1} &= X_{2s+2\nu-1} + X_{2s+2\nu}, && (\nu = 1, \cdots, b);\\
y_{2\nu} &= X_{2s+2\nu-1} - X_{2s+2\nu}, && (\nu = 1, \cdots, b);\\
y_{2\nu-1} &= Y_{2s+2\nu-2b-1} + Y_{2s+2\nu-2b}, && (\nu = b+1, \cdots, h-2s);\\
y_{2\nu} &= Y_{2s+2\nu-2b-1} - Y_{2s+2\nu-2b}, && (\nu = b+1, \cdots, h-2s).
\end{aligned}$$

Then from (2.22) we obtain (2.11) with the restrictions (2.12), (2.13), and (2.14). It is easily seen from (2.21) and (2.23) that distinct solutions of (2.11) give rise to distinct solutions of (2.22) and conversely. Hence the number of solutions of (2.11) with the restrictions (2.12), (2.13), and (2.14) is equal to the number of solutions of (2.22). Equation (2.22), however, has $M(2^{a-1}m, 2h, 2s+2b)$ solutions and the lemma is proved.

The proof of Theorem 1 now proceeds as follows. To obtain the solutions of (2.11) with only the restrictions (2.12) we must add up the number of solutions $M_{ab}(2m, 2h, 4s)$ for $0 \leqq a \leqq 2s$, and $0 \leqq b \leqq h - 2s$, taking into account the fact that a different arrangement of the x's and y's constitutes a different solution. Thus each solution of (2.11) with the restrictions (2.12), (2.13), and (2.14) counts $\binom{2s}{a}\binom{h-2s}{b}$ times as a solution of (2.11) with only the restriction (2.12). Hence by Lemma 1 we have

$$
\begin{aligned}
M(2^a m, 2h, 4s) &= \sum_{a=0}^{2s} \sum_{b=0}^{h-2s} \binom{2s}{a}\binom{h-2s}{b} M_{ab}(2^a m, 2h, 4s) \\
&= \sum_{a=0}^{2s} \binom{2s}{a} \sum_{b=0}^{h-2s} \binom{h-2s}{b} M(2^{a-1}m, 2h, 2s+2b) \qquad (2.24)\\
&= 2^{2s} \sum_{b=0}^{h-2s} \binom{h-2s}{b} M(2^{a-1}m, 2h, 2s+2b).
\end{aligned}
$$

A sum of $2s+2b$ odd squares is congruent to $2s+2b$ (mod 4) and a sum of even squares is congruent to 0 (mod 4). Therefore if $\alpha \geqq 3$ we have $M(2^{a-1}m, 2h, 2s+2b) = 0$ unless $2s+2b \equiv 0 \pmod 4$, that is, $2s+2b = 4v$. Then from (2.24) with $2b$ replaced by $4v-2s$ we obtain (1.11).

The remaining formulas are proved in exactly the same way. The only changes are that we must have $2s+2b = 4v+2$, $4v+1$, and $4v+3$ in (1.12), (1.13), and (1.14), respectively.

3. *Proof of Theorem* 2. From Theorem 1 we have

$$
\begin{aligned}
&\sum_{1 \leqq s \leqq (h-1)/2} A_s(h) M(2^a m, 2h, 4s) \\
&\quad = \sum_{1 \leqq s \leqq (h-1)/2} 2^{2s} \sum_{s/2 \leqq v \leqq (h-s)/2} \binom{h-2s}{2v-s} M(2^{a-1}m, 2h, 4v) \\
&\quad = \sum_{1 \leqq v \leqq (h-1)/2} M(2^{a-1}m, 2h, 4v) \sum_{1 \leqq s \leqq \min(2v, h-2v)} 2^{2s} \binom{h-2s}{2v-s} A_s(h) \\
&\quad = \sum_{1 \leqq v \leqq (h-1)/2} A_v(h, 1) M(2^{a-1}m, 2h, 4v),
\end{aligned}
$$

where

$$
\begin{aligned}
\sum_{1 \leqq v \leqq (h-1)/2} A_v(h, 1) &= \sum_{1 \leqq v \leqq (h-1)/2} \; \sum_{1 \leqq s \leqq \min(2v, h-2v)} 2^{2s} \binom{h-2s}{2v-s} A_s(h) \\
&= \sum_{1 \leqq s \leqq (h-1)/2} 2^{2s} A_s(h) \sum_{s/2 \leqq v \leqq (h-s)/2} \binom{h-2s}{2v-s} \\
&= 2^{h-1} \sum_{1 \leqq s \leqq (h-1)/2} A_s(h).
\end{aligned}
$$

This proves (1.21) and (1.23) for the case $\beta = 1$.

We complete the proof by induction. Assume that we have

$$
\sum_{1 \leqq s \leqq (h-1)/2} A_s(h) M(2^a m, 2h, 4s) = \sum_{1 \leqq v \leqq (h-1)/2} A_v(h, \beta-1) M(2^{a-\beta+1} m, 2h, 4v)
$$

with

$$
\sum_{1 \leqq v \leqq (h-1)/2} A_v(h, \beta-1) = 2^{(h-1)(\beta-1)} \sum_{1 \leqq s \leqq (h-1)/2} A_s(h).
$$

Then, using Theorem 1 with α replaced by $\alpha-\beta+1$ we obtain (1.21), (1.22), and (1.23).

4. *Proof of Theorems* 3 *and* 4. The methods of proof of Theorems 3 and

4 are similar and we give the details only for Theorem 3. The elliptic function $\mathrm{dn}(u, k)$ has the expansion [5]

$$\mathrm{dn}(u,k) = 1 + \sum_{t=1}^{\infty} \frac{(-1)^t u^{2t}}{(2t)!} \left[\sum_{r=0}^{t-1} d_r(t) k^{2t-2r} \right], \tag{4.11}$$

where the $d_r(t)$ are integers > 0. In the usual notation for the theta-functions we have

$$\begin{aligned}
&\vartheta_0(x) = \vartheta_0(x,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx, \\
&\vartheta_2(x) = \vartheta_2(x,q) = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/4} \cos(2n+1)x, \\
&\vartheta_3(x) = \vartheta_3(x,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nx, \\
&\vartheta_0 = \vartheta_0(0), \qquad \vartheta_2 = \vartheta_2(0) = 2\sum_{n=0}^{\infty} q^{(2n+1)^2/4}, \\
&\vartheta_3 = \vartheta_3(0) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.
\end{aligned} \tag{4.12}$$

The function $\mathrm{dn}(u, k)$ is related to the theta quotients by the formulas

$$\frac{\vartheta_3(x)}{\vartheta_0(x)} = \frac{\vartheta_3}{\vartheta_0} \mathrm{dn}(\vartheta_3^2 x, \vartheta_2^2/\vartheta_3^2), \tag{4.13}$$

$$\frac{\vartheta_3(x)}{\vartheta_2(x)} = \frac{\vartheta_3}{\vartheta_2} \mathrm{dn}(i\vartheta_3^2 x, \vartheta_0^2/\vartheta_3^2). \tag{4.14}$$

It follows from (4.11), (4.13), and (4.14) that

$$\frac{\vartheta_3(x)}{\vartheta_0(x)} = \frac{\vartheta_3}{\vartheta_0} \left\{ 1 + \sum_{t=1}^{\infty} \frac{(-1)^t x^{2t}}{(2t)!} \left[\sum_{r=0}^{t-1} d_r(t) \vartheta_3^{4r} \vartheta_2^{4t-4r} \right] \right\}, \tag{4.21}$$

$$\frac{\vartheta_3(x)}{\vartheta_2(x)} = \frac{\vartheta_3}{\vartheta_2} \left\{ 1 + \sum_{t=1}^{\infty} \frac{x^{2t}}{(2t)!} \left[\sum_{r=0}^{t-1} d_r(t) \vartheta_3^{4r} \vartheta_0^{4t-4r} \right] \right\}. \tag{4.22}$$

We can, however, obtain these expansions in another way. It is known that [6]

$$\frac{\vartheta_3(x)}{\vartheta_0(x)} = \frac{1}{\vartheta_0\vartheta_3} \left\{ 1 + 4\sum_{n=1}^{\infty} q^n \left[\sum (-1|\tau) \cos(2nx/\tau) \right] \right\}, \tag{4.31}$$

$$\frac{\vartheta_3(x)}{\vartheta_2(x)} = \frac{1}{\vartheta_2\vartheta_3} \left\{ \sec x + 4\sum_{n=1}^{\infty} q^n \left[\sum (-1|\tau) \cos \tau x \right] \right\}, \tag{4.32}$$

where the summation for τ is over all *odd* divisors of n. By expanding the cosines in (4.31) and (4.32) and then comparing the result with (4.21) and (4.22) we obtain

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[5] See, for example, Tannery and Molk, *Éléments de la Théorie des Fonctions elliptiques*, Paris (1902), Vol. 4, p. 92.

[6] E. T. Bell, *Messenger of Mathematics*, vol. 49 (1919-20), pp. 78-84.

$$(4.33)\quad \vartheta_3{}^2 \sum_{r=0}^{t-1} d_r(t)\vartheta_3{}^{4r}\vartheta_2{}^{4t-4r} = 2^{2t+2}\sum_{n=1}^{\infty} q^n \Big[\, \Sigma\, (-1|\tau)\,(n/\tau)^{2t} \Big],$$

$$(4.34)\quad \vartheta_3{}^2 \sum_{r=0}^{t-1} d_r(t)\vartheta_3{}^{4r}\vartheta_0{}^{4t-4r} = K_{2t} + 4(-1)^t \sum_{n=1}^{\infty} q^n \Big[\, \Sigma\, (-1|\tau)\tau^{2t} \Big].$$

We use these formulas to prove the following result.

LEMMA 2. *For* $\alpha \geqq 0$, $m > 0$, $t \geqq 1$ *we have*

$$(4.41)\quad \sum_{r=0}^{t-1} d_r(t) M(2^{\alpha+2}m, 4t+2, 4t-4r) = 2^{2t(\alpha+1)+2}(-1|m)\xi_{2t}(m),$$

$$(4.42)\quad \sum_{r=0}^{t} h_r(t) M(2^{\alpha+2}m, 4t+2, 4t-4r) = 4\xi_{2t}(m),$$

where

$$(4.43)\qquad h_r(t) = (-1)^r \sum_{s=0}^{\min(r,t-1)} \binom{t-s}{r-s} d_s(t).$$

Proof. From (4.12) it follows that

$$(4.44)\quad \begin{aligned} \vartheta_3{}^{4r+2}\vartheta_2{}^{4t-4r} &= \Big(1 + 2\sum_{n=1}^{\infty} q^{n^2}\Big)^{4r+2} \Big(2\sum_{n=0}^{\infty} q^{(2n+1)^2/4}\Big)^{4t-4r} \\ &= \sum_{n=0}^{\infty} q^{n/4} M(n, 4t+2, 4t-4r). \end{aligned}$$

Since a sum of $4t-4r$ odd squares is congruent to $4t-4r \equiv 0 \pmod 4$ and a sum of even squares is congruent to $0 \pmod 4$, we have $M(n, 4t+2, 4t-4r) = 0$ unless n is divisible by 4. Hence (4.44) becomes

$$(4.45)\qquad \vartheta_3{}^{4r+2}\vartheta_2{}^{4t-4r} = \sum_{n=0}^{\infty} q^n M(4n, 4t+2, 4t-4r).$$

Comparing (4.45) and (4.33) for $n = 2^\alpha m$ we obtain

$$(4.46)\quad \sum_{r=0}^{t-1} d_r(t) M(2^{\alpha+2}m, 4t+2, 4t-4r) = 2^{2t+2}\, \Sigma\, (-1|\tau)\,(2^\alpha m/\tau)^{2t},$$

where the summation for τ is over all *odd* divisors of $2^\alpha m$, that is, over *all* divisors of m. To complete the proof of (4.41) it remains to show that the right sides of (4.41) and (4.46) are equal. We have [7]

$$\begin{aligned} \sum_{\tau|m} (-1|\tau)(m/\tau)^{2t} &= (-1|m) \sum_{\tau|m} (-1|(m/\tau))\,(m/\tau)^{2t} \\ &= (-1|m) \sum_{d|m} (-1|d)\, d^{2t} = (-1|m)\xi_{2t}(m). \end{aligned}$$

To prove (4.42) we first employ the well known relation $\vartheta_0{}^4 = \vartheta_3{}^4 - \vartheta_2{}^4$

[7] It is a property of the Jacobi symbol that $(-1 \mid PQ) = (-1 \mid P)\,(-1 \mid Q)$.

and substitute for ϑ_0^{4t-4r} in (4.34). On expanding and collecting terms we obtain

$$\sum_{r=0}^{t} h_r(t)\vartheta_3^{4r+2}\vartheta_2^{4t-4r} = 4\sum_{n=1}^{\infty} q^n \Big[\sum (-1|\tau)\tau^{2t} \Big],$$

where the $h_r(t)$ are given by (4.43). The proof then proceeds in a similar way to that for (4.41).

Theorem 3 now follows immediately. In Lemma 2 multiply (4.41) by $(-1)^t$ and add the result to (4.42). This gives

$$\sum_{r=0}^{t-1} ((-1)^t d_r(t) + h_r(t))M(2^{\alpha+2}m, 4t+2, 4t-4r)$$
$$+ h_t(t)M(2^{\alpha+2}m, 4t+2, 0) = 4[(-1)^t(-1|m)2^{2t(\alpha+1)} + 1]\xi_{2t}(m).$$

If we write $\beta(t) = h_t(t)$, $\gamma_r(t) = -\{(-1)^t d_r(t) + h_r(t)\}$ we obtain (1.31) of Theorem 3 since $M(2^{\alpha+2}m, 4t+2, 0) = N(2^\alpha m, 4t+2)$. It remains to prove (1.32). We have

$$\begin{aligned}\sum_{r=0}^{t-1} h_r(t) &= \sum_{r=0}^{t-1}\sum_{s=0}^{\min(r,t-1)} (-1)^r \binom{t-s}{r-s} d_s(t)\\ &= \sum_{s=0}^{t-1} d_s(t) \sum_{r=s}^{t-1} (-1)^r \binom{t-s}{r-s}\\ &= \sum_{s=0}^{t-1} d_s(t) \sum_{r=0}^{t-s-1} (-1)^{s-r} \binom{t-s}{r}\\ &= \sum_{s=0}^{t-1} d_s(t)(-1)^s\{(1-1)^{t-s} - (-1)^{t-s}\}\\ &= -\sum_{s=0}^{t-1} (-1)^s d_s(t).\end{aligned}$$

Hence

$$\sum_{r=0}^{t-1} \gamma_r(t) = -\sum_{r=0}^{t-1} (-1)^t d_r(t) - \sum_{r=0}^{t-1} h_r(t) = 0.$$

As remarked above Theorem 4 is proved in a similar way. The difference is that we use the functions $\vartheta'_0(x)/\vartheta_0(x)$ and $\vartheta'_2(x)/\vartheta_2(x)$ instead of $\vartheta_3(x)/\vartheta_0(x)$ and $\vartheta_3(x)/\vartheta_2(x)$.

5. *Proof of Theorem* 5. From Theorem 3 with $t = 3$ we have

$$\begin{aligned}&\beta(3)N(2^\alpha m, 14) + 4[(-1|m)2^{6\alpha+6} - 1]\xi_6(m)\\ &\qquad = \sum_{r=0}^{2} \gamma_r(3)M(2^{\alpha+2}m, 14, 12-4r),\end{aligned} \tag{5.1}$$

with $\sum_{r=0}^{2} \gamma_r(3) = 0$. Then, from Theorem 2 with $h = 7$, α replaced by $\alpha + 2$, $\beta = \alpha$, $A_s(7) = \gamma_{3-s}(3)$, the right side of (5.1) becomes

$$\sum_{v=1}^{3} A_v(7, \alpha) M(4m, 14, 4v), \tag{5.2}$$

where

$$\sum_{v=1}^{3} A_v(7, \alpha) = 2^{6\alpha} \sum_{r=0}^{2} \gamma_r(3) = 0.$$

By applying first (1.12) and then (1.13) or (1.14) of Theorem 1 the expression (5.2) further reduces to

$$8[\{255A_1(7,\alpha) + 256A_2(7,\alpha) + 256A_3(7,\alpha)\}\{M(m,14,5) + M(m,14,9)\} \\ + A_1(7,\alpha)\{M(m,14,1) + M(m,14,13)\}]$$

or

$$16[\{23A_1(7,\alpha) + 24A_2(7,\alpha) + 32A_3(7,\alpha)\}\{M(m,14,3) + M(m,14,11)\} \\ + \{210A_1(7,\alpha) + 208A_2(7,\alpha) + 192A_3(7,\alpha)\}M(m,14,7)]$$

according as $m \equiv 1 \pmod 4$ or $m \equiv 3 \pmod 4$. Finally, with the condition $\sum_{v=1}^{3} A_v(7, \alpha) = 0$, these expressions become

$$8A_1(7,\alpha)[M(m,14,1) - M(m,14,5) - M(m,14,9) + M(m,14,13)],$$
$$-[144A_1(7,\alpha) + 128A_2(7,\alpha)][M(m,14,3) - 2M(m,14,7) + M(m,14,11)]$$

in the respective cases. Hence from (5.1) we obtain

$$\beta(3)N(2^{\alpha}m, 14) + 4[(-1|m)2^{6\alpha+6} - 1]\xi_6(m)$$
$$= [g(\alpha)/g(0)][\beta(3)N(m,14) + 4[(-1|m)2^6 - 1]\xi_6(m)], \tag{5.4}$$

where

$$g(\alpha) = \begin{cases} 8A_1(7,\alpha), & m \equiv 1 \pmod 4, \\ -[144A_1(7,\alpha) + 128A_2(7,\alpha)], & m \equiv 3 \pmod 4. \end{cases}$$

Since it is known that $N(m, 14) = b_6\xi_6(m)$ is impossible[8] with b_6 independent of m it then follows from (5.4) that there is no α for which $N(2^{\alpha}m, 14) = b_6(\alpha)\xi_6(m)$ is possible.

Theorem 6 is proved by precisely similar arguments but using Theorem 4 instead of Theorem 3.

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[8] E. T. Bell, *loc. cit.*

NEW RESULTS FOR THE NUMBER $g(n)$ IN WARING'S PROBLEM.[1]

By H. S. Zuckerman.

I. *Introduction.* Waring's problem deals with the representation of positive integers by sums of positive or zero integral n-th powers. The number $g(n)$ is defined as the minimum s such that

$$(1) \qquad m = \sum_{i=1}^{s} x_i{}^n, \quad x_i \text{ integers}, \quad x_i \geqq 0, \qquad (i = 1, 2, \cdots, s)$$

has at least one solution $x_1, x_2, \cdots, x_s$ for every integer $m > 0$. The number $G(n)$ is the minimum s such that (1) has at least one solution for every sufficiently large integer m. We shall define $G(n, c)$ as the minimum s such that (1) has at least one solution for every integer $m \geqq c$. Then

$$g(n) = G(n, 1), \quad G(n, c+1) \leqq G(n, c), \quad G(n) = \lim_{c=\infty} G(n, c).$$

A decomposition of m is a representation of m as a sum of positive integral n-th powers, in the form (1) with the added restriction $x_i > 0$. The number of n-th powers, s, is the weight of the decomposition.

Let us consider the m in the range $0 < m < 3^n$. Write

$$3^n = q2^n + r, \quad q, r \text{ integers}, \quad 0 < r < 2^n.$$

In any decomposition of m each x_i is either 2 or 1. Then the decompositions of m of minimum weight are given by

$$m = y2^n + z, \quad 0 \leqq z < 2^n \text{ for } 0 \leqq y < q, \quad 0 \leqq z < r \text{ for } y = q$$

of weight $y + z$. Since $r \leqq 2^n - 1$ the maximum weight for any m in the range $0 < m < 3^n$ is

$$I = q + 2^n - 2,$$

which is the weight of the decomposition of $q2^n - 1$. Therefore

$$g(n) \geqq I = q + 2^n - 2$$

and, since 3^n is a decomposition of 3^n of weight 1, I n-th powers suffice to represent all integers in the range $0 < m \leqq 3^n$.

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[1] Presented to the American Mathematical Society April 6, 1935 and December 31, 1935.

I. Vinogradow [2] has proved that

$$G(n) \leqq [n(6 \log n + \log 216 + 4)], \ n > 3$$

by proving the existence of a number V_n such that

$$G(n, V_n) \leqq [n(6 \log n + \log 216 + 4)]. \tag{2}$$

In parts IV and V of the author's doctor's dissertation at the University of California the Vinogradow proof is reproduced and the constants determined as they occur. In some cases proofs are supplied for results which Vinogradow states without proof. In part V we conclude that (2) holds if

$$\log V_n = 18n2^{14n^2/3}, \ n > 4. \tag{3}$$

The upper bound (2) for $G(n, V_n)$ is less than I for $n > 7$.

In part II of this paper we develop a method for dealing with the integers between 3^n and V_n without the use of tables. In part III we apply this method for some values of n to obtain upper bounds for $g(n)$ using the results of part V of the author's dissertation and the fact that I n-th powers suffice to represent all integers between zero and 3^n. In the cases where this upper bound is equal to I we then have $g(n) = I$. It has been known for a long time that $g(n) = I$ when $n = 2$ or 3 but this is the first time the formula $g(n) = I$ has been proved for other values of n. The results are tabulated at the end of part III.

II. *Decomposition of integers between* 3^n *and* V_n. We define the integers $q, r, u, v, w,$ by

$$\begin{gathered} 3^n = q2^n + r, \quad 4^n = u3^n + v2^n + w, \\ 0 < r < 2^n, \quad 0 < w < 2^n, \quad 0 \leqq v \leqq q, \quad u = [4^n/3^n]. \end{gathered} \tag{4}$$

Then we have

$$\begin{aligned} (k - ax)4^n + (l + uax + bx)3^n + (m + vax - qbx)2^n \\ = k4^n + l3^n + m2^n + (br - aw)x. \end{aligned} \tag{5}$$

We take k, l, m, a, b, x to be integers, and $x \geqq 0$. If $k \geqq ax$, $l \geqq -uax - bx$, $m \geqq -vax + qbx$ the left member of (5) is a decomposition of $k4^n + l3^n + m2^n + (br - aw)x$ of weight

[2] I. Vinogradow, "On Waring's problem," *Annals of Mathematics*, vol. 36 (1935), pp. 395-405.

$$k + l + m - ax + uax + bx + vax - qbx.$$

If $br > aw$ we can obtain a decomposition of any integer between

$$k4^n + l3^n + m2^n + (br - aw)x \quad \text{and} \quad k4^n + l3^n + m2^n + (br - aw)(x+1) - 1$$

inclusive by adding 1^n at most $br - aw - 1$ times to the decomposition of $k4^n + l3^n + m2^n + (br - aw)x$. Then the highest weight in this interval is

$$k + l + m + (-a + ua + b + va - qb)x + br - aw - 1.$$

We now let x take the values $0, 1, \cdots, [2^n/(br - aw)]$ and obtain decompositions for every integer between

$$k4^n + l3^n + m2^n \quad \text{and} \quad k4^n + l3^n + (m + 1)2^n + \phi$$

inclusive, where

$$\phi = [2^n/(br - aw)](br - aw) + br - aw - 1 - 2^n \geqq 0.$$

The greatest weight in this interval is

$$k + l + m + \max(0, -a + ua + b + va - qb)[2^n/(br - aw)] + br - aw - 1.$$

If $br < aw$ we can obtain a decomposition of any integer between

$$k4^n + l3^n + m2^n - (aw - br)x \quad \text{and} \quad k4^n + l3^n + m2^n - (aw - br)(x - 1) - 1$$

inclusive by adding 1^n at most $aw - br - 1$ times to our decomposition of

$$k4^n + l3^n + m2^n + (br - aw)x.$$

Then the highest weight in this interval is

$$k + l + m + (-a + ua + b + va - qb)x + aw - br - 1.$$

We now let x take the values $0, 1, \cdots, [2^n/(aw - br)]$ and obtain decompositions for every integer between

$$k4^n + l3^n + (m - 1)2^n + \psi \quad \text{and} \quad k4^n + l3^n + m2^n + aw - br - 1$$

inclusive, where

$$\psi = 2^n - [2^n/(aw - br)](aw - br) \leqq aw - br - 1.$$

The greatest weight in this interval is

$$k + l + m + \max(0, -a + ua + b + va - qb)[2^n/(aw - br)] + aw - br - 1.$$

In both cases we must have $k \geqq ax$, $l \geqq -uax - bx$, $m \geqq -vax + qbx$. Hence we take

$$k = \max(0, a)[2^n/|br - aw|], \quad l = \max(0, -ua - b)[2^n/|br - aw|],$$
$$m = \max(0, qb - va)[2^n/|br - aw|].$$

Let W be the greatest weight in the interval. Then, in both cases, we have

$$W = \{\max(0, a) + \max(0, -ua - b) + \max(0, qb - va)$$
$$+ \max(0, -a + ua + b + va - qb)\}[2^n/|br - aw|] + |br - aw| - 1.$$

We proceed to show that changing the signs of both a and b leaves W and the interval invariant. Let W_1 be the value of W for $a = A$, $b = B$ and let W_2 be its value for $a = -A$, $b = -B$. By choice of notation we can make $Br > Aw$. Let L_1 be the lower end of the interval for $a = A$, $b = B$; L_2 for $a = -A$, $b = -B$. Let U_1 and U_2 be the two upper ends of the intervals. Then

$$W_1 = \{\max(0, A) + \max(0, -uA - B) + \max(0, qB - vA)$$
$$+ \max(0, -A + uA + B + vA - qB)\}[2^n/(Br - Aw)] + Br - Aw - 1,$$
$$W_2 = \{\max(0, -A) + \max(0, uA + B) + \max(0, -qB + vA)$$
$$+ \max(0, A - uA - B - vA + qB)\}[2^n/(Br - Aw)] + Br - Aw - 1.$$

Since $\max(0, \alpha) - \max(0, -\alpha) = \alpha$, we have

$$W_1 - W_2 = (A - uA - B + qB - vA - A + uA + B + vA - qB)$$
$$\times [2^n/(Br - Aw)] = 0.$$

Also, we have

$$L_1 = \max(0, A)[2^n/(Br - Aw)]4^n + \max(0, -uA - B)[2^n/(Br - Aw)]3^n$$
$$+ \max(0, qB - vA)[2^n/(Br - Aw)]2^n,$$
$$L_2 = \max(0, -A)[2^n/(Br - Aw)]4^n + \max(0, uA + B)[2^n/(Br - Aw)]3^n$$
$$+ \max(0, -qB + vA)[2^n/(Br - Aw)]2^n - 2^n + \psi,$$
$$\psi = 2^n - [2^n/(Br - Aw)](Br - Aw),$$

and hence

$$L_1 - L_2 = \{A4^n + (-uA - B)3^n + (qB - vA)2^n\}[2^n/(Br - Aw)]$$
$$+ 2^n - 2^n + [2^n/(Br - Aw)](Br - Aw) = 0$$

by (4). Finally, we have

$$U_1 = L_1 + 2^n + \phi,$$
$$U_2 = L_2 + 2^n - Aw + Br - 1 - \psi,$$
$$\phi = [2^n/(Br - Aw)](Br - Aw) + Br - Aw - 1 - 2^n,$$
$$U_1 - U_2 = L_1 - L_2 + \phi + Aw - Br + 1 + \psi$$
$$= [2^n/(Br - Aw)](Br - Aw) + Br - Aw - 1 - 2^n + Aw - Br + 1 + 2^n$$
$$- [2^n/(Br - Aw)](Br - Aw) = 0.$$

Therefore $L_1 = L_2$, $U_1 = U_2$, $W_1 = W_2$ so that the value of W and the interval remain unchanged when we change the signs of a and b. Hence we can, without loss of generality, choose a and b so that $br > aw$. Then

$$\begin{aligned}(6)\qquad W = \{&\max(0, a) + \max(0, -ua - b) + \max(0, qb - va) \\ &+ \max(0, -a + ua + b + va - qb)\}[2^n/(br - aw)] + br - aw - 1,\end{aligned}$$

$$\begin{aligned}(7)\qquad k = \max(0, a)[2^n/(br - aw)],\quad l = \max(0, -ua - b)\ [2^n/(br - aw)] \\ m = \max(0, qb - va)[2^n/(br - aw)],\end{aligned}$$

and the interval is

$$(8)\qquad k4^n + l3^n + m2^n \text{ to } k4^n + l3^n + (m + 1)2^n + \phi \text{ inclusive,}$$

$$(9)\qquad \phi = [2^n/(br - aw)](br - aw) + br - aw - 1 - 2^n.$$

For each value of n we choose values for a and b to make W small and then determine the interval over which W suffices by (7), (8) and (9). Call this interval I_2. We then make ascents from I_2 using the two following theorems proved by L. E. Dickson.[3]

THEOREM 1. *If every integer* $> l$ *and* $\leqq g$ *is a sum of* $k - 1$ *integral n-th powers* $\geqq 0$, *and if* m *is an integer for which*

$$(m + 1)^n - m^n < g - l$$

then every integer $> l$ *and* $\leqq g + (m + 1)^n$ *is a sum of* k *integral n-th powers* $\geqq 0$.

THEOREM 2. *Let* l *be an integer* $\geqq 0$. *Let*

$$\nu = (1 - l/L_0)/n, \quad L_0 > l, \quad (\nu L_0)^{n/(n-1)} \geqq L_0.$$

Compute L_t *by* $\log L_t = (n/(n - 1))^t(\log L_0 + n \log \nu) - n \log \nu$. *If all integers between* l *and* L_0 *inclusive are sums of* k *integral n-th powers* $\geqq 0$, *then all integers between* l *and* L_t *inclusive are sums of* $k + t$ *integral n-th powers* $\geqq 0$.

To take care of the integers between 3^n and I_2 we use either a result proved by L. E. Dickson[4] or, if possible, choose a pair of values for a and b so that the interval I_1 includes the number 3^n and then ascend from I_1 to I_2.

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[3] L. E. Dickson, "Recent progress on Waring's theorem and its generalizations," *Bulletin of the American Mathematical Society*, vol. 39 (1933), pp. 701-727, Theorems 10 and 12.

[4] *Ibid.*, Theorems 5 and 6.

III. *The special case* $n = 15$. We shall now apply the method of part II to the case $n = 15$. We have

$$2^n = 32\ 768, \quad 3^n = 14\ 348\ 907, \quad 4^n = 1\ 073\ 741\ 824.$$

By division we obtain

$$q = 437, \quad r = 29291, \quad u = 74, \quad v = 363, \quad w = 27922,$$
$$I = q + 2^n - 2 = 33203.$$

Taking $a = 0$, $b = 1$ we find, by (6), (7), (8), and (9),

$$W = 29727, \quad I_1\colon \text{ from } 3^n - 29291 \text{ to } 3^n + 29290.$$

Taking $a = 1$, $b = 1$ we find

$$W = 3093, \quad I_2\colon \text{ from } 23 \cdot 4^n + 3 \cdot 3^n + 388 \cdot 2^n + 10431 \text{ to } 23 \cdot 4^n + 3 \cdot 3^n + 389 \cdot 2^n + 10518.$$

That is, 29727 fifteenth powers suffice to represent every integer between $3^n - 29291$ and $3^n + 29290$ and 3093 suffice to represent every integer between

$$23 \cdot 4^n + 3 \cdot 3^n + 388 \cdot 2^n + 10431 \quad \text{and} \quad 23 \cdot 4^n + 3 \cdot 3^n + 389 \cdot 2^n + 10518.$$

Starting from I_1 we ascend by Theorem 1, applying it 437 times with $m = 1$, 74 times with $m = 2$, and 23 times with $m = 3$. Then

$$29727 + 437 + 74 + 23 = 30261$$

fifteenth powers suffice to represent every integer from I_1 to I_2.

Ascending from I_2 we apply Theorem 1, 436 times with $m = 1$; 73 times with $m = 2$; 27 times with $m = 3$; 14 times with $m = 4$; 9 times with $m = 5$; 6 times with $m = 6$; 5 times with $m = 7$; 4 times with $m = 8$; 3 times with $m = 9$; and 2 times each with $m = 10, 11, 12, 13, 14$. We have then applied the theorem 587 times. Hence $3093 + 587 = 3680$ fifteenth powers suffice to represent every integer from

$$23 \cdot 4^n + 3 \cdot 3^n + 388 \cdot 2^n + 10431 \text{ to}$$
$$2 \cdot 15^n + 23 \cdot 4^n + 3 \cdot 3^n + 388 \cdot 2^n + 10431.$$

We now apply Theorem 2 with

$$l = 23 \cdot 4^n + 3 \cdot 3^n + 388 \cdot 2^n + 10431, \quad L_0 = l + 2 \cdot 15^n, \quad k = 3680,$$

and

$$\nu = (1 - (l/L_0))/15 = 0.066.$$

Then

$$\log L_t = (15/14)^t(\log L_0 + 15 \cdot \log \nu) - 15 \cdot \log \nu$$
$$\log\log L_t > .02996t - .521 \text{ (logarithms to the base 10)}$$

and all integers between l and L_t are sums of $3680 + t$ integral fifteenth powers $\geqq 0$. If we choose $t = I - k = 29523$ we have

$$\log\log L_t > 883 \text{ (logarithms to the base 10).}$$

In the introduction we showed that I n-th powers suffice to represent every positive integer $\leqq 3^n$. Hence, since I_1 includes 3^n, we have that every integer m,

$$0 < m \leqq 10^{10^{883}},$$

is a sum of $I = 33203$ integral fifteenth powers $\geqq 0$.

By the results of part V of the author's dissertation we have the following inequality for the Vinogradow constant when $n = 15$.

$$\log\log V_{15} < 319 \text{ (logarithms to the base 10).}$$

If we choose $t = 10665$ we have

$$\log\log L_t > 319 \text{ (logarithms to the base 10)}$$

so all integers between l and V_{15} are sums of $10665 + 3680 = 14345$ integral fifteenth powers $\geqq 0$. For $n = 15$ the inequality (2) is

$$G(15, V_{15}) \leqq 384.$$

We now have the following results.

$$G(15, 10^{10^{319}}) \leqq 384, \qquad G(15, 24 \cdot 4^n) \leqq 14345,$$
$$G(15, 3^n) \leqq 30261, \qquad g(15) \leqq 33203.$$

The second inequality follows from the facts that all integers between l and V_{15} are sums of 14345 and all integers greater than V_{15} are sums of 384. We have replaced the value of l by a larger value.

Since $I = 33203$ we have $g(15) = I = 33203$.

Let L be the value of L_t obtained by setting $t = I - k$. By treating other values of n as we did $n = 15$ we obtain the following results. In the cases $n = 17$ and $n = 20$ we cannot find a suitable interval I_1 so we use L. E. Dickson's result referred to above.

List of results for various n from 14 *to* 20.

$$n = 14.$$

Use $a = 0,\ b = 1$ and $a = -1,\ b = -1.$

$$\log\log L > 249,\quad \log\log V_{14} < 278,\quad g(14) \leqq 17555,\quad g(14) \geqq I = 16673,$$
$$G(14, V_{14}) \leqq 352.$$

$$n = 15.$$

Use $a = 0,\ b = 1$ and $a = 1,\ b = 1.$

$$\log\log L > 883,\quad \log\log V_{15} < 319,\quad g(15) = I = 33203,$$
$$G(15, 3^n) \leqq 30261,\quad G(15, 24 \cdot 4^n) \leqq 14345,\quad G(15, V_{15}) \leqq 384.$$

$$n = 16.$$

Use $a = 0,\ b = 1$ and $a = 1,\ b = 1.$

$$\log\log L > 1630,\quad \log\log V_{16} < 363,\quad g(16) = I = 66190,$$
$$G(16, 3^n) \leqq 56526,\quad G(16, 12 \cdot 4^n) \leqq 20974,\quad G(16, V_{16}) \leqq 416.$$

$$n = 17.$$

Use $a = 0,\ b = 1.$

$$\log\log L > 2464,\quad \log\log V_{17} < 409,\quad g(17) = I = 132055,$$
$$G(17, 3 \cdot 3^n) \leqq 53984,\quad G(17, V_{17}) \leqq 448.$$

$$n = 18.$$

Use $a = 0,\ b = 1$ and $a = 2,\ b = 1.$

$$\log\log L > 2472,\quad \log\log V_{18} < 459,\quad g(18) = I = 263619,$$
$$G(18, 3^n) \leqq 236932,\quad G(18, 3 \cdot 4^n) \leqq 182522,\quad G(18, V_{18}) \leqq 480.$$

$$n = 19.$$

Use $a = 0,\ b = 1$ and $a = 1,\ b = 1.$

$$\log\log L > 5339,\quad \log\log V_{19} < 511,\quad g(19) = I = 526502,$$
$$G(19, 3^n) \leqq 443926,\quad G(19, 2 \cdot 4^n) \leqq 320864,\quad G(19, V_{19}) \leqq 513.$$

$$n = 20.$$

Use $a = 1,\ b = 1.$

$$\log\log L > 20042,\quad \log\log V_{20} < 566,\quad g(20) = I = 1051899,$$
$$G(20, 8 \cdot 4^n) \leqq 177571,\quad G(20, V_{20}) \leqq 546.$$

The logarithms above are all to base 10.

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ON WARING'S PROBLEM WITH POLYNOMIAL SUMMANDS.

By Loo-keng Hua.[1]

1. *Introduction and notation.* Let

$$P(h) = \alpha_0 h^k + \alpha_1 h^{k-1} + \cdots + \alpha_k$$

be an integral-valued polynomial with $\alpha_0 > 0$;

$r_{P,s}(n) = r(n) =$ the number of solutions of the diophantine equation

$$(1) \qquad n = P(h_1) + P(h_2) + \cdots + P(h_s), \quad h_i \geqq 0 \quad (i = 1, 2, \cdots, s);$$

$k =$ integer $\geqq 3$; $s \geqq K(k-2) + 5$; $K = 2^{k-1}$;

$d =$ the least common denominator of $\alpha_0, \alpha_1, \cdots, \alpha_k$;
the canonical product of $n = p_1^{l_1} \cdots p_t^{l_t}$;
$\theta_i =$ a positive integer or zero such that $p_i^{\theta_i} \mid d$, $p_i^{\theta_i+1} \nmid d$;

$$n^* = p_1^{l_1+\theta_1} \cdots p_t^{l_t+\theta_t};$$
$$\rho = e^{2\pi i(l/q)}, \qquad (l, q) = 1;$$
$$S_\rho = \sum_{z=1}^{q^*} \rho^{P(z)};$$

$A_{P,s}(q, j) = A(q) = (1/q^{*s}) \sum_\rho S_\rho^s \rho^{-j}$, where the summation is taken over all q-th primitive roots ρ of unity;

$$\mathfrak{S}(j, P, s, \lambda) = \mathfrak{S}(j, \lambda) = \sum_{q=1}^{\lambda} A_{P,s}(q, j);$$
$$\mathfrak{S}_j = \mathfrak{S}(j, \infty);$$

$j_w =$ the greatest real root of $P(x) = j$ if there is any;

$$f(x) = \sum_{h=0}^{n_w} x^{P(h)};$$
$$\psi_\rho(x) = \frac{\Gamma(1+a)}{\alpha_0^a} \frac{S_\rho}{q^*} \left(1 + \sum_{j=1}^{n} \frac{a(a+1)\cdots(a+j-1)}{j!} (x/\rho)^j\right);$$
$$\phi_\rho(x) = \frac{\Gamma(1+a)}{\alpha_0^a} \frac{S_\rho}{q^*} \left(\sum_{j=n+1}^{\infty} \frac{a(a+1)\cdots(a+j-1)}{j!} (x/\rho)^j\right);$$
$$\Psi_\rho(x) = \psi_\rho(x) + \phi_\rho(x) = \frac{P(1+a)}{\alpha_0^a} \frac{S_\rho}{q^*} (1 - x/\rho)^a.$$

[1] Research fellow of the China Foundation for the Promotion of Education and Culture.

Throughout the paper ϵ denotes an arbitrarily small positive number and $C_1, C_2, \cdots,$ denote positive numbers depending at most on P, s and ϵ.

It is to be noticed that if $P(x)$ is a polynomial with integral coefficients, then $\mathfrak{S}_j$ reduces to that treated by Landau[2] (1), (2).

We shall prove the following result:

THEOREM 1. *We have*

$$| r_{P,s}(n) - \alpha_0^{-sa} \frac{P^s(1+a)}{P(sa)} \mathfrak{S}_n n^{sa-1} | < C_1 n^{sa-1-a/K+\epsilon}.$$

The method used is due to Gelbcke (3) for $P(x) = x^k$. We shall omit proofs of results when the method of proof differs only slightly from that given by Gelbcke.

2. *Lemmas for polynomial $P(x)$.*

LEMMA 1. *The necessary and sufficient condition that a polynomial $P(x)$ of k-th degree in the rational field be integral-valued is*

$$(2) \qquad P(x) = a_0 P_k(x) + \cdots + a_k P_0(x)$$

where

$$P_i(x) = \frac{x(x-1)\cdots(x-i+1)}{i!}, \qquad (i = 1, \cdots, k);\ P_0(x) = 1,$$

and the a_i's are integers. [Hilbert (7)].

LEMMA 2. *The necessary and sufficient condition that there do not exist integers u and l (> 1) such that*

$$P(x) \equiv u \pmod{l}$$

for all values of x, is

$$(a_0, a_1, \cdots, a_{k-1}) = 1,$$

where $P(x)$ is given by (2).

LEMMA 3. *If all sufficiently large integers are sums of s values of $P(h)$, $h \geqq 0$, then they are also sums of s values of the polynomial $Q(h) = P(h-t)$, where t is a positive integer.*

Proof. If

$$n = P(h_1) + \cdots + P(h_s), \qquad h_i \geqq 0,$$

then evidently

$$n = Q(h_1 + t) + \cdots + Q(h_s + t), \qquad h_i + t \geqq 0.$$

[2] See the list of references at the end of the paper, page 562.

LEMMA 4. *There exists a positive integer t such that the coefficients of $P(x+t)$ are all positive.*

It follows from Lemmas 3 and 4 that the study of Waring's problem for a general integral-valued polynomial $P(h)$ depends on the same problem for a polynomial with positive coefficients.

Henceforth we shall assume that all α's are positive and the constant term is zero. Under such an assumption j_w is uniquely determined and is greater than zero for $j > 0$.

3. *Well known lemmas.*

LEMMA 5. *Let $|y| \leqq 1/2$ and $a_0 \geqq a_1 \geqq \cdots \geqq 0$, then*

$$\left|\sum_{j=1}^{N} a_j e^{2\pi i y j}\right| \leqq \frac{a_0}{\sin \pi |y|}.$$

[Landau (6), Theorem 140 with $R(w) = \sum_{j=1}^{w} e^{2\pi i y j}$].

LEMMA 6. *We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{j=0}^{N} a_j e^{2\pi i y j}\right|^2 dy = \sum_{j=0}^{N} |a_j|^2.$$

[Landau (6), Theorem 223].

LEMMA 7. *For $\beta > 0$ and j an integer > 0,*

$$\left|\frac{\Gamma(1+\beta+j)}{j!} - j^\beta\right| < \gamma(\beta) j^{\beta-1},$$

where $\gamma(\beta)$ depends on β only.

LEMMA 8. *We have*

$$\sum_{j=0}^{w} r_2{}^2(j) < C_2 w^{2a+\epsilon}.$$

Proof. Landau, *Mathematische Zeitschrift*, vol. 31 (1929-30), p. 149.

LEMMA 9. *Let ρ be a primitive q-th root of unity, $m > 0$, r integers, then*

$$\left|\sum_{h=r+1}^{r+m} \rho^{P(h)}\right|^K < C_6 q^\epsilon m^\epsilon (m^{K-1} + m^K/q + q m^{K-k}).$$

Proof. By the same method used by Landau (6), Theorem 267, we can prove that

$$\left|\sum_{h=r+1}^{r+m} \rho^{Q(h)}\right|^K < C_7 q^\epsilon m^\epsilon (m^{K-1} + m^K/q + q m^{K-k}),$$

where $Q(h)$ is a polynomial with real coefficients and its first coefficient $= 1$.

Suppose that

$$\rho^{a_0} = e^{2\pi i(l/q)a_0} = e^{2\pi i(la_0/qk!)}$$

is a primitive q'-th root of unity. If $(l, k!) = 1$, then also $(l, qk!) = 1$ and

$$q' = qk! \quad \text{if} \quad (a_0, qk!) = 1,$$

$$\left.\begin{array}{r} q' < qk! \\ q' > (qk!)/a_0 \end{array}\right\} \quad \text{if} \quad (a_0, qk!) > 1.$$

From the first part we have

$$\begin{aligned} \Big|\sum_{h=r+1}^{r+m} \rho^{P(h)}\Big|^K &< C_7 q'^{\epsilon} m^{\epsilon}(m^{K-1} + m^K/q' + q' m^{K-k}) \\ &< C_6 q^{\epsilon} m^{\epsilon}(m^{K-1} + m^K/q + q m^{K-k}). \end{aligned}$$

If $(l, k!) \neq 1$, then there exists an integer t such that $(l + tq, k!) = 1$. In fact, since $(l, q) = 1$, the arithmetic progression $l + tq$ contains a prime which is not a divisor of k.

Therefore

$$\begin{aligned} \Big|\sum_{h=r+1}^{r+m} e^{2\pi i(l/q)P(h)}\Big|^K &= \Big|\sum_{h=r+1}^{r+m} e^{2\pi i[(l+qt)/q]P(h)}\Big|^K \\ &< C_6 q^{\epsilon} m^{\epsilon}(m^{K-1} + m^K/q + q m^{K-k}), \end{aligned}$$

since $P(x)$ is an integral-valued polynomial and therefore $e^{2\pi i t P(h)} = 1$.

LEMMA 10. *We have*

$$|S_\rho| < C_8 q^{1-1/K+\epsilon}.$$

LEMMA 11. *We have*

$$|A(q)| < C_1 q^{-(5/4)+\epsilon}.$$

Proof. These two lemmas follow at once from Lemma 9 in the same way that Landau's Theorems 269 and 270 follow from his Theorem 267. We use the fact that $q \leqq q^* \leqq k!q$.

5. *Farey cut.* On the unit circle $|x| = 1$, we take points $\rho = e^{2\pi i(l/q)}$ which correspond to Farey fractions l/q with denominators less than or equal to $n^{1-\alpha}$, and divide the circumference into sub-arcs by means of mediants. Let us write

$$x = \rho e^{2\pi i y}.$$

Then on the arc associated with ρ, we shall have

$$-y_1 \leqq y \leqq y_2,$$

where

$$\frac{1}{2qn^{1-a}} \leqq y_1 < \frac{1}{qn^{1-a}}, \qquad \frac{1}{2qn^{1-a}} \leqq y_2 < \frac{1}{qn^{1-a}}.$$

We consider minor and major arcs given by

minor arcs $\mathfrak{m}$: $\qquad n^a \leqq q \leqq n^{1-a},$

major arcs $\mathfrak{M}$: $\qquad 1 \leqq q < n^a.$

Each arc $\mathfrak{M}$ is further divided into two parts. That part of $\mathfrak{M}$ for which

$$|y| \leqq \frac{1}{2q^{1/2}n^{1-a/2}}$$

is called an arc of type $\mathfrak{M}_1$ and the remaining part of $\mathfrak{M}$ in which

$$|y| > \frac{1}{2q^{1/2}n^{1-a/2}}$$

is called an arc of type $\mathfrak{M}_2$. There are always arcs of type $\mathfrak{M}_2$, since

$$\frac{1}{2q^{1/2}n^{1-a/2}} \leqq \frac{1}{2qn^{1-a}},$$

where $q < n^a$.

6. *Further lemmas.*

LEMMA 12. *On the whole circle* $|x| = 1$ *we have*

$$|\psi_\rho(x)| < C_{10}n^a q^{-1/K+\epsilon}.$$

Proof. From the definition of $\psi_\rho(x)$ we have

$$\psi_\rho(x) = \frac{S_\rho}{\alpha_0{}^a q^*}\left(\Gamma(1+a) + a\sum_{j=1}^{n}\frac{\Gamma(1+a+j)}{j!\,\Gamma(a+j)}(x/\rho)^j\right).$$

From Lemmas 10 and 7, it follows that

$$\begin{aligned}|\psi_\rho(x)| &< C_8 q^{-1/K+\epsilon}(\Gamma(1+a) + a\sum_{j=1}^{n}(j^{a-1}+\gamma(a)j^{a-2}))\\ &< C_{10}n^a q^{-1/K+\epsilon}.\end{aligned}$$

LEMMA 13. *If* $|y| \leqq 1/2$,

then

$$|\phi_\rho(x)| < C_{11}n^{a-1}q^{-1/K+\epsilon}|y|^{-1},$$
$$|\Psi_\rho(x)| < C_{12}q^{-1/K+\epsilon}|y|^{-a},$$

and

$$|\psi_\rho(x)| < C_{13}q^{-1/K+\epsilon}\,\mathrm{Min}(n^a, |y|^{-a}).$$

The proof of these inequalities is the same as that of Theorems 7, 8 of Gelbcke (3).

LEMMA 14. *We have*

$$\sum_{\mathfrak{M}_1} \int_{\mathfrak{K}-\mathfrak{M}_1} |\psi_\rho{}^s(x)|\ |dx| < C_{14} n^{sa-1-(s/K-2)a+\epsilon},$$

where $\psi_\rho(x)$ *corresponds to the arc* $\mathfrak{M}_1$, *and the path of integration is the whole circle excluding the arc* $\mathfrak{M}_1$.

The proof of this lemma is similar to that of Theorem 9 of Gelbcke (3).

7. *Lemmas for the arcs* $\mathfrak{m}$.

LEMMA 15. *On* $\mathfrak{m}$ *we have*

$$|f(x)| < C_{15} n^{a-a/K+\epsilon}.$$

Proof. Let

$$\tau(j) = \sum_{h=0}^{j_w} \rho^{P(h)} \quad \text{for} \quad j \geqq 0, \qquad \tau(-1) = 0.$$

Then

$$\begin{aligned} f(x) &= \sum_{h=0}^{n_w} \rho^{P(h)} (x/\rho)^{P(h)} \\ &= \sum_{j=0}^{n} [\tau(j) - \tau(j-1)] (x/\rho)^j \\ &= \sum_{j=0}^{n-1} \tau(j) [(x/\rho)^j - (x/\rho)^{j+1}] + \tau(n)(x/\rho)^n \\ &= (1 - x/\rho) \sum_{j=0}^{n-1} \tau(j)(x/\rho)^j + \tau(n)(x/\rho)^n. \end{aligned}$$

Now

$$|\arg (x/\rho)| < 2\pi/qn^{1-a},$$

therefore the length of the chord

$$|1 - x/\rho| < 2\pi/qn^{1-a}.$$

Further from Lemma 9 it follows that

$$\begin{aligned} |\tau(j)|^K &< C_6 q^\epsilon n_w{}^\epsilon (n_w{}^{K-1} + n_w{}^K/q + qn_w{}^{K-k}) \\ &< C_{16} q^\epsilon n^{a\epsilon} (n^{a(K-1)} + n^{aK}/q + qn^{a(K-k)}) \end{aligned}$$

since $j_w < C_{17} j^a$. Then

$$|\tau(j)| < C_{18} n^{a-a/K+\epsilon}.$$

Therefore

$$|f(x)| < [(2\pi/qn^{1-a})n + 1]\, C_{18} n^{a-a/K+\epsilon} < C_{15} n^{a-aK/+\epsilon}.$$

LEMMA 16. *We have*

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} | f^s(x) | \; | dx | < C_{19} n^{(s-4)(a-a/K)+2a+\epsilon}.$$

Proof. See Gelbcke (3).

8. *Lemmas for the arcs* $\mathfrak{M}_1$.

LEMMA 17. *If* $| y | \leqq 1/2$, *then*

$$| f(x) - \psi_\rho(x) | < C_{22} q^{1-1/K+\epsilon} \operatorname{Max} (n | y |, 1).$$

Proof. As in Lemma 15 we may write

$$f(x) = (1 - x/\rho) \sum_{j=1}^{n-1} \tau(j) (x/\rho)^j + \tau(n) (x/\rho)^n,$$

$$\psi_\rho(x) = \frac{\Gamma(1+a)}{\alpha_0{}^a} \frac{S_\rho}{q^*} \left(1 + \sum_{j=1}^{n} \frac{a(a+1) \cdots (a+j-1)}{j!} (x/\rho)^j\right)$$

$$= \frac{S_\rho}{\alpha_0{}^a q^*} \left((1 - x/\rho) \sum_{j=1}^{n-1} \frac{\Gamma(1+a+j)}{j!} (x/\rho)^j + \frac{\Gamma(1+a+n)}{n!} (x/\rho)^n\right)$$

$$= (1 - x/\rho) \sum_{j=1}^{n-1} v(j) (x/\rho)^j + v(n) (x/\rho)^n,$$

where

$$v(j) = \frac{S_\rho}{\alpha_0{}^a q^*} \frac{\Gamma(1+a+j)}{j!}, \qquad (j = 1, \cdots, n).$$

For every $j \geqq 0$, $\tau(j)$ has $[j_w] + 1$ terms. Divide $\tau(j)$ into partial sums, each $= S_\rho$, plus $[j_w] + 1 - [j_w/q^*]q^* \leqq q^*$ terms. By Lemma 9, we have

$$| \tau(j) - [j_w/q^*] S_\rho | < C_{23} q^{1-1/K+\epsilon},$$

$$| \tau(j) - (j_w/q^*) S_\rho | < C_{23} q^{1-1/K+\epsilon} + | S | < C_{24} q^{1-1/K+\epsilon}.$$

Now

$$\left| \frac{\Gamma(1+a+j)}{j!} - j^a \right| < C_{25} j^{a-1} < C_{26},$$

and

$$| j^a - \alpha_0{}^a j_w | = | (\alpha_0 j_w{}^k + \cdots + \alpha_k)^a - \alpha_0{}^a j_w |$$

$$\leqq j_w | (\alpha_0 + \alpha_1/j_w + \cdots + \alpha_k/j_w{}^k)^a - \alpha_0{}^a | < C_{27}.$$

Therefore

$$| v(j) - j_w (S_\rho/q^*) | < C_{28} | S_\rho/q^* | < C_{29}.$$

Thus we have

$$| \tau(j) - v(j) | < C_{30} q^{1-1/K+\epsilon},$$

and consequently

$$|f(x)-\psi_\rho(x)|=|(1-x/\rho)\sum_{j=0}^{n-1}[\tau(j)-v(j)](x/\rho)^j+[\tau(n)-v(n)](x/\rho)^n|$$
$$< C_{31}(n\,|\,y\,|+1)q^{1-1/K+\epsilon} < C_{22}q^{1-1/K+\epsilon}\operatorname{Max}(n\,|\,y\,|,1).$$

LEMMA 18. *We have*

$$\sum_{\mathfrak{M}_1}\int_{\mathfrak{M}_1}|f^s(x)-\psi_\rho{}^s(x)|\;|\,dx\,| < C_{32}n^{sa-1-(s/K-2)a+\epsilon}$$
$$+C_{33}n^{sa-1-a+\epsilon}\begin{cases}n^{(3/4)a} & \text{for } k=3,\\ n^{(3/8)a} & \text{for } k=4,\\ 1 & \text{for } k=5.\end{cases}$$

Proof. See Gelbcke (3).

9. *Lemmas for the arcs* $\mathfrak{M}_2$.

LEMMA 19. *If* $|\,y\,|\leqq 1/2$ *then for* ρ *corresponding to* $\mathfrak{M}_2$ *we have*

$$|\,f(x)\,| < C_{39}n^{a-1/K+\epsilon}\,q^{-1/K-\epsilon}\,|\,y\,|^{-1/K-\epsilon}.$$

Proof. See Gelbcke (3).

LEMMA 20. *We have*

$$\sum_{\mathfrak{M}_2}\int_{\mathfrak{M}_2}|f^s(x)|\;|\,dx\,| < C_{43}n^{sa-1-(s/K-1)(a/2)+\epsilon}\begin{cases}n^{(3/8)a} & \text{for } k=3,\\ n^{(3/4)a} & \text{for } k=4,\\ 1 & \text{for } k\geqq 5.\end{cases}$$

Proof. See Gelbcke (3).

10. *Proof of Theorem* 1. It is evident that the coefficient of the n-th power of x in $f^s(x)$ is equal to $r_{P,s}(n)$. By Cauchy's Theorem we have

$$r_{P,s}(n)=\frac{1}{2\pi i}\int_{\mathfrak{R}}\frac{f^s(x)}{x^{n+1}}\,dx$$
$$=\frac{1}{2\pi i}\left(\sum_{\mathfrak{M}}\int_{\mathfrak{M}}\frac{f^s(x)}{x^{n+1}}\,dx+\sum_{\mathfrak{M}_2}\int_{\mathfrak{M}_2}\frac{f^s(x)}{x^{n+1}}\,dx+\sum_{\mathfrak{M}_1}\int_{\mathfrak{M}_1}\frac{f^s(x)}{x^{n+1}}\,dx\right).$$

By Lemmas 16, 20, we have

$$\left|\,r_{P,s}(n)-\sum_{\mathfrak{M}_1}\int_{\mathfrak{M}_1}\frac{f^s(x)}{x^{n+1}}\,dx\,\right| < C_{19}n^{(s-4)(a-a/K)+2a+\epsilon} \tag{5}$$
$$+C_{43}n^{sa-1-(s/K-1)(a/2)+\epsilon}\begin{cases}n^{(3/8)a} & \text{for } k=3,\\ n^{(3/16)a} & \text{for } k=4,\\ 1 & \text{for } k\geqq 5.\end{cases}$$

By Lemma 18, we have

$$(6)\qquad \left| \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} \frac{f^s(x)}{x^{n+1}}\,dx - \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} \frac{\psi_\rho{}^s(x)}{x^{n+1}}\,dx \right| < C_{32} n^{sa-1-(s/K-2)a+\epsilon}$$

$$+ C_{33} n^{sa-1-a+\epsilon} \begin{cases} n^{(3/4)a} & \text{for } k = 3, \\ n^{(3/8)a} & \text{for } k = 4, \\ 1 & \text{for } k \geqq 5. \end{cases}$$

By Lemma 14,

$$(7)\qquad \left| \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} \frac{\psi_\rho{}^s(x)}{x^{n+1}}\,dx - \sum_{\mathfrak{M}_1} \int_{\mathfrak{K}} \frac{\psi_\rho{}^s(x)}{x^{n+1}}\,dx \right| < C_{14} n^{sa-1-(s/K-2)a+\epsilon}.$$

Further

$$\frac{1}{2\pi i} \sum_{\mathfrak{M}_1} \int_{\mathfrak{K}} \frac{\psi_\rho{}^s(x)}{x^{n+1}} = \alpha_0{}^{-sa} \frac{\Gamma(sa+n)}{n!}\,\frac{\Gamma^s(1+a)}{\Gamma(sa)} \sum_{\mathfrak{M}_1} (S_\rho/q^*)^s \rho^{-n}$$

$$= \alpha_0{}^{-sa}\,\frac{(sa+n)}{n!}\,\frac{\Gamma^s(1+a)}{\Gamma(sa)} \sum_{q=1}^{n^a} A_{P,s}(q,n).$$

Since by Lemma 7,

$$\left| \frac{\Gamma(sa+n)}{n!} - n^{sa-1} \right| < C_{46} n^{sa-2},$$

and by Lemma 10,

$$| A_{P,s}(q,n) | < C_{47} q^{1-s/K+\epsilon};$$

thus

$$(8)\qquad \left| \frac{1}{2\pi i} \sum_{\mathfrak{M}_1} \int_{\mathfrak{K}} \frac{\psi_\rho{}^s(x)}{x^{n+1}}\,dx - \alpha_0{}^{-sa} n^{sa-1} \frac{\Gamma^s(1+a)}{\Gamma(sa)} \sum_{q=1}^{n^a} A_{P,s}(q,n) \right|$$

$$< C_{48} n^{sa-2} \sum_{q=1}^{n^a} q^{1-s/K+\epsilon} < C_{48} n^{sa-2}.$$

Since

$$\left| \sum_{q>n^a}^{\infty} A_{P,s}(q,n) \right| < C_{49} \sum_{q>n^a}^{\infty} q^{1-s/K+\epsilon} < C_{50} n^{-(s/K-2)a+\epsilon},$$

we have

$$\left| \alpha_0{}^{-sa} n^{sa-1} \frac{\Gamma^s(1+a)}{\Gamma(sa)} \sum_{q=1}^{n^a} A_{P,s}(q,n) - \alpha_0{}^{-sa} n^{sa-1} \frac{\Gamma^s(1+a)}{\Gamma(sa)} \mathfrak{S}_n \right|$$

$$< C_{51} n^{-(s/K-2)a+sa-1+\epsilon}.$$

Then from (4), (5), (6), (7), and (8) the desired result follows.

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POLYNOMIALS FOR THE n-ARY COMPOSITION OF NUMERICAL FUNCTIONS.

By D. H. Lehmer.

In discussing the general binary associative and commutative composition of numerical functions it has been essential to deal with a function $\psi(x_1, x_2)$ satisfying (for $n = 2$) the three postulates given below.[1] In this paper we find all polynomials $\psi(x_1, x_2, \cdots, x_n)$ satisfying these postulates and discuss some of their properties.

The postulates are as follows:

Postulate A. *If each x is a positive integer, $\psi(x_1, x_2, \cdots, x_n)$ is a positive integer.*

Postulate B. *For each positive integer N the equation $N = \psi(x_1, x_2, \cdots, x_n)$ has a solution in positive integers.*

Postulate C. *$\psi(x_1, x_2, \cdots, x_{n-1}, \psi(x_n, x_{n+1}, \cdots, x_{2n-1}))$ is a symmetric function.*

Postulate C is a generalization of the commutative and associative laws.[2] The function appearing in this postulate will for reasons of brevity be called the iterate of ψ, and will be designated by $\psi^* = \psi^*(x_1, x_2, \cdots, x_{2n-1})$. We begin by assuming Postulate C alone. From this point on ψ will be assumed to be a polynomial.[3]

Theorem 1. *If $\psi(x_1, x_2, \cdots, x_n)$ is a polynomial such that*

$$\psi^* = \psi(x_1, x_2, \cdots, x_{n-1}, \psi(y_1, y_2, \cdots, y_n))$$

[1] D. H. Lehmer, *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 945-957, *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 723-726; G. Pall, *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 56-58; E. T. Bell, *Bulletin of the American Mathematical Society*, vol. 41 (1935), pp. 798, 801.

[2] This postulate could have been weakened at the expense of simplicity in a manner analogous to that used in following papers: W. A. Hurwitz, *Annals of Mathematics* (2), vol. 15 (1913-14), p. 93; D. H. Lehmer, *American Journal of Mathematics*, vol. 54 (1932), p. 329.

[3] Abel, *Journal für Mathematik*, vol. 1, pp. 11-15; *Oeuvres*, vol. 1, pp. 61-66, discussed functions of two variables satisfying Postulate C and stated that such functions must be symmetric. Examples to the contrary can be given however. For instance let $\psi(1,2) = 3$, while $\psi(x, y) = 1$ otherwise. Here $\psi(x, \psi(y, z))$ is symmetric whereas $\psi(1,2) \neq \psi(2,1)$.

Lémeray, *Nouvelles Annales Mathématiques* (4), vol. 1 (1901), pp. 163-167 used symmetric functions satisfying Postulate C to generalize the Dedekind inversion formula.

is a symmetric function of its $2n-1$ variables, then ψ is symmetric and linear in each of its n variables.

Proof. Let $\psi(x_1, x_2, \cdots, x_n)$ be of degree $d_k > 0$ in x_k $(k=1,2,\cdots,n)$. Then ψ^* is of degree d_1 in x_1 and of degree $d_k d_n$ in y_k. But it is symmetric by hypothesis. Hence $d_1 = d_k d_n$. Since $d_1 \neq 0$, it follows with $k=1$, that $d_n = 1$. Hence $d_1 = d_2 = \cdots = d_n = 1$, so that ψ is linear in each variable. The symmetry of ψ now follows from the symmetry of ψ^* since ψ^* is linear in ψ.

If we write

$$\sigma_{n,k} = \sigma_{n,k}(x_1, x_2, \cdots, x_n) = \Sigma x_1 x_2 \cdots x_k$$

for the k-th elementary symmetric function of $x_1, x_2, \cdots, x_n$ we may (by Theorem 1) write our ψ function in the form

$$\psi(x_1, x_2, \cdots, x_n) = a_0 + a_1\sigma_{n,1} + \cdots + a_n\sigma_{n,n}. \tag{2}$$

Our problem is to find necessary and sufficient conditions on the a's so that ψ^* will be symmetric.

If in (2) we collect coefficients of all products containing x_n we get

$$\psi(x_1, x_2, \cdots, x_n) = \sum_{\nu=0}^{n-1} (a_{\nu+1}x_n + a_\nu)\sigma_{n-1,\nu}(x_1, x_2, \cdots, x_{n-1}).$$

Hence

$$\begin{aligned} \psi^* &= \psi(x_1, x_2, \cdots, x_{n-1}, \psi(y_1, y_2, \cdots, y_n)) \\ &= \sum_{\nu=0}^{n-1} (a_{\nu+1}\psi(y_1, y_2, \cdots, y_n) + a_\nu)\sigma_{n-1,\nu}(x_1, x_2, \cdots, x_{n-1}). \end{aligned} \tag{3}$$

Now the coefficient of any product occurring in (3) and consisting of precisely k x's and $r-k$, y's $(0 \leqq k \leqq n-1;\ r-k \leqq n)$ is given by

$$\begin{aligned} &a_{k+1}a_{r-k} && \text{if} \quad 0 \leqq k < r \\ &a_r + a_{r+1}a_0 && \text{if} \quad 0 \leqq k = r. \end{aligned} \tag{4}$$

For ψ^* to be symmetric it is necessary that for each r and for different values of $k \leqq r$, the coefficients (4) should be equal. Two cases now present themselves according as $a_0 = 0$ or not.

Case 1. $a_0 \neq 0$. Let $R = (a_1 - 1)/a_0$. Let $r = 1$, and $k = 0, 1$ in (4) and equate. We have

$$a_1^2 = a_1 + a_2 a_0.$$

That is

$$a_2 = a_1 R.$$

By induction, if $a_i = a_1 R^{i-1}$, for some $i < n$, we can determine a_{i+1} by setting $r = i$, and $k = i-1$ and i in (4) and equating. We have

whence

$$a_1 a_i = a_i + a_{i+1} a_0,$$

$$a_{i+1} = R\, a_i = a_1 R^i.$$

Hence in general

$$a_\nu = a_1 R^{\nu-1} \qquad (\nu = 1, 2, \cdots, n). \tag{5}$$

Case II. $a_0 = 0$. In this case the coefficients (4) become

$$\begin{array}{lll} a_{k+1} a_{r-k} & \text{if} & 0 \leqq k < r \\ a_r & \text{if} & k = r. \end{array} \tag{4'}$$

Setting $r = 1$, and $k = 0, 1$ we get on equating

$$a_1^2 = a_1.$$

Hence two subcases arise according as $a_1 = 0$ or 1.

Case II$_1$. $a_0 = 0$, $a_1 = 1$. In this case no condition on a_2 is obtained. To determine a_3 we set $r = 3$ and $k = 0, 1$ in (4′) and get

$$a_3 = a_2^2.$$

By induction we find that

$$a_\nu = a_2^{\nu-1}, \qquad 2 \leqq \nu \leqq n, \qquad a_0 = 0, \quad a_1 = 1.$$

Case II$_2$. $a_0 = a_1 = 0$. Setting $k = r - 1$ and r in (4′) with $r < n$, we find

$$a_r a_1 = a_r = 0.$$

For $r = n$, however, we get no condition on a_n. Hence in this case

$$\psi(x_1, x_2, \cdots, x_n) = a_n x_1 x_2 \cdots x_n.$$

We may sum up the results obtained thus far in the following

THEOREM 2. *A polynomial* $\psi(x_1, x_2, \cdots, x_n)$, *whose iterate*

$$\psi^*(x_1, x_2, \cdots, x_{2n-1}) = \psi(x_1, x_2, \cdots, x_{n-1}, \psi(x_n, \cdots, x_{2n-1}))$$

is symmetric, is necessarily of one of the forms

$$\begin{aligned}\psi(x_1, x_2, \cdots, x_n) = a_0 + a_1 \Sigma x_i + \frac{a_1(a_1 - 1)}{a_0} \Sigma x_i x_j \\ + \frac{a_1(a_1-1)^2}{a_0^2} \Sigma x_i x_j x_k + \cdots + \frac{a_1(a_1-1)^{n-1}}{a_0^{n-1}} x_1 x_2 \cdots x_n.\end{aligned} \tag{6}$$

$$\psi(x_1, x_2, \cdots, x_n) = \Sigma x_i + a_2 \Sigma x_i x_j + a_2^2 \Sigma x_i x_j x_k + \cdots + a_2^{n-1} x_1 \cdots x_n. \tag{7}$$

$$\psi(x_1, x_2, \cdots, x_n) = a_n x_1 x_2 \cdots x_n. \tag{8}$$

We now show that all polynomials of the form (6), (7) or (8) have symmetric iterates. In fact this is obvious in case ψ is of the form (8). The same is true of (7) when $a_2 = 0$. If $a_2 \neq 0$, we can write (7) in the form

$$(7') \qquad a_2\psi(x_1, x_2, \cdots, x_n) + 1 = \prod_{i=1}^{n} (a_2 x_i + 1).$$

Hence

$$(9) \qquad \begin{aligned} a_2\psi^* + 1 &= a_2\psi(x_1, x_2, \cdots, x_{n-1}, \psi(x_n, x_{n+1}, \cdots, x_{2n-1})) + 1 \\ &= \prod_{i=1}^{n-1} (a_2 x_i + 1)\{a_2\psi(x_n, \cdots, x_{2n-1}) + 1\} \\ &= \prod_{i=1}^{2n-1} (a_2 x_i + 1). \end{aligned}$$

Therefore ψ^* is symmetric. If ψ is of the form (6) and $a_1(a_1 - 1) \neq 0$ (the symmetry of ψ^* being evident otherwise), we may write (6) in the form

$$(6') \qquad R\psi(x_1, x_2, \cdots, x_n) + 1 = a_1 \prod_{i=1}^{n} (Rx_i + 1), \quad R = (a_1 - 1)/a_0 \neq 0.$$

As before we find

$$(10) \qquad R\psi^* + 1 = a_1^2 \prod_{i=1}^{2n-1} (Rx_i + 1)$$

so that ψ^* is symmetric in this case also. *Hence the class of all polynomials whose iterates are symmetric coincides with* (6), (7) *and* (8).

Converse propositions regarding product representation may also be proved.

THEOREM 3. *Let $\psi(x_1, x_2, \cdots, x_n)$ be a symmetric multilinear form*

$$a_0 + a_1 \Sigma x_i + a_2 \Sigma x_i x_j + \cdots + a_n x_1 x_2 \cdots x_n$$

in which $a_0 a_1 (a_1 - 1) \neq 0$. Suppose further that $1 + \psi(a_1 - 1)/a_0$ be expressible as a product of linear factors. Then the iterate of ψ is also symmetric and ψ is of type (6).

Proof. Let $R = (a_1 - 1)/a_0$. By hypothesis α_i and β_i exist such that

$$R\psi(x_1, x_2, \cdots, x_n) + 1 = \prod_{i=1}^{n} (\alpha_i x_i + \beta_i).$$

If we set all the variables equal to zero, we obtain

$$Ra_0 + 1 = \prod_{i=1}^{n} \beta_i = a_1 \neq 0.$$

Hence we can set $\alpha_i/\beta_i = \gamma_i$ and write

$$R\psi(x_1, x_2, \cdots, x_n) + 1 = a_1 \prod_{i=1}^{n} (\gamma_i x_i + 1).$$

But ψ is a symmetric function so that the γ's are equal. Hence

$$R\psi + 1 = a_1 \prod_{i=1}^{n} (\gamma x_i + 1). \tag{11}$$

Setting $x_1 = 1$ and $x_i = 0$ for $i > 1$, we have

$$Ra_0 + Ra_1 + 1 = a_1(\gamma + 1). \tag{12}$$

Recalling the definition of R we obtain from (12)

$$a_1(R + 1) = a_1(\gamma + 1).$$

Hence $R = \gamma$, and (11) becomes (6′). Therefore ψ^* is symmetric and ψ is of type (6).

THEOREM 4. *Let $\psi(x_1, x_2, \cdots, x_n)$ be of the form*

$$a_0 + \Sigma x_i + a_2 \Sigma x_i x_j + \cdots + a_n x_1 x_2 \cdots x_n$$

where $a_2 \neq 0$. Suppose further that $a_2\psi + 1$ is expressible as a product of linear factors. Then ψ^* is symmetric and ψ is of the type (7).

Proof. By hypothesis we have

$$a_2\psi(x_1, x_2, \cdots, x_n) + 1 = \prod_{i=1}^{n} (\alpha_i x_i + \beta_i).$$

Setting all the variables equal to zero we have

$$a_2 a_0 + 1 = \prod_{i=1}^{n} \beta_i.$$

We show now that each β is different from zero. In fact if one β were zero, all would have to be zero since $a_2\psi(x_1, x_2, \cdots, x_n) + 1$ is a symmetric function. Moreover we would have

$$a_0 a_2 + 1 = 0 \tag{13}$$

$$a_2\psi(x_1, x_2, \cdots, x_n) + 1 = \prod_{i=1}^{n} \alpha_i x_i. \tag{14}$$

Setting $x_1 = 1$, and $x_i = 0$ for $i > 1$ in (14) we would have

$$a_2 a_0 + a_2 + 1 = 0.$$

This contradicts (13). Hence $\Pi\beta_i = B \neq 0$, and we can write

$$a_2\psi(x_1, x_2, \cdots, x_n) + 1 = B \prod_{i=1}^{n} (\gamma_i x_i + 1) \tag{15}$$

where $\gamma_i = \alpha_i/\beta_i$. By symmetry the γ's are equal.

Expanding (15) with this fact in mind we have

$$1 + a_2a_0 + a_2\sigma_{n,1} + a_2^2\sigma_{n,2} + \cdots = B + B\gamma\sigma_{n,1} + B\gamma^2\sigma_{n,2} + \cdots.$$

Identifying coefficients of the σ's we have

$$B = 1 + a_2a_0, \qquad B\gamma = a_2, \qquad B\gamma^2 = a_2^2.$$

Hence, $B = 1$, $a_0 = 0$, $a_2 = \gamma$. Therefore ψ is of the type (7′) and hence of type (7). The symmetry of ψ^* follows as before.

THEOREM 5. *If the iterate ψ^* of a polynomial ψ is symmetric so also is the iterate of ψ^*.*

Proof. In case ψ is a simple sum or product, the theorem is obvious. If not, ψ will be of type (6) or (7). By (9) and (10) ψ^* will be of type (6) or (7) also. Hence the iterate of ψ^* will be symmetric.

THEOREM 6. *Every polynomial ψ in n variables, whose iterate ψ^* is symmetric may be obtained by iterating $n-2$ times such a polynomial in 2 variables.*

Proof. In case ψ is a simple sum or product the theorem is obvious. Otherwise ψ will be of the type (6) or (7) with $a_1 \neq 1$. In the first case ψ is the result of $n-2$ successive iterations of the function

$$\psi(x_1, x_2) = A_0 + A_1(x_1 + x_2) + \frac{A_1(A_1-1)}{A_0}x_1x_2$$

where

$$A_1^{n-1} = a_1 \tag{16}$$

$$A_0 = a_0(A_1-1)/(a_1-1). \tag{17}$$

In fact the result of iterating $\psi(x_1, x_2)$ a certain number of times is in every case a function of the type (6), the ratio of consecutive coefficients being always $(A_1-1)/A_0$ by (9). Hence (17) must hold. The coefficients of x_1 in the successive iterates of $\psi(x_1, x_2)$ are the successive powers of A_1. Hence (16). In case ψ is of type (7), it is seen to be the result of iterating

$$\psi(x_1, x_2) = x_1 + x_2 + a_2x_1x_2.$$

Hence the theorem.

We now turn our attention to Postulates A and B. To simplify the discussion we introduce a function f by

(18) $f(x_1, x_2, \cdots, x_n) = -1 + \psi((x_1+1), (x_2+1), \cdots, (x_n+1))$.

By Theorem 1, f will be a symmetric multilinear form

(19) $f(x_1, x_2, \cdots, x_n) = c_0 + c_1\Sigma x_i + c_2\Sigma x_i x_j + \cdots + c_n x_1 x_2 \cdots x_n$.

If Postulates A and B hold for ψ, f must, by (18), satisfy the conditions.

(A′). *If each x is an integer $\geqq 0$, so also is $f(x_1, x_2, \cdots, x_n)$.*

(B′). *The equation $f(x_1, x_2, \cdots, x_n) = N$ has a solution $(x_1, x_2, \cdots, x_n)$ in integers $\geqq 0$ for each $N \geqq 0$.*

THEOREM 7. *A function f of the form* (19) *satisfies* (A′) *and* (B′) *if and only if $c_0 = 0$, $c_1 = 1$ while $c_2, c_3, \cdots, c_n$ are integers $\geqq 0$.*

Proof. The sufficiency of the condition as far as (A′) is concerned is obvious, and (B′) is easily seen to follow from the identity

$$f(N, 0, 0, \cdots, 0) = N.$$

As to necessity, we first show that if f satisfies (A′) then the c_ν are integers $\geqq 0$. In fact if we set $x_1 = x_2 = \cdots = x_\nu = x$ and $x_{\nu+1} = x_{\nu+2} = \cdots = x_n = 0$, we obtain from (19)

$$f = c_0 + \binom{\nu}{1} c_1 x + \binom{\nu}{2} c_2 x^2 + \cdots + c_\nu x^\nu. \tag{20}$$

If c_k is an integer for all $k < \nu$, it follows by setting $x = 1$ that c_ν is an integer in view of (A′). But $c_0 = f(0, 0, \cdots, 0)$ is an integer. Hence by induction each c is an integer. However if c_ν were negative, we could choose x in (20) so large that f would be negative contrary to (A′). Hence the c's are non-negative integers.

To show that $c_0 = 0$ and $c_1 = 1$ we introduce (B′) and note first that not all the c's are zero since $f =$ constant fails to satisfy (B′). For this reason

$$f(x_1, x_2, \cdots, x_n) > f(0, 0, \cdots, 0) = c_0 \geqq 0,$$

provided the x's are non-negative integers not all zero. By (B′) f must represent zero. Hence $c_0 = 0$. Similarly the next largest value of f is

$$f(1, 0, 0, \cdots, 0) = c_0 + c_1 = c_1.$$

Hence $c_1 = 1$ and the theorem follows. Another statement of it is

THEOREM 8. *All symmetric multilinear forms ψ satisfying Postulates* A *and* B *are*

$$\psi(x_1, x_2, \cdots, x_n) = 1 + \Sigma(x_i - 1) + c_2\Sigma(x_i - 1)(x_j - 1) + \cdots + c_n(x_1 - 1)(x_2 - 1)\cdots(x_n - 1) \tag{21}$$

where the c's are not-negative integers.

Writing ψ in the form

$$\psi(x_1, x_2, \cdots, x_n) = a_0 + a_1\Sigma x_i + a_2\Sigma x_i x_j + \cdots + a_n x_1 x_2 \cdots x_n$$

and identifying the coefficients with those obtained after expanding the right side of (21), we obtain the relation

$$a_\nu = \sum_{\lambda=0}^{n-\nu} c_{\nu+\lambda}(-1)^\lambda \binom{n-\nu}{\lambda} \tag{22}$$

or what is the same

$$c_\nu = \sum_{\lambda=0}^{n-\nu} a_{\nu+\lambda} \binom{n-\nu}{\lambda}. \tag{23}$$

It follows from (23) by a simple induction that the a's are integers.

Thus far we have not made full use of Postulate C. To do this we require that ψ be of the form (6), (7) or (8).

In the first of these three cases $a_k = R^{k-1}a_1$, $(k = 1, 2, \cdots, n)$. Putting these values in (23) we have for $\nu = 1$

$$c_1 = 1 = \sum_{\lambda=0}^{n-1} a_1 R^\lambda \binom{n-1}{\lambda} = a_1(1 + R)^{n-1}.$$

Recalling that $R = (a_1 - 1)/a_0$, we have

$$a_0^{n-1} = a_1(a_0 + a_1 - 1)^{n-1}. \tag{24}$$

Since the a's are integers it follows that a_1 is the $(n-1)$-st power of an integer, say $a_1 = a^{n-1}$. Substituting this in (24) and taking $(n-1)$-st roots we find for every n

$$a_0 = (a^n - a)/(1 - a), \quad R = (1 - a)/a, \quad a_\nu = (1 - a)^{\nu-1}a^{n-\nu}. \tag{25}$$

In case $n - 1$ is even we obtain another solution of (24)

$$a_0 = -(a^n - a)/(1 + a), \qquad R = -(1 + a)/a, \qquad a_\nu = (1 + a)^{\nu-1}a^{n-\nu}(-1)^{\nu-1}. \tag{26}$$

Substituting these values of a_ν in (23) we have for $\nu > 1$

$$c_\nu = (1 \pm a)^{\nu-1}$$

according as (26) or (25) is taken. Since $c_\nu \geqq 0$ we find in these respective cases

$$a \geqq 1 \quad \text{or} \quad a \leqq 1.$$

The polynomials found thus far are therefore

$$(27) \quad \psi(x_1, x_2, \cdots, x_n) = (a^n - a)/(1-a) + a^{n-1}\Sigma x_i + (1-a)a^{n-2}\Sigma x_i x_j + (1-a)^2 a^{n-3}\Sigma x_i x_j x_k + \cdots + (1-a)^{n-1} x_1 x_2 \cdots x_n$$

with a an integer $\leqq 1$, and

$$(28) \quad \psi(x_1, x_2, \cdots, x_n) = -(a^n - a)/(1+a) + a^{n-1}\Sigma x_i - (1+a)a^{n-2}\Sigma x_i x_j + (1+a)^2 a^{n-3}\Sigma x_i x_j x_k + \cdots + (1+a)^{n-1} x_1 x_2 \cdots x_n$$

where n is odd and a is an integer $\geqq 1$.

Returning now to polynomials of type (7) and (8) we find that those which satisfy Postulates A and B are of the type (27) or (28).

In fact if ψ is of type (7) we may substitute $a_k = a_2^{k-1}$ $(k = 1, 2, \cdots, n)$ in (23) and with $\nu = 1$ obtain

$$(29) \qquad c_1 = 1 = \sum_{\lambda=0}^{n-1} a_2^{\lambda} \binom{n-1}{\lambda} = (1 + a_2)^{n-1}$$

If n is even (29) has the single solution $a_2 = 0$, which leads to the polynomial

$$\psi(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$$

which fails to satisfy Postulate B with $N = 1$. If n is odd (29) has the additional solution $a_2 = -2$. This leads to the same polynomial as the special case of (27) in which $a = -1$.

If ψ is of the type (8) namely

$$\psi(x_1, x_2, \cdots, x_n) = a_n x_1 x_2 \cdots x_n,$$

it is clear that Postulate B will be satisfied if and only if $a_n = 1$. But this is (27) with $a = 0$. Finally we observe that (28) is a duplication of (27) since every form obtained from (28) with $a \geqq 1$ is identical with the form obtained from (27) with $-a$.

Hence we may sum up the investigation by the following

THEOREM 9. *All polynomials which satisfy Postulates* A, B *and* C *are given by* (27), *where a is an integer* $\leqq 1$.

For $n = 2$, (27) becomes[4]

$$\psi(x_1, x_2) = -a + a(x_1 + x_2) + (1 - a)x_1x_2.$$

By (16) and (17) this polynomial if iterated $n - 2$ times produces (27). Hence

THEOREM 10. *There exist no polynomials satisfying Postulates* A, B *and* C *which are not iterates of a polynomial in two variables, satisfying these postulates.*

The effect of Theorem 10 on the theory of n-ary composition of numerical functions may be described in a few lines.

Let $f_\nu(N)$ $(\nu = 1, 2, \cdots, n)$ be n arbitrary numerical functions defined for all positive integers N. The composite (or symbolic product) of these functions with respect to a function $\psi(x_1, x_2, \cdots, x_n)$ is a function $F(N)$ which may be defined for example by

$$F(N) = \Sigma f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

where the sum extends over all solutions $(x_1, x_2, \cdots, x_n)$ of $\psi(x_1, x_2, \cdots, x_n) = N$ and where the multiplication indicated under the sign of summation may be ordinary multiplication for example. If ψ satisfies Postulates A, B and C, then $F(N)$ will be a numerical function with the maximum degree of symmetry. As a result of Theorem 10 any composition of numerical functions n at a time with respect to a polynomial can be achieved by repeated applications of the ordinary binary composition.

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[4] This polynomial is identical with that of Pall, *loc. cit.*, with $a - 1 = \alpha$.

MINIMUM PARTITIONS INTO SPECIFIED PARTS.

By Hansraj Gupta.

1. Dickson[1] has studied "Minimum Decompositions" of numbers into "N-th powers." In this paper, I consider the problem of partitions of a positive integral number n into parts $1, a, b$; where a, b are positive integers such that

$$1 < a < b. \tag{1}$$

The problem is here attacked directly and in an elementary way.

2. If

$$n = x + ay + bz, \tag{2}$$

where x, y, z are positive integers ≥ 0; then $(x + y + z)$ shall be called in general the weight of n. The least value of $(x + y + z)$ shall be termed after Dickson the "Minimum weight" of n, and written Min. (n).

If

$$n = X + aY + bZ, \quad \text{where} \quad 0 \leq X < a,\ 0 \leq X + aY < b; \tag{3}$$

then $(X + Y + Z)$ shall be termed the "Absolute weight" of n, and written Ab. (n). Evidently

$$n \geq \text{Ab. } (n) \geq \text{Min. } (n). \tag{4}$$

3. Let

$$b = qa + r, \qquad 0 \leq r < a. \tag{5}$$

If $r = 0$, then Min. $(n) =$ Ab. (n). If

$$r > 0, \quad \text{let} \quad X + jr = ia + m,\ 0 \leq m < a,\ 0 \leq j \leq Z; \tag{6}$$

then

$$n = m + a(Y + jq + i) + b(Z - j). \tag{7}$$

Now for some value of j, (7) must represent the partition with minimum weight. Subtracting the weight of (7) from the absolute weight, we get

$$\Delta \equiv X - (m + i) - j(q - 1), \tag{8.1}$$

or

$$\Delta(i, j) \equiv i(a - 1) - j(q + r - 1). \tag{8.2}$$

[1] *American Journal of Mathematics*, vol. 55 (1933).

If Min. $(n) \neq$ Ab. (n), we must have

$$(9) \qquad 1 \leq \frac{j}{i} < \frac{a-1}{q+r-1}.$$

Hence

$$(10) \qquad \text{Min. } (n) = \text{Ab. } (n), \text{ if } q+r \geq a; \text{ in particular if } b \geq a.$$[2]

Moreover as j takes the values $1, 2, 3, \cdots$ in (6), m decreases only when i increases. Hence to find the maximum value of Δ, we need give the values $1, 2, 3, \cdots$ to i only.

For any value of i, j will be the least if $0 \leq m < r$. Hence

$$(11) \qquad j = \left[\frac{ia - X + r - 1}{r}\right],$$

with the condition stated in (6).

We can now express $\Delta(i, j)$ as a function of i alone. Thus

$$(12) \qquad \Delta(i) = i(a-1) - \left[\frac{ia - X + r - 1}{r}\right](q + r - 1).$$

4. $\Delta(i)$. In (12) if i is increased by t, then we have either

$$(13.1) \qquad \delta_1(t) \equiv \Delta(i+t) - \Delta(i) = t(a-1) - \left[\frac{ta}{r}\right](q+r-1);$$

or

$$(13.2) \qquad \delta_2(t) \equiv \Delta(i+t) - \Delta(i) = t(a-1) - \left[\frac{ta+r}{r}\right](q+r-1).$$

The latter is always negative.

If $s > 0$ be a value of t, for which $\delta_1(t)$ is negative, then

$$(14) \qquad \Delta(i+s) < \Delta(i).$$

This result is independent of X.

Hence to find the maximum value of $\Delta(i)$, i need not be given values $> s$ in (12).

5. Search for s. Change a/r and $(a-1)/(q+r-1)$ into simple continued fractions, and find an odd convergent c_{2l+1} of the former $\geq$ an even convergent C_{2k} of the latter. If d_h denote the denominator of the h-th convergent of a/r, then a value of s will be found among the members of the A. P. whose first term is $(d_{2l} + d_{2l-1})$ and common difference is d_{2l}.

An example will make the method clear. Let

$$a = 2^{22} = 4194304, \qquad b = 3^{22};$$

then

$$q = 7481, \quad \text{and} \quad r = 3471385.$$

Changing a/r and $(a-1)/(q+r-1)$ into s. c. fractions, we get the convergents

(c) $$\frac{1}{1}, \frac{5}{4}, \frac{6}{5}, \frac{29}{24}, \frac{586}{485}, \cdots$$

and

(C) $$\frac{1}{1}, \frac{5}{4}, \frac{6}{5}, \frac{41}{34}, \cdots.$$

We find $$\frac{586}{485} > \frac{41}{34}.$$

We now search for s among the numbers 29, 53, 77, 101, $\cdots$; and find $\delta_1(29) < 0$. Hence $s = 29$. We notice that $\delta_1(t) \not< 0$ for any value of $t < 29$.

6. The following table gives s, when

$$a = 2^u, \quad b = 3^u; \qquad u \leq 36.$$

The values of q and r, for these values of u are given in Dickson's paper cited above.

u	1	2	3	4	5	6	7	8	9	10
s	1	1	1	1	2	1	1	2	1	2
u	11	12	13	14	15	16	17	18	19	20
s	1	3	5	14	9	16	5	17	21	7
u	21	22	23	24	25	26	27	28	29	30
s	31	29	23	7	12	64	4	70	4	6
u	31	32	33	34	35	36				
s	53	53	245	73	20	9				

7. The numbers X from 0 to $(a-1)$ can be divided into a number of groups; all numbers for which Δ has the same maximum value being placed in one group. Thus X will be said to belong to the group (c, e), if for the

given X, $\Delta(i, j)$ is maximum when $i = c$, and $j = e$. The inferior limits of these groups are easily obtained. Thus the inferior limit of (c, e) is

$$ac - re. \tag{15}$$

Since $ac - re < a$, therefore $e > a(c-1)/r$. Moreover

$$e < (a-1)c/(q+r-1).$$

Hence

$$\left[\frac{a(c-1)}{r}\right] < e \leq \left[\frac{(a-1)c}{q+r-1}\right]. \tag{16}$$

As an example, consider $a = 2^{13} = 8192$; $b = 3^{13} = 1594323$; then $q = 194$, $r = 5075$; and $s = 5$. The limits for the various groups are tabulated below.

Group (c, e)	Inferior Limit $ac - re$	Superior Limit	Maximum Δ $(a-1)c - (q+r-1)e$
(0, 0)	0	1158	0
(1, 1)	3117	4275	2923
(2, 2)	6234	7392	5846
(2, 3)	1159	2317	578
(3, 4)	4276	5434	3501
(4, 5)	7393	8191	6424
(4, 6)	2318	3116	1156
(5, 7)	5435	6233	4079

The superior limits are ascertained after the inferior limits have been calculated.

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DIVISIBILITY SEQUENCES OF THIRD ORDER.

By MARSHALL HALL.

1. *Introduction.* By a divisibility sequence of k-th order will be meant a sequence of rational integers $u_0, u_1, u_2, \cdots u_n, \cdots$ satisfying the linear recurrence

$$(1) \qquad u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n$$

where the a's are rational integers, and such that $u_n \mid u_{mn}$ (read u_n divides u_{mn}) for any m and n not zero.

It will be shown (a) that there are two types of divisibility sequences which may be distinguished according as $u_0 \neq 0$ or $u_0 = 0$. If $u_0 \neq 0$, the totality of primes dividing terms of the sequence is finite and the sequence is said to be degenerate. If $u_0 = 0$, all but a finite number of primes will appear as divisors of the terms, and we call the sequence regular. Furthermore this paper shows (b) that the factorization properties of divisibility sequences are similar to the factorization properties of the Lucas[1] sequences, and (c) that there is no regular divisibility sequence of third order whose associated cubic is irreducible. Here a_2 and a_3 are assumed to be co-prime.

Divisibility sequences are of particular interest because of their remarkable factorization properties. Lucas was the first to discover the striking relations in second order sequences and give a coherent theory, though some of his results were implied by earlier work on the theory of quadratic forms. Among other results, he developed the tests for primality applicable to the Mersenne numbers. Other special types of divisibility sequences have been investigated by Lehmer,[2] Pierce,[3] and Poulet.[4]

2. *Properties of General Linear Recurrences.* There will be occasion to use the following properties of recurring sequences, whether divisibility sequences or not. Let the sequence (u_n) be determined by the recurrence

[1] E. Lucas, "Théorie des Fonctions Numériques Simplement Périodiques," *American Journal of Mathematics*, vol. 1 (1875), pp. 184-240, 289-321.

[2] D. H. Lehmer, "An extended theory of Lucas' functions," *Annals of Mathematics* (2), vol. 31 (1930), pp. 419-448.

[3] T. A. Pierce, "The numerical factors of the arithmetic forms $\prod_{i=1}^{n} (1 \pm a_i{}^m)$," *Annals of Mathematics* (2), vol. 18 (1916-17), pp. 53-64.

[4] Poulet, *L'Intermédiare des Mathématiciens*, vol. 27, pp. 86-87; (2), vol. 1, p. 47; vol. 3, p. 61.

(1) and by an initial set of values $u_0, u_1, u_2, \cdots u_{k-1}$. With the recurrence (1) is associated its characteristic polynomial,

$$f(x) = x^k - a_1 x^{k-1} \cdots - a_k = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k).$$

If the roots of $f(x)$ are distinct, then

$$u_n = c_1 \alpha_1{}^n + c_2 \alpha_2{}^n \cdots + c_k \alpha_k{}^n \tag{2}$$

where $c_1, c_2, \cdots c_k$ are constants which may be determined from the initial values $u_0, u_1, \cdots u_{k-1}$.

The sequence (u_n) is periodic [5] for an arbitrary modulus m. That is to say there exists a period τ of (u_n) modulo m, depending on m and $a_1, a_2, \cdots a_k$ such that

$$u_{n+\tau} \equiv u_n \pmod{m} \tag{3}$$

for all $n \geqq n_0[m, a_1, a_2, \cdots a_k]$. In particular $n_0 = 0$ if $(a_k, m) = 1$. The period τ is taken to be the least number satisfying such a relation. All other numbers with this property are multiples of the period. If p be a prime not dividing the discriminant of $f(x)$, and if $f(x) \equiv f_1(x) f_2(x) \cdots f_s(x) \pmod{p}$ be the decomposition of $f(x)$ into irreducible factors modulo p, whose degrees are $k_1, k_2, \cdots k_s$ respectively, then τ divides the least common multiple of $p^{k_i} - 1$, $i = 1, 2, \cdots s$. Moreover (u_n) has a restricted period [6] $\mu \pmod{m}$. μ is defined to be the least integer for which there is a b such that

$$u_{n+\mu} \equiv b u_n \pmod{m} \tag{4}$$

for all $n \geqq n_0$. If e is the exponent to which b belongs $\pmod{m}$, then $\mu e = \tau$. If $f(x)$ is irreducible modulo p, p a prime, then $\mu \,\Big|\, \dfrac{p^k - 1}{p - 1}$.

3. *Properties of General Divisibility Sequences.* References have been given above to investigations of certain types of divisibility sequences. This paper, however, is the first to treat them in general. It is the first attempt to find what the general characteristics of a divisibility sequence are, and what types exist. In this section the fundamental difference between regular and degenerate divisibility sequences is given by Theorem II. Theorem III is the key to the factorization properties of all divisibility sequences. In § 4 these theorems are applied to third order divisibility sequences.

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[5] H. T. Engstrom, "On sequences defined by linear recurrence relations," *Transactions of the American Mathematical Society*, vol. 33 (1931), pp. 210-218.

[6] R. Carmichael, "On sequences of integers defined by recurrence relations," *Quarterly Journal of Mathematics*, vol. 48 (1920), pp. 343-372. See page 354 for reference to the restricted period. In particular $(b, m) = 1$ if $(a_k, m) = 1$.

THEOREM I. *If* (u_n) *is a divisibility sequence and some* u_r *has a factor* m *relatively prime to* a_k, *then* $u_0 \equiv 0 \pmod{m}$.

As (u_n) is a divisibility sequence $u_r \mid u_{\tau r}$, and hence $u_{\tau r} \equiv 0 \pmod{m}$. Since $(a_k, m) = 1$, relation (3) holds with $n = 0$. This yields $u_{\tau r} \equiv u_0 \pmod{m}$ and hence $u_0 \equiv 0 \pmod{m}$ as was to be proved.

It is on the basis of this theorem that divisibility sequences have been separated into two categories, viz., degenerate if $u_0 \neq 0$, regular if $u_0 = 0$.

If u_n be any term of a degenerate divisibility sequence (u_n), it may be written as the product of two factors, $u_n = A_n B_n$, where $A_n \mid u_0$, and B_n is divisible only by primes dividing a_k. The totality of primes dividing the terms of (u_n) will be finite. Degenerate divisibility sequences will be excluded from consideration in this paper, but will be treated further elsewhere.

If (u_n) is a regular divisibility sequence satisfying (1) and p is any prime not dividing a_k, $u_{s\tau} \equiv u_0 \equiv 0 \pmod{p}$ where τ is the period of (u_n) modulo p. Hence every prime not dividing a_k will divide the terms of a subsequence of (u_n) if (u_n) is a regular divisibility sequence. Furthermore, we may take $u_1 = 1$ without loss of generality since $(u_n) = (v_n/v_1)$ is a divisibility sequence satisfying (1) if (v_n) is a divisibility sequence satisfying (1). (u_n) will, of course, be a sequence of integers as $v_1 \mid v_n$ for all n, including $n = 0$, as $v_0 = 0$.

It is convenient to state these results as a theorem.

THEOREM II. *The totality of primes dividing the terms of a degenerate sequence* (u_n) *is contained in the set of primes dividing* u_0 *and* a_k. *The totality of primes dividing the terms of a regular sequence* (u_n) *includes every prime not dividing* a_k.

Consider the factorization of u_n, a particular term of a regular divisibility sequence. By the divisibility property, any prime dividing u_r where $r \mid n$ is a divisor of u_n. The remaining primes belong essentially to the term u_n itself.

Definition. A prime p is said to be a primitive divisor of u_n if $p \mid u_n$, $p \nmid u_r$ for $r \mid n$, $r \neq n$, and if $p \nmid a_k$.

The following theorem on the factorization of terms of a divisibility sequence is fundamental.

THEOREM III. *If* p *is a primitive divisor of* u_n, *and if* μ *is the restricted period of* (u_n) *modulo* p, *then* $n \mid \mu$.

Proof. Let $(n, \mu) = r$. Then there exist positive integers x and y such that $nx - \mu y = r$.

Since $u_n \equiv 0 \pmod p$, we have $u_{nx} \equiv 0 \pmod p$ (divisibility) $u_{nx} \equiv b^y u_{nx-\mu y} \pmod p$ (restricted period) or $u_{nx} \equiv b^y u_r \equiv 0 \pmod p$, whence $u_r \equiv 0 \pmod p$ as $b \not\equiv 0 \pmod p$ if $(a_k, p) = 1$. But as p is a primitive divisor of u_n, $u_r \equiv 0 \pmod p$ for $r | n$ implies $r = n$. Hence $(n, \mu) = r = n$, and $n | \mu$ as was to be shown.

Combining this with the information on μ given in § 2, it is seen that p is restricted to certain arithmetic progressions $tn + r_i$. For example, if the sequence is of second order $\mu | p - 1$ or $p + 1$, whence $p = tn \pm 1$.

4. *Divisibility Sequences of Third Order.* The condition $u_0 = 0$ makes it easy to find the regular divisibility sequences of first and second order. There is no regular sequence of first order unless the trivial sequence of zeros be considered a divisibility sequence. For second order we have $u_n = t(\alpha_1^n - \alpha_2^n)/(\alpha_1 - \alpha_2)$ or tna^{n-1} according as the roots of the associated polynomial are distinct or equal. The first of these is the well known Lucas sequence.

The consideration of third order sequences is by no means so simple. We may construct formally $u_n = \left(\dfrac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right)^2 = \dfrac{\alpha_1^{2n} + \alpha_2^{2n} - 2\alpha_1^n\alpha_2^n}{\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2}$ which satisfies a third order sequence whose characteristic polynomial has roots α_1^2, α_2^2, and $\alpha_1\alpha_2$. This will be a sequence of integers if $v_n = (\alpha_1^n - \alpha_2^n)/(\alpha_1 - \alpha_2)$ is of either the Lucas or Lehmer type. Such a sequence is essentially of quadratic type and there is nothing to be gained by considering it as a third order sequence. It is probable that there are no regular third order sequences of any other type.[7]

It is easily seen that we cannot obtain a divisibility sequence of third order satisfying an arbitrary recurrence merely by an appropriate choice of initial values. Consider $u_{n+3} = u_{n+1} + u_n$. From § 3 we must take $u_0 = 0$, and may take $u_1 = 1$. The condition $u_2 | u_4$ implies $u_2 = \pm 1$, but in neither case does $u_4 | u_8$.

If a sequence is of type v_n^2 as given above, its characteristic cubic $f(x)$ has a rational root $a = \alpha_1\alpha_2$. Hence if there is a third order divisibility sequence whose $f(x)$ is irreducible, it is certainly not of type v_n^2. This possibility is considered in the following theorem.

THEOREM IV. *There is no regular divisibility sequence (u_n), whose*

[7] Since completing this paper I have learned from Dr. Morgan Ward that he has been able to show that this is the only type if $f(x)$, the characteristic polynomial, has a linear and an irreducible quadratic factor. As this paper covers the case $f(x)$ irreducible, the only doubtful possibility is that $f(x)$ is the product of three linear factors.

characteristic polynomial is an irreducible cubic whose last two coefficients are relatively prime.

As the proof of this theorem is quite long, it will be subdivided into Lemmas. Lemma 4 gives the first of the equations which lead to the contradiction of the assumption that there is a divisibility sequence satisfying the requirements of the theorem.

Assume that there is a regular divisibility sequence (u_n), whose characteristic is $f(x) = x^3 - a_1x^2 - a_2x - a_3 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ where $(a_2, a_3) = 1$ and let $f(x)$ be irreducible. (u_n) satisfies the recurrence

$$u_{n+3} = a_1u_{n+2} + a_2u_{n+1} + a_3u_n. \tag{5}$$

As $f(x)$ is irreducible α_1, α_2, and α_3 are distinct and

$$u_n = c_1\alpha_1{}^n + c_2\alpha_2{}^n + c_3\alpha_3{}^n. \tag{6}$$

We note that as (u_n) is a regular divisibility sequence $u_0 = 0$ or

$$c_1 + c_2 + c_3 = 0. \tag{7}$$

Moreover we take $u_1 = 1$, as is permissable.

LEMMA 1. *If $p \mid a_3$, and $p \mid u_n$, then n has a factor r, $1 < r < \bar{n}$, $\bar{n}$ a fixed number.*

For if p divides any terms of (u_n), let u_m be the first. It evidently suffices to show $(m, n) \neq 1$. Then take $\bar{n}$ greater than m. As there are only a finite number of primes dividing a_3, there is one value for $\bar{n}$ which will do for all divisors of a_3. In fact, it can be shown that $a_3{}^3$ will suffice. Now if $(m, n) = 1$, there are positive integers x and y such that $mx = ny + 1$. By the divisibility property $u_{mx} \equiv 0 \pmod p$ and $u_{ny} \equiv 0 \pmod p$. From (5)

$$u_{mx} = a_1u_{mx-1} + a_2u_{mx-2} + a_3u_{mx-3}.$$

Now $p \mid u_{mx}$, $p \mid u_{mx-1} = u_{ny}$, $p \mid a_3$, but $p \nmid a_2$ as $(a_2, a_3) = 1$. Hence $p \mid u_{mx-2}$. Similarly as

$$u_{mx-1} = a_1u_{mx-2} + a_2u_{mx-3} + a_3u_{mx-4},$$

we have $p \mid u_{mx-3}$. Proceeding thus we finally obtain $p \mid u_1 = 1$, which is a contradiction. Hence $(m, n) \neq 1$.

LEMMA 2. *If $p \mid u_n$ and p is a divisor of the discriminant of $f(x)$, n has a factor less than a finite limit $\bar{n}$.*

If p also divides a_3 then Lemma 1 proves this. If $p \nmid a_3$, then p is either a primitive divisor of u_n or of u_r where $r|n$. In this case $r|\mu$ the restricted period of (u_n) modulo p, by reason of Theorem III, and $r \neq 1$ as $u_1 = 1$. As $f(x)$ is irreducible, its discriminant is not zero and has only a finite number of divisors. The restricted periods of these primes will lie below a finite limit $\bar{n}$. Hence $r < \bar{n}$ and so n has a factor less than $\bar{n}$.

LEMMA 3. *If q is a prime greater than $\bar{n}$, then* $u_q{}^6 \equiv u_1{}^6 \pmod{q}$, $u_{q^2}{}^6 \equiv u_1{}^6 \pmod{q}$.

By Lemma 1, u_q has no prime factor dividing a_3. As $u_1 = 1$, every prime dividing u_q is a primitive divisor of u_q. Hence if $p|u_q$ and μ is the restricted period of (u_n) modulo p, then $q|\mu$ by Theorem III. As p does not divide the discriminant of $f(x)$ by Lemma 2, we have $\mu|p-1$, p^2-1, or p^3-1, and hence $q|p^6-1$. Since $p^6 \equiv 1 \pmod{q}$ for every prime p dividing u_q, it follows by multiplication that $u_q{}^6 \equiv 1 \pmod{q}$ or $u_q{}^6 \equiv u_1{}^6 \pmod{q}$ as $u_1 = 1$. Now $p^6 \equiv 1 \pmod{q^2}$ for the primitive divisors of u_{q^2}, and hence à fortiori $p^6 \equiv 1 \pmod{q}$. Since all the divisors of u_{q^2} are primitive divisors of either u_q or u_{q^2}, we have $u_{q^2}{}^6 \equiv 1 \pmod{q}$ or $u_{q^2}{}^6 \equiv u_1{}^6 \pmod{q}$ as before.

LEMMA 4.

$$c_1\alpha_2 + c_2\alpha_3 + c_3\alpha_1 = \epsilon_1(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)$$

and

$$c_1\alpha_3 + c_2\alpha_1 + c_3\alpha_2 = \epsilon_2(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)$$

where $\epsilon_1{}^6 = \epsilon_2{}^6 = 1$.

For if q is a prime greater than $\bar{n}$, by Lemma 3 we have

$$(8) \qquad (c_1\alpha_1{}^q + c_2\alpha_2{}^q + c_3\alpha_3{}^q)^6 \equiv (c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)^6 \pmod{q}.$$

Now if $f(x)$ is irreducible $\pmod{q}$ then

$$(9) \qquad \alpha_1{}^q \equiv \alpha_2, \ \alpha_2{}^q \equiv \alpha_3, \ \alpha_3{}^q \equiv \alpha_1 \pmod{Q}$$

where Q is a prime ideal dividing q in $K(\alpha_1, \alpha_2, \alpha_3)$. Hence from (8)

$$(10) \qquad (c_1\alpha_2 + c_2\alpha_3 + c_3\alpha_1)^6 \equiv (c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)^6 \pmod{Q}.$$

Now if $f(x)$ is irreducible there are infinitely many primes q for which $f(x)$ is irreducible $\pmod{q}$.[8] Hence the difference of the two sides of (10) is an

[8] Hasse, "Bericht über Neuere Untersuchungen und Probleme aus der Theorie der Algebraischen Zahlkörper," Part II, p. 127, *Jahresbericht Ergänzungsbände*, vol. 6 (1930). Here $K(\alpha_1, \alpha_2, \alpha_3)$ is a cyclic extension of either the rational field or a quadratic field.

algebraic number divisible by infinitely many prime ideals, and consequently must be zero. Hence

$$c_1\alpha_2 + c_2\alpha_3 + c_3\alpha_1 = \epsilon_1(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3) \tag{11}$$

where $\epsilon_1{}^6 = 1$. Similarly since $\alpha_1{}^{q^2} \equiv \alpha_3$, $\alpha_2{}^{q^2} \equiv \alpha_1$, $\alpha_3{}^{q^2} \equiv \alpha_2 \pmod{Q}$ and reasoning on u_{q^2} we have

$$c_1\alpha_3 + c_2\alpha_1 + c_3\alpha_2 = \epsilon_2(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3) \tag{12}$$

where $\epsilon_2{}^6 = 1$.

Combining (7), (11) and (12) we have the system of equations:

$$\begin{aligned} c_1 \qquad\qquad &+ c_2 \qquad\qquad &+ c_3 \qquad\qquad &= 0 \\ c_1(\alpha_2 - \epsilon_1\alpha_1) &+ c_2(\alpha_3 - \epsilon_1\alpha_2) &+ c_3(\alpha_1 - \epsilon_1\alpha_3) &= 0 \\ c_1(\alpha_3 - \epsilon_2\alpha_1) &+ c_2(\alpha_1 - \epsilon_2\alpha_1) &+ c_3(\alpha_2 - \epsilon_2\alpha_3) &= 0 \end{aligned} \tag{13}$$

If the c's all vanish, then the sequence (u_n) will consist merely of 0's. If not, the determinant of the c's

$$-(1 + \epsilon_1 + \epsilon_2)(\alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2 - \alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\alpha_3)$$

must vanish.

If $\alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2 - \alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\alpha_3 = 0$, then $a_1{}^2 = 3a_2$ and the roots of $f(x)$ are

$$\begin{aligned} \alpha_1 &= -a_1/3 + (a_1{}^3/27 - a_3)^{1/3} \\ \alpha_2 &= -a_1/3 + \rho(a_1{}^3/27 - a_3)^{1/3} \\ \alpha_3 &= -a_1/3 + \rho^2(a_1{}^3/27 - a_3)^{1/3} \end{aligned} \tag{14}$$

where ρ is a primitive cube root of unity. Here for primes $q = 3k + 2$, $\alpha_1{}^q \equiv \alpha_1$, $\alpha_2{}^q \equiv \alpha_3$, $\alpha_3{}^q \equiv \alpha_2 \pmod{q}$ and reasoning as before

$$c_1 + c_2\rho^2 + c_3\rho = \epsilon_3(c_1 + c_2\rho + c_3\rho^2).$$

Trying the six possible values of ϵ_3, we find that two of the c's must be equal, or one must vanish. In no one of these cases can the sequence (u_n) be a sequence of rational integers.

If $1 + \epsilon_1 + \epsilon_2 = 0$, we have $\epsilon_1 = \rho$, $\epsilon_2 = \rho^2$. Solving (13) with

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = u_1 = 1,$$

we obtain

$$c_1 = \frac{1}{\alpha_1 + \rho\alpha_2 + \rho^2\alpha_3}, \quad c_2 = \frac{\rho}{\alpha_1 + \rho\alpha_2 + \rho^2\alpha_3}, \quad c_3 = \frac{\rho^2}{\alpha_1 + \rho\alpha_2 + \rho^2\alpha_3}. \tag{15}$$

Here the vanishing of the denominators implies the vanishing of the second factor of the determinant, a possibility which has just been excluded. Here again the field is of the type $K(\sqrt[3]{d})$; for from the fact that u_2 is rational it is easily shown that $(\alpha_1 + \rho^2\alpha_2 + \rho\alpha_3)^3$ is rational. Hence for

$$q = 3k + 2, \quad \alpha_1^q \equiv \alpha_1, \quad \alpha_2^q \equiv \alpha_3, \quad \alpha_3^q \equiv \alpha_2 \pmod{q}$$

and reasoning as before

$$c_1\alpha_1 + c_2\alpha_3 + c_3\alpha_2 = \epsilon_4(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3).$$

Combining these six possibilities with (15) we have one of

$$\begin{array}{ll} \alpha_1 = \alpha_2 & 2\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ \alpha_1 = \alpha_3 & 2\alpha_2 - \alpha_1 - \alpha_3 = 0 \\ \alpha_2 = \alpha_3 & 2\alpha_3 - \alpha_1 - \alpha_2 = 0 \end{array}$$

Each one of these contradicts the irreducibility of $f(x)$. For an irreducible polynomial has no equal roots, and if (say) $2\alpha_1 - \alpha_2 - \alpha_3 = 0$ then $3\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 = -a_1$, and the root α_1 is rational. This completes the proof of Theorem IV.

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THE CONSTRUCTION OF A NORMAL BASIS IN A SEPARABLE NORMAL EXTENSION FIELD.

By Ruth Stauffer.

1. *Introduction.* If K is a separable normal extension field over k and if the characteristic of k is not a divisor of the degree of K/k, then the group ring defined by the Galois Group, $\mathfrak{G}$, of K/k and the field k is a direct sum of simple algebras. These different simple algebras determine corresponding components for every element of K. The components of the elements of a basis of K/k can thus be arranged in matric form such that the components of one element form one column. From every column of this matrix it is possible to choose a non-vanishing component. The sum of these components, then, is an element of K which generates a *normal basis* of K/k, that is, it together with its conjugates forms a basis of K/k.

E. Noether (10), and A. Speiser (12) obtained results assuming the existence [1] of a normal basis. Noether formulated the problem in terms of operator isomorphisms. Speiser employed the theory of group representations. The formulae used by Speiser are obtained in this paper in some preliminary work on complementary bases. These same methods are then used to determine the discriminant of the centrum of the integral group ring. Since this discriminant is integral, a relation is obtained between the order of the group, the degree of the irreducible representations, and the class numbers.

2. *Complementary bases.* Two bases $a_i, \bar{a}_k$ of the semi-simple algebra $\mathfrak{o}$ (3), (5), (8), (13), (14) are said to be *complementary* if the trace matrix is the unit matrix, that is, $(Tr(a_i\bar{a}_k)) = E$. The following two theorems are direct consequences of this definition and of the linear property of the trace.

Theorem 1. *If $a_i\bar{a}_k$ are complementary bases and $b_i, \bar{b}_k$ are also bases of $\mathfrak{o}$, then there exist non-singular matrices P, Q in Ω the fundamental field of $\mathfrak{o}$, such that*

$$(\cdot\ \cdot \bar{a}_k \cdot\ \cdot) = (\cdot\ \cdot \bar{b}_k \cdot\ \cdot)P, \qquad \begin{bmatrix} \cdot \\ a_i \\ \cdot \end{bmatrix} = Q \begin{bmatrix} \cdot \\ b_i \\ \cdot \end{bmatrix},$$

and b_i and b_k are complementary if and only if $QP = E$.

[1] The existence of a normal basis for Galois fields was proved by Hensel in 1888, (7). Deuring (4), Hasse (6), and Brauer (1) have proved the existence of a normal

THEOREM 2. *If both $a_i, \bar{a}_k$ and $b_i, \bar{b}_k$ are complementary bases of* $\mathfrak{o}$, *and* Γ, Λ *are any two representations of* $\mathfrak{o}$, *then*

$$\sum_i Tr_\Gamma(a_i)Tr_\Lambda(\bar{a}_i) = \sum_i Tr_\Gamma(b_i)Tr_\Lambda(\bar{b}_i), \tag{2.1}$$

where $Tr_\Gamma(a_i)$ means the trace with respect to the representation Γ of a_i.

We note from Theorem 1 that every basis has a complementary basis. The following are outstanding examples of such bases: 1) the group elements $(\cdot\cdot S_i \cdot\cdot)$ and $(\cdot\cdot S_i^{-1}/g \cdot\cdot)$ of the group ring (13) where g is the order of the group, 2) the matric units, (8), (13), $(\cdot\cdot c_{ik} \cdot\cdot)$ and $(\cdot\cdot c_{ki}/n \cdot\cdot)$ of a total matric ring of degree n, 3) the matric units $(\cdot\cdot c_{ik} \cdot\cdot)$ and $(\cdot\cdot c_{ki}^{(l)}/f_l \cdot\cdot)$ of a semi-simple ring in which the coefficient field is algebraically closed, and f_l is the degree of the simple algebra defined by $\{c_{ik}^{(l)}\}$.

3. *Orthogonality relations.* We shall denote an irreducible representation of the semi-simple algebra $\mathfrak{o}$, by Γ_μ, and the trace with respect to Γ_μ, by χ_μ. If, then, a_i and $\bar{a}_k$ are complementary bases of $\mathfrak{o}$, the equality (2.1) may be written as

$$\sum_i \chi_\nu(a_i)\chi_\mu(\bar{a}_k) = \sum_{i,k,l} \chi_\nu(c_{ik}^{(l)})\chi_\mu(c_{ki}^{(l)}/f_l).$$

Since $\chi_\nu(c_{ik}^{(l)}) = 0$ for $\nu \neq l$ and $= \delta_{ik}$ for $\nu = l$, we conclude that

$$\sum_i \chi_\nu(a_i)\chi_\mu(\bar{a}_i) = \delta_{\nu\mu}, \tag{3.1}$$

our *first set of orthogonality relations.*

We note that if $\mathfrak{o}$ is a group ring and the elements of the group are S_i, then the complementary basis is S_i^{-1}/g and

$$\sum_i \chi_\mu(S_i)\chi_\nu(S_i^{-1}) = g\delta_{\mu\nu}, \tag{3.2}$$

if the characteristic of the fundamental field does not divide g.

In the case of a group ring there is also an orthogonality property for the sums of the different irreducible representations of a fixed element of the group and its inverse. Consider the *classes of conjugates of a group.* We shall denote by K_S the sum of the elements of a class generated by S. The elements K_S are commutative with all the group elements, and, moreover, generate the centrum, (8), p. 692. Furthermore, if $ASA^{-1} = T$, then $AS^{-1}A^{-1} = T^{-1}$ and if $S = T$, then $S^{-1} = T^{-1}$. Therefore the number of elements in K_S is

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basis of K/k when k has a zero characteristic. In Deuring's paper the proof has been extended to the general case that k have any characteristic.

Numbers in parentheses refer to the bibliography on page 597.

equal to the number of elements in $K_{S^{-1}}$. If, then, h_i denotes the number of elements of the class of S_i,

$$\sum_i h_i \chi_\mu(T_i) h_i \chi_\nu(T_i^{-1}/h_i g) = \sum_i \chi_\mu(S_i)\chi_\nu(S_i^{-1}/g) = \delta_{\mu\nu},$$

where T_i is a representative of K_{S_i}. Therefore

$$\sum_i \chi_\mu(K_{S_i})\chi_\nu(K_{S_i}{}^{-1}/h_i g) = \delta_{\mu\nu}, \tag{3.3}$$

or

$$(\chi_\mu(K_{S_i}))(\chi_\nu K_{S_k}{}^{-1}/h_k g)) = E,$$

where μ, k denote rows, i, ν denote columns. Therefore

$$(\chi_\nu(K_{S_k}{}^{-1}/h_k g))(\chi_\mu(K_{S_i})) = E.$$

That is,

$$\sum_\mu \chi_\mu(K_{S_k}{}^{-1}/h_k g)\,\chi_\mu(K_{S_i}) = \delta_{ik}, \tag{3.4}$$

our *second set of orthogonality relations.*

4. *Discriminant of the centrum of the group ring with integral coefficients.* If the irreducible representation Γ_ν is of degree f_ν, then

$$\chi_\nu(K_{S_k}{}^{-1}/h_k g)\chi_\nu(K_{S_i}) = f_\nu \chi_\nu(K_{S_k}{}^{-1}/h_k g \cdot K_{S_i})$$

and therefore

$$\sum_\nu f_\nu \chi_\nu(K_{S_k}{}^{-1}/h_k g \cdot K_{S_i}) = \delta_{ik}.$$

Hence, if Γ is the regular representation of the group ring,

$$Tr_\Gamma(K_{S_i} K_{S_k}{}^{-1}/h_k g) = \sum_\nu f_\nu \chi_\nu(K_{S_i} K_{S_k}{}^{-1}/h_k g) = \delta_{ik}. \tag{4.1}$$

That is, the elements $K_{S_i}{}^{-1}/h_i g$ form a basis of the centrum which is "complementary" to the basis K_{S_i}, "complementary" in the sense that the traces are traces with respect to the regular representation of the group.

Let the group ring $\mathfrak{G}(\Omega)$ be defined by the group $\mathfrak{G}$ and the field Ω. Then to determine the *discriminant, d, of the centrum,* write $\mathfrak{G}(\Omega)$ as the direct sum of two-sided simple ideals $\mathfrak{a}^{(i)}$. Let e_i be the corresponding components of the unit element, e, of $\mathfrak{G}(\Omega)$. Then $e_1, \cdots, e_r$ form a basis for the centrum, and the discriminant with respect to this basis is $|Tr(e_i e_k)| = 1$. The trace, of course, is taken with respect to the regular representation of the centrum. The determinant of the traces with respect to the regular representation of the group ring is

$$(4.2) \qquad | Tr_\Gamma(e_i e_k) | = \prod_{i=1}^{r} f_i^2.$$

If $\begin{bmatrix} \cdot \\ K_{s_i} \\ \cdot \end{bmatrix} = P \begin{bmatrix} \cdot \\ e_i \\ \cdot \end{bmatrix}$, and $(\cdot \cdot K_{s_i}{}^{-1}/h_i g \cdot \cdot) = (\cdot \cdot e_i \cdot \cdot)Q$, then

$$| Tr_\Gamma(K_{s_i} K_{s_k}{}^{-1}/h_k g) | = P \, | Tr_\Gamma(e_i e_k) | \, Q,$$

and

$$| Tr(K_{s_i} K_{s_k}{}^{-1}/h_k g) | = P \, | Tr(e_i e_k) | \, Q.$$

We conclude, from (4.1) and (4.2) that

$$\frac{d}{g^r \Pi h_i} = \frac{| Tr(K_{s_i} K_{s_k}{}^{-1}/h_k g) |}{| Tr(e_i e_k) |} = \frac{| Tr_\Gamma(K_{s_i} K_{s_k}{}^{-1}/h_k g) |}{| Tr_\Gamma(e_i e_k) |} = \frac{1}{\Pi f_i^2}.$$

Moreover, d is an integer, for the elements K_{s_i} are group ring elements with integral coefficients. It follows that $g^r \Pi h_i$ is divisible by Πf_i^2.

5. *Orthogonality relations of the coefficients of the representations.* If $\mathfrak{o}$ is a semi-simple algebra, a_i, $\bar{a}_k$ complementary bases of $\mathfrak{o}$, b_i, $\bar{b}_k$ also complementary bases of $\mathfrak{o}$, and if Γ, Λ are any two representations of $\mathfrak{o}$, then it follows from the linear property of traces and from the definition of complementary bases that

$$(5.1) \qquad \sum \alpha^{(i)}_{lm,\gamma} \alpha^{(i)}_{sr,\lambda} = \sum \beta^{(i)}_{lm,\gamma} \beta^{(i)}_{sr,\lambda}$$

where

$$\Gamma(a_i) = (\alpha^{(i)}_{lm,\gamma}), \ \Lambda(a_i) = (\alpha^{(i)}_{lm,\lambda}), \text{ and } \Gamma(b_i) = (\beta^{(i)}_{lm,\gamma}), \ \Lambda(b_i) = (\beta^{(i)}_{lm,\lambda}).$$

On calculating the sum (5.1) for the representation of the complementary bases $(c^{(l)}_{ik})$, $(c^{(l)}_{ki}/f_l)$ of $\mathfrak{o}$, we see that

$$\sum \alpha^{(i)}_{lm,\gamma} \alpha^{(i)}_{sr,\lambda} = 0, \qquad \lambda \neq \gamma$$

and for $\lambda = \gamma$,

$$(5.2) \qquad \sum \alpha^{(i)}_{lm,\gamma} \alpha^{(i)}_{sr,\lambda} = \begin{cases} 0, & (l, m) \neq (r, s) \\ 1/f, & (l, m) = (r, s) \end{cases}$$

We note that the orthogonality property of the coefficients of the representations of the group elements for the group ring is a particular case of relations (5.2).

6. *Calculation of the matric units by means of complementary bases.* Another problem to which complementary bases may be applied is that of calculating the matric units of the group ring $\mathfrak{G}(\Omega)$, where Ω is an algebraically closed field for which the characteristic is not a divisor of the order

of the group. We write $\mathfrak{G}(\Omega) = \mathfrak{a}^{(1)} + \cdots + \mathfrak{a}^{(r)}$, where $\mathfrak{a}^{(l)}$ are two-sided simple ideals. Let the r distinct irreducible representations of $\mathfrak{G}$ be

$$s \to S^{(l)} = (\sigma_{ik}^{(l)}), \qquad t \to T^{(l)} = (\tau_{ik}^{(l)}), \cdots, \qquad (l = 1, 2, \cdots, r).$$

It follows from Theorem 1 that

$$(e, s, t, \cdots) = (\cdots c_{ik}^{(l)} \cdots)P, \text{ and } \begin{bmatrix} \cdot \\ c_{ki}^{(l)}/f_l \\ \cdot \end{bmatrix} = P \begin{bmatrix} \cdot \\ s^{-1}/g \\ \cdot \end{bmatrix},$$

where g is the order of the group, and f_l the degree of the representation defined by $\mathfrak{a}^{(l)}$. Therefore, in order to determine the matric units we need only determine P. From the theory of representations we know

$$s = \sum_{ik} c_{ik}^{(1)} \sigma_{ik}^{(1)} + \cdots + \sum_{ik} c_{ik}^{(r)} \sigma_{ik}^{(r)}.$$

Hence

$$P = \begin{bmatrix} \cdot & \cdot & \cdot & \\ \epsilon_{ik}^{(l)}, & \sigma_{ik}^{(l)}, & \tau_{ik}^{(l)}, & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

and

$$P \begin{bmatrix} \cdot \\ s^{-1} \\ \cdot \end{bmatrix} = g \begin{bmatrix} \cdot \\ c_{ki}^{(l)}/f_l \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \epsilon_{ik}^{(l)} e^{-1} + \sigma_{ik}^{(l)} s^{-1} + \tau_{ik}^{(l)} t^{-1} + \cdots \\ \cdot \end{bmatrix}.$$

In other words, if

$$M^{(l)} = E^{(l)} e^{-1} + S^{(l)} s^{-1} + T^{(l)} t^{-1} + \cdots = (\mu_{ik}^{(l)}), \tag{6.1}$$

then

$$g c_{ki}^{(l)}/f_l = \mu_{ik}^{(l)}. \tag{6.2}$$

This discussion can be applied to an arbitrary semi-simple algebra, if instead of the basis $(e, s, \cdots)$ and its complementary basis, one considers any set of complementary bases a_i and $\bar{a}_k$. In the general case the matrix is

$$M^{(l)} = \sum_i \bar{A}_i^{(l)} a_i. \tag{6.3}$$

7. *Complementary bases for commutative normal fields. Idempotents.* If K and K are isomorphic separable normal extension fields of degree n over k, the direct product ring K_{K} consisting of the set of all elements $\epsilon = \Sigma d_i \delta_i$ where d_i belongs to K and δ_i belongs to K, is directly decomposible [(8), p. 683] into a sum of n simple ideals, $e_i K$,[2] where e_i are indecomposible components of unity. That is

$$e_i e_k = \delta_{ik} e_i \quad \text{and} \quad \sum_i e_i = e, \tag{7.1}$$

[2] $e_i K$ is a field with unit element e_i and isomorphic to K.

where e is the unit element of K_{K}. Let $\alpha = \Sigma e_i a_i$, where $\alpha \in \mathrm{K}$ and $a_i \in K$. Then

$$e_i \alpha = e_i a_i. \tag{7.2}$$

That is e_i defines a representation of the first degree which maps α on a_i. Furthermore if e is any indecomposible component of unity, say e_1, then the set $\{e_i\}$ is actually the set of conjugates $\{e^S\}$ (4), p. 141.

We now choose $\bar{S}$, an element of the Galois group of K over k such that

$$e\alpha^{\bar{S}} = ea^S. \tag{7.3}$$

That is $\bar{S}$ is the automorphism of K which takes α into the element corresponding to a^S. It follows from (7.2) that $e^S\alpha = e^S a^S$, and from (7.3) that $e^{\bar{S}^{-1}}\alpha = e^{\bar{S}^{-1}}a^S$. In other words, e^S maps α on a^S and $e^{\bar{S}^{-1}}$ maps α on a^S. However, a was an arbitrary element of K. We may conclude, therefore, that $e^S = e^{\bar{S}^{-1}}$. We note that if $E, S, T, \cdots$ are the elements of the Galois Group, $\mathfrak{G}$, of K/k and $\bar{E}, \bar{S}, \bar{T}, \cdots$ are the elements of the Galois Group $\bar{\mathfrak{G}}$ of K/k, then $e^E, e^S, e^T, \cdots$ are distinct and hence $e^{\bar{E}}, e^{\bar{S}}, e^{\bar{T}}, \cdots$ are also distinct. Therefore the correspondence defined between $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ is a one to one correspondence.

If z_i is a basis of K/k, hence also a basis of $K_{\mathrm{K}}/\mathrm{K}$ and if the idempotent $e = \Sigma z_i \alpha_i$, $e^{\bar{S}} = \Sigma z_i \alpha_i^{\bar{S}}$, then $\alpha_i^{\bar{S}}$ is that representation of the complementary basis of z_i defined by $e^{\bar{S}}$, (9), p. 538. This theorem is a key to the first step in the proofs of the existence of a normal basis, namely the proof that $\mathfrak{G}(\mathrm{K})$ is operator isomorphic to $K(\mathrm{K})$, where $\mathfrak{G}$, K, and K are as previously defined, and $K(\mathrm{K})$ is the operator module consisting of the elements of the ring K_{K}, the operators being the elements of the Galois Group $\mathfrak{G}$ of K applied as automorphisms of K. The proof is as follows. Given any basis $z_1, \cdots, z_n$ of K/k, we form the product ring K_{K} of K with an isomorphic extension field K of k. Then the matrix,

$$\begin{bmatrix} \alpha_1^{\bar{E}^{-1}}, & \alpha_1^{\bar{S}^{-1}}, & \cdot\ \cdot \\ \cdot & \cdot & \cdot\ \cdot \\ \cdot & \cdot & \cdot\ \cdot \\ \alpha_n^{\bar{E}^{-1}}, & \alpha_n^{\bar{S}^{-1}}, & \cdot\ \cdot \end{bmatrix}$$

where α_i are as defined above, transforms $z_1, \cdots, z_n$ into a set of linearly independent conjugates $e^E, \cdots, e^S, \cdots$, the indecomposible components of the unit element of K_{K}. The correspondence $S \leftrightarrow e^S$ then, defines the operator isomorphism

$$\mathfrak{G}(\mathrm{K}) \simeq K(\mathrm{K}).$$

This is essentially the proof given both by Deuring and by Hasse. Deuring works with the idempotents without stating the explicit formulae, Hasse works with the set of coefficients of the idempotents. These coefficients are, in reality, the columns of the matrix of transformation $(\alpha_i^{\bar{S}^{-1}})$.

8. *Construction of a normal basis.* We first construct a set of linearly independent elements which are isomorphic to the matric units of the group ring $\mathfrak{G}(\Omega)$, where Ω is an algebraically closed field over k, and $\mathfrak{G}$ is the Galois group of K/k. By so choosing Ω every irreducible representation of $\mathfrak{G}$ may be expressed by means of matrices with coefficients in Ω. Let $E, A, B, \cdot\cdot, S, \cdot\cdot$ be the elements of $\mathfrak{G}$, $z^E, z^A, \cdot\cdot, z^S \cdot\cdot$, the conjugates of z, and Γ an absolute irreducible representation of $\mathfrak{G}$ defined as follows:

$$(8.1) \qquad E = (\epsilon_{ik}), \quad A = (\alpha_{ik}), \cdots S = (\sigma_{ik}), \cdots.$$

Let

$$(8.2) \qquad M(z) = Ez^{E^{-1}} + Az^{A^{-1}} + \cdots + Sz^{S^{-1}} + \cdots = (\sum_{\mathfrak{G}} \sigma_{ik} z^{\ -1}),$$

where z is any element of K. If z generates a normal basis of K/k, the matrix $M(z)$ corresponds to the matrix (6.1) in the isomorphism between $\mathfrak{G}(k)$ and $K(k)$. For convenience let

$$(8.3) \qquad \sum_{\mathfrak{G}} \sigma_{ik} z^{S^{-1}} = \xi_{ik}(z).$$

Thus $M(z) = (\xi_{ik}(z))$. We first show that there exists an element z of K such that the $\xi_{ik}(z)$ are linearly independent. For this purpose we consider representations of $\mathfrak{G}$ defined by the $\xi_{ik}(z)$. If $S^{-1} = (\bar{\sigma}_{ik})$,[3]

$$(8.4) \qquad (\xi_{i1}(z^{S^{-1}}), \cdot\cdot, \xi_{if}(z^{S^{-1}})) = (\xi_{i1}(z), \cdot\cdot, \xi_{if}(z))(\sigma_{ik}),$$

This equality is true for $i = 1, 2, \cdot\cdot, f$, where f is the degree of Γ. Thus $(\bar{\sigma}_{ik})$ is a representation of S^{-1} on Ω defined by the representation module $(\{\xi_{i1}(z)\}, \cdot\cdot, \{\xi_{if}(z)\})$, where the sets of columns $\{\xi_{ik}(z)\}$, $i = 1, \cdot\cdot, f$ are considered as elements of the module. Similarly,

$$(8.4') \qquad (\xi_{1i}(z)^{S^{-1}}, \cdot\cdot, \xi_{fi}(z)^{S^{-1}}) = (\xi_{1i}(z), \cdot\cdot, \xi_{fi}(z))(\bar{\sigma}_{ik}).$$

Hence $(\bar{\sigma}_{ik})$ is a representation of S^{-1} on Ω defined by the representation module $(\{\xi_{1i}(z)\}, \cdot\cdot, \{\xi_{fi}(z)\})$. However, $(\bar{\sigma}_{ik})$ is an irreducible representation of S^{-1} on Ω defined by Γ. Therefore the representation modules $(\{\xi_{i1}(z)\}, \cdot\cdot, \{\xi_{if}(z)\})$ and $(\{\xi_{1i}(z)\}, \cdot\cdot, \{\xi_{fi}(z)\})$ must be zero or simple, in fact, in the latter case, $\{\xi_{i1}(z)\}, \cdot\cdot, \{\xi_{if}(z)\}$ and also $\{\xi_{1i}(z)\}, \cdot\cdot, \{\xi_{fi}(z)\}$ must be linearly independent. Furthermore,

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[3] $(\bar{\sigma}_{k_i})$ is the adjoint representation, $\bar{\Gamma}(S)$, see (11), p. 163.

LEMMA 1. *There exists an element z of K such that $\xi_{ik}(z) \neq 0$ for every (i, k).*

For suppose that $\xi_{ik}(z) = 0$ for every z belonging to K, and for (i, k) fixed, then if $z_1, \cdot \cdot, z_n$ is a basis of K/k, the discriminant of K with respect to this basis is

$$D_z = \begin{vmatrix} z_1 & \cdots & z_n \\ \cdots & \cdots & \cdots \\ z_1^S & \cdots & z_n^S \\ \cdots & \cdots & \cdots \end{vmatrix}^2 = 0,$$

but $D_z \neq 0$ since K is separable over k. We may conclude, therefore, that there exists a z of the z_i such that $\xi_{ik}(z) \neq 0$, (i, k) fixed. Then $(\xi_{i1}(z), \cdot \cdot, \xi_{if}(z)) \neq 0$, and therefore $\xi_{i1}(z), \cdot \cdot \xi_{if}(z)$ are linearly independent. This demands that they all be different from zero, hence that all the modules $(\xi_{1i}(z), \cdot \cdot, \xi_{fi}(z))$ be different from zero. Thus all $\xi_{ik}(z) \neq 0$. Also,

LEMMA 2. *If $\mathfrak{G}(\Omega) = \mathfrak{a}^{(1)} + \cdot \cdot + \mathfrak{a}^{(r)}$, $\mathfrak{a}^{(i)}$ two-sided simple ideals, then the basis of every simple right ideal of $\mathfrak{a}$*[4] *can be written as a linear combination of the bases of simple right ideals $\mathfrak{r}_i$ defined by the matric units c_{ik},*

$$\{\mathfrak{r}_i\} = (c_{i1}, \cdot \cdot \cdot, c_{if}).$$

That is there exist α in Ω such that

$$\{\mathfrak{r}\} = \alpha\{c_{1i}\} + \cdot \cdot \cdot + \alpha_f\{c_{fi}\}. \tag{8.5}$$

For every simple right ideal of $\mathfrak{a}$ is isomorphic to $\mathfrak{r}_1$, and, henc there exists an element a in the left ideal $(c_{11}, \cdot \cdot, c_{f1})$ such that

$$\{\mathfrak{r}\} = a\{\mathfrak{r}_1\} = (\sum_i \alpha_{i1}c_{i1})\,\{\mathfrak{r}_1\} = \sum \alpha_{i1}\{\mathfrak{r}_i\}.$$

THEOREM 3. *The elements $\xi_{ik}(z)$ defined by* (8.3) *and which satisfy Lemma* 1, *are linearly independent.*

Let $\mathfrak{G}(\Omega) = \mathfrak{a}^{(1)} + \cdot \cdot + \mathfrak{a}^{(r)}$, and let $\mathfrak{r}$, a simple right ideal in $\mathfrak{a} = \mathfrak{r}_1 + \cdot \cdot + \mathfrak{r}_f$, define the irreducible representation Γ, (8.1). Thus $\mathfrak{a}$, is a representation module defining the representation

[4] $\mathfrak{a}$ may be any of the two sided-simple ideals $\mathfrak{a}^{(i)}$, and c_{ik} the matric units of $\mathfrak{a}$.

$$\begin{bmatrix} \cdot & & 0 \\ & \cdot \quad \cdot & \\ & (\sigma_{ik}) & \\ & \cdot \quad \cdot & \\ 0 & & \cdot \end{bmatrix} \text{ of } S, \text{ and } \begin{bmatrix} \cdot & & 0 \\ & \cdot & \\ & (\bar{\sigma}_{ik}) & \\ & & \cdot \\ 0 & & \cdot \end{bmatrix} \text{ of } S^{-1}.$$

On the other hand, M is a representation module defining the same representation of S^{-1}. Therefore $\mathfrak{a}$ is operator homomorphic to M, i. e., $\mathfrak{a} \rightarrow M$, and

$$\begin{aligned} c_{ik} &\rightarrow \xi_{ki}, \\ c_{ik}\Omega &\rightarrow \xi_{ki}\Omega, \\ c_{ik}S &\rightarrow \xi^{S}_{ki}. \end{aligned} \tag{8.6}$$

Furthermore, M is operator isomorphic to a factor group $\mathfrak{a}/\mathfrak{r}$ where $\mathfrak{r}$ is a right ideal in $\mathfrak{a}$ and corresponds to the zero element in M. Since the ring is completely reducible $\mathfrak{r} = \Sigma \mathfrak{r}_i^*$, where $\mathfrak{r}_i^*$ are simple right ideals. It follows from Lemma 2, (8. 5), that $\mathfrak{r}_1^* = \Sigma \alpha_{i1}\{\mathfrak{r}_i\}$, and if we apply the homomorphism (8. 6) to this equality, we may conclude that

$$0 = \Sigma \alpha_{k1}\{\xi_{ik}\} = \alpha_{11}\{\xi_{i1}\} + \cdot \cdot \cdot + \alpha_{f1}\{\xi_{if}\}.$$

The sets $\{\xi_{i1}\}, \cdot \cdot, \{\xi_{if}\}$, however, define a non-zero simple module and are linearly independent with respect to Ω. Therefore $\alpha_{i1} \equiv 0$. Similarly it can be shown that the coefficients of $\mathfrak{r}_i^*$ all vanish. Hence

$$\mathfrak{r} = \mathfrak{r}_1^* + \cdot \cdot \cdot + \mathfrak{r}_s^* = 0,$$

and the homomorphism is an isomorphism. We may conclude, therefore, that $\xi_{ik}(z)$ are linearly independent with respect to Ω.

For each of the r distinct irreducible representations of $\mathfrak{G}$ defined by the two-sided simple ideals $\mathfrak{a}^{(l)}$ of $\mathfrak{G}(\Omega)$, we now define a matrix

$$M^{(l)} = (\xi^{(l)}_{ik}(z_l)),$$

where $\xi^{(l)}_{ik}(z_l) \neq 0$ for all i, k. It remains to determine an element w belonging to K such that $\xi^{(l)}_{ik}(w) \neq 0$ for all i, k, l. Let the centrum idempotents of $\mathfrak{G}(\Omega)$ in $\mathfrak{a}^{(1)}, \cdot \cdot, \mathfrak{a}^{(r)}$ be $E^{(1)}, \cdot \cdot, E^{(r)}$ respectively, and

$$z^{TE^{(i)}} = \sum_{\mathfrak{G}} \alpha_s z^{TS}, \quad \text{if} \quad E^{(i)} = \sum_{\mathfrak{G}} \alpha_S S.$$

We construct

$$w = z_1^{E^{(1)}} + \cdot \cdot \cdot + z_r^{E^{(r)}}, \tag{8.7}$$

where z_l have been chosen such that $\xi_{ik}^{(l)}(z_l) \neq 0$. Furthermore, if $\mathfrak{a}^{(l)}$ is conjugate to $\mathfrak{a}^{(m)}$ with respect to k, we take $z_l = z_m$. This is possible since $\xi_{ik}^{(m)}(z_l)$ is conjugate to $\xi_{ik}^{(l)}(z_l)$ and therefore is not equal to zero if $\xi_{ik}^{(l)}(z_l) \neq 0$. If k contains the n-th roots of unity the case that $\mathfrak{a}^{(l)}$ and $\mathfrak{a}^{(m)}$ are conjugate $(l \neq m)$ does not occur.

Does w, defined as the sum of these components, belong to K? If k contains the n-th roots of unity, $z^{E^{(l)}}$ and hence w belong to K. If k does not contain the n-th roots of unity, we consider the sums of the conjugate idempotents. These sums give us elements of $\mathfrak{G}(k)$. That is,

$$e^{(i)} = \sum_{\text{Conjugates}} E^{(i)} = \sum_{\mathfrak{G}} \alpha_{si} S, \qquad \alpha_{si} \in k. \tag{8.8}$$

Furthermore, since we have the same z_i for conjugate $E^{(i)}$, we can write w as the sum $\sum_i z^{e^{(i)}}$. Hence w belongs to K and is defined as a *rational sum.*

It must now be shown that $\xi_{ik}^{(l)}(w) \neq 0$ for every i, k, l. To do this we apply the homomorphisms of (8.6). It follows from the construction of w, (8.7), that

$$\begin{aligned} \xi_{ik}^{(l)}(w) &= \sum_{\mathfrak{G}} \sigma_{ik}^{(l)} w^S = \sum_{\mathfrak{G}} \sigma_{ik}^{(l)} \left(\sum_m z_m{}^{E^{(m)}}\right)^S \\ &= \sum_{\mathfrak{G}} \sigma_{ik}^{(l)} (z^{E^{(1)}})^S + \cdots + \sum_{\mathfrak{G}} \sigma_{ik}^{(l)} (z_l{}^{E^{(l)}}) + \cdots + \sum_{\mathfrak{G}} \sigma_{ik}^{(l)} (z_r{}^{E^{(r)}})^S. \end{aligned} \tag{8.9}$$

The general term of the right hand side of (8.9) may be expressed

$$\sum_{\mathfrak{G}} \sigma_{ik}^{(l)} (z_\mu{}^{E^{(\mu)}})^S = {}^{E^{(\mu)}}\xi_{ik}^{(l)}(z_\mu),$$

and corresponds, in the homomorphism defined by

$$c_{ki}^{(l)} \rightarrow \xi_{ik}^{(l)}(z_\mu), \tag{8.10}$$

to $E^{(\mu)} c_{ki}^{(l)} = 0$ if $\mu \neq l$. Thus

$$\sum_{\mathfrak{G}} \sigma_{ik}^{(l)} (z_\mu{}^{E^{(\mu)}}) = 0, \qquad \mu \neq l.$$

If, however, $\mu = l$, $E^l c_{ki}^{(l)} = c_{ki}^{(l)}$. Therefore, in this case,

$${}^{E^{(l)}}\xi_{ik}^{(l)}(z_l) = \xi_{ik}^{(l)}(z_l).$$

We conclude that

$$\xi_{ik}^{(l)}(w) = \xi_{ik}^{(l)}(z_l) \neq 0, \tag{8.11}$$

for any i, k, l. It follows, as shown previously, that $\xi_{ik}^{(l)}(w)$ are linearly independent with respect to Ω. Moreover, $\xi_{ik}^{(l)}(w)$ are expressed linearly by means

of the set of elements $\{w^S\}$. Therefore this set $\{w^S\}$ forms a *normal basis* of K/k.

9. *A rational construction.* In 8 w was defined as a *rational sum*, but the z_l were chosen to satisfy the *irrational condition* $\xi^{(l)}_{ik}(z_l{}^{E^{(l)}}) \neq 0$. This irrational condition, however, may be replaced by a rational condition, $z^{e^{(l)}} \neq 0$, where $e^{(l)}$ is as defined in equation (8.8). Since the regular representation $\Gamma = \Sigma f_l \Gamma_l$,

$$\sum_{i,l} (f_l/n)\sigma^{(l)}_{ii} = Tr(S/n) = \delta_{SE},$$

and therefore

$$\sum_{i,l} (f_l/n)\xi^{(l)}_{ii}(z) = \sum_{\mathfrak{G}} (\sum_{i,l} (f_l/n)\sigma^{(l)}_{ii}) z^S = z.$$

This relation, together with the fact (8.11) that

$$\xi^{(l)}_{ik}(z^{e^{(l)}}) = \xi^{(l)}_{ik}(z^{E^{(l)}}),$$

gives us the equivalence of the two conditions

$$\xi^{(l)}_{ik}(z^{E^{(l)}}) \neq 0, \quad \text{and} \quad z^{e^{(l)}} \neq 0.$$

First suppose

$$\xi^{(l)}_{ik}(z^{E^{(l)}}) = \xi^{(l)}_{ik}(z^{e^{(l)}}) \neq 0.$$

Hence in the isomorphism defined by

$$c^{(l)}_{ii} \leftrightarrow \xi^{(l)}_{ii}(z^{e^{(l)}}),$$

$$z^{e^{(l)}} = \sum_{i,l} (f_l/n)\xi^{(l)}_{ii}(z^{e^{(l)}}) \leftrightarrow \sum_{i,l} (f_l/n)c^{(l)}_{ii} = f_l/n \neq 0.$$

Therefore

$$z^{e^{(l)}} \neq 0.$$

Conversely, if $z^{e^{(l)}} \neq 0$, one of the summands of

$$\sum_{i,l} (f_l/n)\xi^{(l)}_{ii}(z^{e^{(l)}})$$

must be different from zero. We have shown, however, that if one $\xi^{(l)}_{ik}(z^{e^{(l)}}) \neq 0$, then $\xi^{(l)}_{ik}(z^{e^{(l)}}) \neq 0$, l fixed. Hence the two conditions are equivalent and the following theorem is proved.

THEOREM 4. *If z_l is any element of K such that its component $z^{e^{(l)}} \neq 0$, then $w = \Sigma z_l{}^{e^{(l)}}$ will generate a normal basis of K/k.*

10. *The most general normal basis.* The final problem of this paper is to determine from a given normal basis the most general normal basis. We

note first that if $\mathfrak{G}(k)$ is operator isomorphic to $K(k)$ with operator region $\mathfrak{G}$ and if $\alpha \leftrightarrow a$, $\beta \leftrightarrow b$, where $\alpha, \beta \in G(k)$ and $a, b \in K(k)$, then $a^{(\beta)} = {}^{(\alpha)}b$, where $a^{(\beta)}$ means that a is operated on by β from the right and ${}^{(\alpha)}b$ means that b is operated on by α from the left.

THEOREM 5. *If* $(w^{S_1}, \cdots, w^{S_n})$ *is any normal basis of* K/k, *then the most general normal basis of* K/k *is of the form* $(w^{\nu S_1}, \cdots, w^{\nu S_n})$ where ν is *any element of* $\mathfrak{G}(k)$ *which is not a zero divisor in* $\mathfrak{G}(k)$.

Let $E \leftrightarrow w$, then $S \leftrightarrow w^S$. Suppose that $v^{S_1}, \cdots, v^{S_n}$ is any other normal basis of K/k. In the isomorphism $\mathfrak{G}(k) \simeq K(k)$ let $\nu \leftrightarrow v$. Then we know from the above note that

$$(\nu S_1, \cdots, \nu S_n) \leftrightarrow (v^{S_1}, \cdots v^{S_n}) = (w^{\nu S_1}, \cdots, w^{\nu S_n}).$$

On the other hand, if ν is any element of $\mathfrak{G}(k)$, then

$$\nu(S_1, \cdots, S_n) \leftrightarrow (w^{\nu S_1}, \cdots, w^{\nu S_n}).$$

Furthermore suppose there exist $c_i \not\equiv 0$ such that

$$\Sigma \nu c_i S_i \leftrightarrow \Sigma c_i w^{\nu S_i} = 0,$$

then if ν is not a divisor of zero, $\nu = 0$ since $\Sigma c_i S_i \neq 0$. Hence if ν is not a divisor of zero the set $(w^{\nu S_1}, \cdots, w^{\nu S_n})$ is a normal basis of K/k.

Note. R. Brauer has told me of another proof of Theorem 5. It is the following: Let w^G be a normal basis, then

$$(w^G)^T = \Sigma \delta_{GT,H} w^H. \tag{1*}$$

The matrix $S_T = (\delta_{GT,H})$, G gives the row, H the column, is the matrix corresponding to T in the regular representation. (1*) may then be written

$$w^T = S_T(w). \tag{2*}$$

Let z^G be a second normal basis and let

$$w = A(z),$$

that is, $w^G = \Sigma a_{GH} z^H$ if $A = (a_{GH})$. Then

$$z^T = A^{-1}(w^T) = A^{-1} S_T(w) = A^{-1} S_T A(z).$$

The condition for A is therefore $A^{-1} S_T A = S_T$, $|A| \neq 0$. The most general

matrix commutative with every S_T is a matrix of the second regular representation. If α is an element of the group ring, this representation is defined by

$$\alpha G = \Sigma a_{GH} H.$$

It is a reciprocal representation (A' is a direct representation). Then $w = A(z)$ or $w^G = \Sigma a_{GH} z^H$ reads $w^G = z^{\alpha G}$. The element α must not be a divisor of zero because otherwise $|A| = 0$.

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A REMARK ON THE AREA OF SURFACES.[1]

By Tibor Radó.

Introduction. Let Σ be a continuous surface given by equations

$$\Sigma: x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad 0 \leqq u \leqq 1, \quad 0 \leqq v \leqq 1.$$

One of the fundamental problems in the theory of the area is to find conditions under which the Lebesgue area $L[\Sigma]$ of Σ is given by the classical formula

$$L[\Sigma] = \int_0^1 \int_0^1 (EG - F^2)^{1/2} du dv,$$

where E, F, G have the familiar meaning.[2] In the special case $x = u$, $y = v$ Tonelli found that the absolute continuity, in the sense defined by him, of the function $z(u, v)$ is a necessary and sufficient condition for the validity of the classical formula for the area.[3] In the general case, only sufficient conditions were established so far. The most general result in this direction was obtained by McShane and by Morrey.[4] To make our remarks more definite, we shall consider the work of Morrey. Morrey defines a class L of surfaces by requiring the existence of representations where the coördinate functions $x(u, v)$, $y(u, v)$, $z(u, v)$ satisfy two conditions (i) and (ii).[5] Condition (ii) is concerned with the approximation of the coördinate functions by integral means. Condition (i) requires that the coördinate functions be absolutely continuous in the sense of Tonelli. Both McShane and Morrey observe that in the special case $x = u$, $y = v$ these conditions are equivalent to the necessary and sufficient condition of Tonelli. It might be therefore of some interest to point out that condition (i) can be replaced by the weaker condition that the coördinate

[1] Presented to the American Mathematical Society at the meeting in Chicago, April, 1936.

[2] For general information and for references to the literature concerning continuous surfaces and the Lebesgue area the reader may consult the paper of C. B. Morrey, "A class of representations of manifolds, part I," *American Journal of Mathematics*, vol. 55 (1933), pp. 683-707.

[3] L. Tonelli, "Sulla quadratura delle superficie," *Atti della Reale Accademia dei Lincei*, series 6, vol. 3 (1926), pp. 357-362, 445-450, 633-638, 714-719. S. Saks gave a very elegant presentation of the results of Tonelli in his paper "Sur l'aire des surfaces $z = f(x, y)$," *Acta Szeged*, vol. 3 (1927), pp. 170-176.

[4] E. J. McShane, "Integrals over surfaces in parametric form," *Annals of Mathematics*, vol. 34 (1933), pp. 815-838. C. B. Morrey, *loc. cit.* [2].

[5] *Loc. cit.* [2], p. 701.

functions be of bounded variation in the sense of Tonelli. If we denote this weaker condition by (i*), and if we use $\mathfrak{K}$ to denote the class of surfaces defined by conditions (i*) and (ii), then our main result states that the classical formula for the Lebesgue area $L[\Sigma]$ holds for every surface Σ of class $\mathfrak{K}$ (provided, of course, that we use a representation satisfying conditions (i*) and (ii)). In the special case $x = u$, $y = v$ conditions (i*) and (ii) are still equivalent to the necessary and sufficient condition of Tonelli. Our argument will be based upon certain simple facts concerning the topological index which were established in a previous paper of the author [6] and which we are going to state presently.

Let C_k be a sequence of closed continuous curves, in the (x, y)-plane, which converge in the sense of Fréchet to a closed continuous curve C. This means that these curves admit of simultaneous representations

$$\begin{aligned} &C\colon\ x = x(t), \quad y = y(t), \quad 0 \leqq t \leqq 1, \quad x(0) = x(1), \quad y(0) = y(1), \\ &C_k\colon\ x = x_k(t), \quad y = y_k(t), \quad 0 \leqq t \leqq 1, \quad x_k(0) = x_k(1), \quad y_k(0) = y_k(1), \end{aligned}$$

where the functions $x(t)$, $y(t)$, $x_k(t)$, $y_k(t)$ are continuous in $0 \leqq t \leqq 1$ and $x_k(t) \to x(t)$, $y_k(t) \to y(t)$ uniformly in $0 \leqq t \leqq 1$. Let $n(x, y)$, $n_k(x, y)$ be the index-functions relative to C, C_k respectively (see 2.3). Let us use $T[\lambda]$ to denote the total variation of a function $\lambda(t)$ defined in $0 \leqq t \leqq 1$, where $T[\lambda] = \infty$ if $\lambda(t)$ is not of bounded variation. It is well known that if at least one of the functions $x(t)$, $y(t)$ is of bounded variation, then the index-function $n(x, y)$ is summable.[7] Using these notations, we have the following lemmas.

LEMMA 1. *If $C_k \to C$ in the sense of Fréchet, and if $T[x_k] \to T[x] < \infty$ (or $T[y_k] \to T[y] < \infty$), then*

$$\iint | n(x, y) - n_k(x, y) | \, dxdy \to 0.$$

LEMMA 2. *If* 1) *$C_k \to C$ in the sense of Fréchet,* 2) *the functions $x(t)$, $y(t)$, $x_k(t)$, $y_k(t)$ are of bounded variation,* 3) *$T[x_k] < M$, $T[y_k] < M$, where M is some finite constant independent of k, then*

$$\iint n_k(x, y) \, dxdy \to \iint n(x, y) \, dxdy.$$

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[6] "A lemma on the topological index," submitted for publication to the editors of the *Fundamenta Mathematicae*. An abstract appeared in the *Bulletin of the American Mathematical Society*, vol. 42 (1936), p. 187.

[7] J. Schauder, "Über stetige Abbildungen," *Fundamenta Mathematicae*, vol. 12 (1928), pp. 47-74, in particular pp. 64-66.

Lemma 2 is a consequence of lemma 1, but can be proved independently also. The reader will note that the conclusion is much weaker in lemma 2 than in lemma 1. In the situation which we shall consider, the assumptions of the strong lemma 1 will be amply satisfied, while we shall actually need only the conclusion of the weak lemma 2. In a general way, despite the generality of our main theorem it will be quite apparent that the definition of the class $\mathfrak{K}$ implies considerably more than what is actually needed in the proof. With regard to possible use in investigations suggested by this remark, we develop certain inferences from our assumptions beyond the absolute minimum needed for our present purposes. We also took the liberty of stating condition (ii) of Morrey in an obviously equivalent but somewhat more convenient form, by splitting it into conditions II and III of section 2.1.

INDEX OF NOTATIONS AND DEFINITIONS.

1. *On the approximation by integral means.*

1.1. Let $x(u, v)$ be a continuous function in the closed square S_0: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. For $0 < h < \frac{1}{2}$, the closed square $h \leqq u \leqq 1 - h$, $h \leqq v \leqq 1 - h$ will be denoted by S_h. In S_h, we define

$$x^{(h)}(u, v) = (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} x(u + \alpha, v + \beta)\, d\alpha d\beta.$$

As it is well known, $x^{(h)}(u, v)$ is continuous in S_h together with its partial derivatives of the first order.[8]

[8] For a systematic presentation of the properties of this approximation see H. E. Bray, "Proof of a formula for an area," *Bulletin of the American Mathematical Society*, vol. 29 (1923), pp. 264-270.

1.2. Let $R\colon a \leqq u \leqq b,\ c \leqq v \leqq d$ be a rectangle completely interior to S_0. For small values of h, R will be completely interior to S_h also and clearly $x^{(h)}(u, v) \to x(u, v)$ for $h \to 0$ uniformly in R.

1.3. Suppose that $x(u, v)$, considered as a function of v alone, is of bounded variation on the interval $u = \text{const.}$, $v_1 \leqq v \leqq v_2$. The total variation on this interval will be denoted by ${}_vT_{v_1}{}^{v_2}[u; x]$. If $x(u, v)$ is not of bounded variation on this interval, we put ${}_vT_{v_1}{}^{v_2}[u; x] = \infty$. The symbols ${}_uT_{u_1}{}^{u_2}[v; x]$, ${}_vT_{v_1}{}^{v_2}[u; x^{(h)}]$, ${}_uT_{u_1}{}^{u_2}[v; x^{(h)}]$ are defined in a similar fashion.

1.4. Let $R\colon a \leqq u \leqq b,\ c \leqq v \leqq d$ be a rectangle completely interior to S_0. We define

$$T[R; x] = {}_uT_a{}^b[c; x] + {}_vT_c{}^d[b; x] + {}_uT_a{}^b[d; x] + {}_vT_c{}^d[a; x]$$

if all the terms on the right-hand side are finite. Otherwise we put $T[R; x] = \infty$. For small values of h, R will be completely interior to S_h also, and we define then the symbol $T[R; x^{(h)}]$ in a similar way.

1.5. The continuous function $x(u, v)$ is of bounded variation in the sense of Tonelli if ${}_vT_0{}^1[u; x]$, ${}_uT_0{}^1[v; x]$ are summable in the intervals $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$ respectively. To describe this situation, we shall say that $x(u, v)$ is B. V. T. in S_0.

1.6. Let the continuous function $x(u, v)$ be B. V. T. in S_0. The function ${}_vT_0{}^1[u; x]$ being then summable, the function ${}_vT_{v_1}{}^{v_2}[u; x]$, where $0 \leqq v_1 < v_2 \leqq 1$, is *a fortiori* summable in the interval $0 \leqq u \leqq 1$ for fixed v_1, v_2. By a well-known theorem of the Lebesgue theory, we have therefore in the interval $0 \leqq u \leqq 1$ a set ${}_uE[v_1, v_2; x]$ of measure zero, such that ${}_vT_{v_1}{}^{v_2}[u; x]$ is finite and

$$(1/2h) \int_{-h}^{h} {}_vT_{v_1}{}^{v_2}[u + \alpha; x]\, d\alpha \underset{h \to 0}{\to} {}_vT_{v_1}{}^{v_2}[u; x]$$

for u not in ${}_uE[v_1, v_2; x]$. Similarly, we have in the interval $0 \leqq v \leqq 1$ a set ${}_vE[u_1, u_2; x]$ of measure zero such that ${}_uT_{u_1}{}^{u_2}[v; x]$ is finite and

$$(1/2h) \int_{-h}^{h} {}_uT_{u_1}{}^{u_2}[v + \beta; x]\, d\beta \underset{h \to 0}{\to} {}_uT_{u_1}{}^{u_2}[v; x]$$

for v not in ${}_vE[u_1, u_2; x]$.

1.7. We denote by ${}_uE[x]$ the sum of all the sets ${}_uE[v_1, v_2; x]$ which correspond to all pairs of *rational* numbers v_1, v_2 such that $0 \leqq v_1 < v_2 \leqq 1$. The set ${}_vE[x]$ is defined in a similar way. The sets ${}_uE[x]$, ${}_vE[x]$ are both

sums of a denumerable infinity of sets of measure zero, and are therefore of measure zero.

1.8. A rectangle R: $a \leqq u \leqq b$, $c \leqq v \leqq d$ will be called *admissible* with respect to $x(u, v)$ if a, b are not in ${}_uE[x]$, c, d are not in ${}_vE[x]$, and if R is completely interior to S_0.

1.9. LEMMA. *If the continuous function $x(u, v)$ is B. V. T. in S_0, then*

$$T[R; x^{(h)}] \underset{h\to 0}{\rightarrow} T[R; x]$$

for every admissible rectangle R.[9]

To prove this, it is sufficient to discuss any one of the four terms which make up $T[R; x]$. Let us show, for instance, that

$$(1) \qquad {}_uT_a{}^b[c; x^{(h)}] \underset{h\to 0}{\rightarrow} {}_uT_a{}^b[c; x].$$

Since the interval $v = c$, $a \leqq u \leqq b$ is completely interior to S_0, we have $x^{(h)} \rightarrow x$ uniformly on this interval. Hence obviously

$$(2) \qquad \underline{\lim}_{h\to 0}\, {}_uT_a{}^b[c; x^{(h)}] \geqq {}_uT_a{}^b[c; x].$$

Let now u', u'' be any two *rational* numbers such that

$$(3) \qquad 0 < u' < a < b < u'' < 1.$$

Let us take any system of numbers

$$u_0 = a < u_1 < \cdots < u_{k-1} < u_k < \cdots < u_n = b.$$

We have then, for $h < u'' - b$, $h < a - u'$,

$$\begin{aligned}
&\sum_{k=1}^{n} | x^{(h)}(u_k, c) - x^{(h)}(u_{k-1}, c) | \\
&\leqq (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} \sum_{k=1}^{n} | x(u_k + \alpha, c + \beta) - x(u_{k-1} + \alpha, c + \beta) | \, d\alpha d\beta \\
&\leqq (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} {}_uT_{a+\alpha}^{b+\alpha}[c + \beta; x] d\alpha d\beta \\
&\leqq (1/4h^2) \int_{-h}^{h} \int_{-h}^{h} {}_uT_{u'}{}^{u''}[c + \beta; x] d\alpha d\beta = (1/2h) \int_{-h}^{h} {}_uT_{u'}{}^{u''}[c + \beta; x] d\beta.
\end{aligned}$$

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[9] The idea of the proof was suggested by a similar argument used by S. Saks, *loc. cit.* [8]. Morrey, *loc. cit.* [2], p. 703, shows (in our notations) that

$$\overline{\lim_{h\to 0}}\, T[R; x^{(h)}] < \infty$$

by a reasoning which makes use of absolute continuity of the functions involved.

Consequently

(4) $$ {}_uT_a{}^b[c\,;x^{(h)}] \leqq (1/2h)\int_{-h}^{h} {}_uT_{u'}{}^{u''}[c+\beta\,;x]\,d\beta. $$

Since u', u'' are rational and c is not in ${}_vE[x]$, we infer from (4) that

(5) $$ \overline{\lim_{h\to 0}}\, {}_uT_a{}^b[c\,;x^{(h)}] \leqq {}_uT_{u'}{}^{u''}[c\,;x]. $$

But $x(u,v)$ is continuous and consequently, for fixed c, ${}_uT_{u'}{}^{u''}[c\,;x]$ is a continuous function of u', u''. Hence (5) yields, for $u'\to a$, $u''\to b$, the relation

(6) $$ \overline{\lim_{h\to 0}}\, {}_uT_a{}^b[c\,;x^{(h)}] \leqq {}_uT_a{}^b[c\,;x]. $$

(2) and (6) imply (1), and the proof is complete.

2. *On transformations of class $\mathfrak{K}$.*

2.1. If $x(u,v)$, $y(u,v)$ are continuous in S_0, then the equations $x=x(u,v)$, $y=y(u,v)$ define a continuous transformation. We shall say that this transformation is of class $\mathfrak{K}$ if the following conditions are satisfied.[10]

I. $x(u,v)$, $y(u,v)$ are B. V. T. in S_0.

II. The Jacobian [11]

$$ \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} $$

is summable in S_0.

III. For every rectangle R: $a\leqq u\leqq b$, $c\leqq v\leqq d$, which is completely interior to S_0, we have

$$ \iint_R \left| \frac{\partial(x,y)}{\partial(u,v)} - \frac{\partial(x^{(h)},y^{(h)})}{\partial(u,v)} \right| dudv \underset{h\to 0}{\longrightarrow} 0. $$

In the last relation, $y^{(h)}$ is defined in the same way in terms of y as $x^{(h)}$ was defined in terms of x.

2.2. A rectangle R: $a\leqq u\leqq b$, $c\leqq v\leqq d$ will be called admissible with respect to the transformation if it is admissible with respect to both $x(u,v)$ and $y(u,v)$, in the sense of 1.8.

[10] Conditions II and III are obviously equivalent to condition (ii) of Morrey, while condition I replaces the more restrictive condition (i) of Morrey which requires absolute continuity in the sense of Tonelli. See Morrey, *loc. cit.* [2], p. 701.

[11] On account of condition I, the partial derivatives of the first order exist almost everywhere in S_0.

2. 3. Under the continuous transformation $x = x(u, v)$, $y = y(u, v)$ the boundary B of a rectangle R (comprised in S_0) is carried into a closed continuous curve C in the (x, y)-plane. If (x, y) is a point not on C, we define $n_R(x, y)$ as the topological index [12] of (x, y) with respect to C, this curve being described in the sense which corresponds to the counter-clockwise sense around R. If (x, y) is on C, we put $n_R(x, y) = 0$.

2. 4. If the rectangle R is completely interior to S_0, then $x^{(h)}, y^{(h)}$ will be both defined on R for small values of h. The symbol ${}_h n_R(x, y)$ is then defined in the same way in terms of $x^{(h)}(u, v)$, $y^{(h)}(u, v)$ as $n_R(x, y)$ was defined in terms of $x(u, v)$, $y(u, v)$.

2. 5. LEMMA. *If the continuous transformation $x = x(u, v)$, $y = y(u, v)$ is of class $\mathfrak{K}$ in S_0, then*

$$\iint |\, n_R(x, y) - {}_h n_R(x, y) \,|\, dxdy \underset{h \to 0}{\longrightarrow} 0 \tag{7}$$

and

$$\iint_R \frac{\partial(x, y)}{\partial(u, v)}\, dudv = \iint n_R(x, y)\, dxdy \tag{8}$$

for every admissible rectangle R.[13]

Proof. We have $x^{(h)} \to x$, $y^{(h)} \to y$ uniformly on the boundary of R, and since R is admissible, we have, by 1. 9,

$$T[R; x^{(h)}] \underset{h \to 0}{\longrightarrow} T[R; x], \tag{9}$$

$$T[R; y^{(h)}] \underset{h \to 0}{\longrightarrow} T[R; y]. \tag{10}$$

The relation (7) is thus a direct consequence of lemma 1 (see the introduction). Since $x^{(h)}, y^{(h)}$ have continuous derivatives of the first order, we have [14]

$$\iint_R \frac{\partial(x^{(h)}, y^{(h)})}{\partial(u, v)} = \iint {}_h n_R(x, y)\, dxdy. \tag{11}$$

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[12] See, for instance, Kerékjártó, *Vorlesungen über Topologie*, vol. 1, section 2, § 2.

[13] Formula (8) is a generalization of Lemma 4 of Morrey, *loc. cit.* [2], p. 702. For the case when $x(u, v)$, $y(u, v)$ satisfy the Lipschitz condition, formula (8) was established by Schauder, *loc. cit.* [7]. Formula (7) is independent of conditions II and III in 2. 1 and expresses a property of the approximation by integral means which apparently was not yet noticed.

[14] See Schauder, *loc. cit.* [7].

Condition III in 2.1 implies that

$$\lim_{h\to 0}\iint_R \frac{\partial(x^{(h)}, y^{(h)})}{\partial(u,v)}\,dudv = \iint_R \frac{\partial(x,y)}{\partial(u,v)}\,dudv, \tag{12}$$

while (7) implies that

$$\lim_{h\to 0}\iint {}_h n_R(x,y)\,dxdy = \iint n_R(x,y)\,dxdy. \tag{13}$$

Clearly, (11), (12) and (13) imply (8).

3. *Conclusion.*

3.1. We shall say that a continuous surface Σ is of class $\mathfrak{K}$ if it admits of a representation

$$\Sigma\colon\ x = x(u,v),\quad y = y(u,v),\quad z = z(u,v),\quad (u,v) \text{ in } S_0,$$

such that each of the three transformations

$$\begin{aligned} y &= y(u,v), & z &= z(u,v),\\ z &= z(u,v), & x &= x(u,v),\\ x &= x(u,v), & y &= y(u,v) \end{aligned}$$

is of class $\mathfrak{K}$ in S_0, in the sense of 2.1. Every representation with this property will be called a typical representation of Σ.

3.2. If in the definition of transformations of class $\mathfrak{K}$ we replace condition I (see 2.1) by the more restrictive condition that the defining functions be absolutely continuous in the sense of Tonelli, and if we modify the definition of surfaces of class $\mathfrak{K}$ accordingly, then we obtain the surfaces of class L studied by McShane and Morrey.[15] To make the following remarks more concise, we shall consider the work of Morrey. The assumption of absolute continuity is used by Morrey only to establish formula (8) (see 2.5). We derived that formula under the assumption of bounded variation in the sense of Tonelli. This being so, it is clear that we can proceed in exactly the same way in the case of surfaces of class $\mathfrak{K}$ as Morrey did in the case of surfaces of class L.[16] As a result, we obtain the following

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[15] The remarks of McShane, *loc. cit.* [4] and of Morrey, *loc. cit.* [2], concerning the generality of the class L apply *a fortiori* to the class $\mathfrak{K}$.

[16] We are referring to the proof of Theorem I of Morrey, *loc. cit.* 2, pp. 703-704.

THEOREM. *If Σ is a surface of class $\mathfrak{K}$ given in typical representation, then*

$$L[\Sigma] = \iint_{S_0} (EG - F^2)^{1/2} du dv, \tag{14}$$

where $L[\Sigma]$ is the Lebesgue area of Σ and

$$E = x_u^2 + y_u^2 + z_u^2, \quad F = x_u x_v + y_u y_v + z_u z_v, \quad G = x_v^2 + y_v^2 + z_v^2.$$

3. 3. We conclude with a few remarks to the effect that even the weaker assumption of bounded variation (in the sense of Tonelli) implies considerably more than what we actually need to derive (14). Inspection of the proof shows that we needed bounded variation only to establish formula (8) in 2. 5. To establish that formula it would be sufficient to know that

$$\iint {}_h n_R(x, y)\, dx dy \underset{h \to 0}{\longrightarrow} \iint n_R(x, y)\, dx dy, \tag{15}$$

while bounded variation implies (see 2. 5) the much stronger relation

$$\iint | n_R(x, y) - {}_h n_R(x, y) |\, dx dy \underset{h \to 0}{\longrightarrow} 0. \tag{16}$$

In order to establish (15) it would be sufficient to know that

$$\overline{\lim_{h \to 0}}\, T[R; x^{(h)}] < \infty, \quad \overline{\lim_{h \to 0}}\, T[R; y^{(h)}] < \infty, \tag{17}$$

as it follows from lemma 2 (see the introduction), while bounded variation yields (see 1. 9) the more precise information

$$\lim_{h \to 0} T[R; x^{(h)}] = T[R; x], \quad \lim_{h \to 0} T[R; y^{(h)}] = T[R; y]. \tag{18}$$

It follows further from lemma 1 (see the introduction) that only one of the two relations (18) is needed to establish formula (8) in 2. 5. These remarks suggest the possibility of further generalizations.

THE OHIO STATE UNIVERSITY.

ON THE POINCARÉ GROUP OF RATIONAL PLANE CURVES.

By Oscar Zariski.

Introduction. In a paper dealing with the Poincaré group of an algebraic hypersurface V_{n-1} in a projective complex space S_n (Zariski [9]), the following theorem is proved: *the Poincaré group of the residual space $S_n - V_{n-1}$ coincides with the Poincaré group of the residual space of a generic plane section of V_{n-1}.* In this paper we apply this theorem toward the determination of the Poincaré group of any rational plane curve with nodes and cusps only, and, more generally, of any plane curve which admits such a rational curve as a limiting case. It would seem that the sense of the quoted theorem in applications would be to reduce the apparently more difficult problem of the Poincaré group of an hypersurface to that of the Poincaré group of a plane algebraic curve. However, in this paper we apply the theorem in the opposite sense, using a convenient hypersurface in order to solve the problem in the plane. The advantage of transforming the plane problem into a problem in a space of higher dimension seems due to the fact (at least it is so in the present case) that if a curve C is a generic plane section of V_{n-1}, then, everything else being equal, the hypersurface V_{n-1} supplies a more intrinsic picture of the Poincaré group of C than C itself. The essential features of the Poincaré group of C should be revealed best on the hypersurface V_{n-1} lying in a space of the highest possible dimension.

In the present paper the starting point is supplied by the class of rational *maximal cuspidal curves* of even order $2n - 2$. These curves are generic plane sections of what may be referred to as the discriminant hypersurface D: $D(a_0, \cdots, a_n) = 0$, where D is the discriminant of the polynomial $a_0 z^n + \cdots + a_n$, and where $a_0, a_1, \cdots, a_n$ are interpreted as homogeneous coördinates in an S_n. The corresponding class of Poincaré groups G_n practically coincides with the "Zopfgruppe" of Artin (Artin,[1] see also Reidemeister,[4] p. 42). G_n is also the group of automorphism classes of a sphere with n holes (Magnus [3]). We give a proof of the completeness of the generating relations of G_n which is simpler than the previous proofs.

The case of an arbitrary rational curve with nodes and cusps is treated by applying the notion of virtually non-existent nodes or cusps (Severi,[6] Anhang F, B. Segre,[5] Zariski,[7] p. 168), and by studying the effect on the

Poincaré group of a plane curve of the removal of a cusp or of a node. This can be done, since the type of generating relations at a cusp or at a node is known (Zariski [8]). The result is largely negative: *every rational curve with nodes and cusps, other than the maximal cuspidal curve of even order* $2n-2$, *has, with one exception* (see Section 6), *a cyclic Poincaré group. The same is true of any plane algebraic curve which admits a rational curve with nodes and cusps as a limiting case.*

1. *Preliminary remarks on continuous systems of rational curves.* A rational curve with order n with k cusps shall be denoted by the symbol (n,k). Applying the formulae of Plücker, it is seen that the dual of a curve (n,k) is a curve (n',k'), where

$$(1) \qquad n' = 2n - 2 - k, \qquad k' = 3(n-2) - 2k.$$

Hence $k \leqq \frac{3}{2}(n-2)$. If n is even, the rational maximal cuspidal curves $(n, \frac{3}{2}(n-2))$ are dual to the curves $((n+2)/2, 0)$ (rational curves possessing only nodes). If n is odd, the maximal cuspidal curves $(n, (3n-7)/2)$ are dual to the curves $((n+3)/2, 1)$. From this it follows that *the maximal cuspidal curves form in either case a single irreducible continuous system* (Severi,[6] Anhang F). Since the characteristic series of this system is non-special, any number of cusps of the general curve of the system can be converted into nodes (*virtual nodes*), *and hence curves* (n,k) *exist for every integral value of* k *satisfying the inequality* $k \leqq 3(n-2)/2$ (see, for instance, B. Segre [5]).

For any given k, *satisfying the above inequality, the curves* (n,k) *form a single irreducible continuous system.* For the proof we observe that it is indifferent whether the statement is proved for the curves (n,k) or for the dual curves (n',k'). From (1) follows the relation $(n-2)-k = k'-(n'-2)$, and hence either $n-2-k$ or $n'-2-k'$ is a non-negative integer. We may assume therefore $k \leqq n-2$. A curve (n,k) is the projection of a normal rational curve Γ_n^n of order n, in S_n, the center of projection being an S_{n-3} meeting in k points the ruled surface F of the tangents of Γ_n^n. If $k \leqq n-2$, any k points belong to an S_{n-3} and hence the k intersections of S_{n-3} with F can be taken arbitrarily on F. Our statement now follows from the irreducibility of the system of k-ads of points of F (F is irreducible!) and from the irreducibility of the system of all S_{n-3}'s on k fixed points.

Since *a* system of curves (n,k) can be obtained from the system of maximal cuspidal curves by regarding a certain number of cusps of the general curve of this system as virtual nodes, we conclude that the complete system of curves

(n, k) contains for any $k \leqq 3/2(n-2)$, the complete system of rational maximal cuspidal curves. In other words, *any rational cuspidal curve of order n possesses the maximal cuspidal curve of order n as a limit case.*

2. *The Poincaré group of the maximal cuspidal curve of even order.* Let C_{2n-2} be a maximal cuspidal curve $(2n-2, 3(n-2))$, of even order $2n-2$, the dual of a rational plane curve Γ_n of order n, possessing nodes only. For the general curve Γ_n we have the parametric equations:

$$x_i = f_i(t), \qquad (i = 1, 2, 3), \tag{2}$$

where f_1, f_2, f_3 are arbitrary polynomials of degree n in t. The equation of C_{2n-2} is $D(\lambda_1, \lambda_2, \lambda_3) = 0$, where $D(\lambda_1, \lambda_2, \lambda_3)$ is the discriminant of the polynomial $F(t) = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$. If we interpret the coefficients a_i of the general polynomial $f(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$ of degree n as homogeneous coördinates of a point in a complex projective space S_n, we see that C_{2n-2} is the intersection of the plane $F = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ with the hypersurface $D(a_0, a_1, \cdots, a_n) = 0$, where D is the discriminant of $f(t)$. We denote this hypersurface by Δ and we shall refer to Δ as the *discriminant hypersurface.* By the theorem on the Poincaré group of an algebraic hypersurface, quoted in the introduction, we have that the Poincaré group of C_{2n-2} coincides with the Poincaré group of Δ (i. e. of the residual space $S_n - \Delta$). We shall denote this group by G_n, and by g an element of G_n.

We interpret G_n as *the group of motions of n distinct points on the sphere H of the complex variable t.* Each point f $(= f(t))$ of S_n represents an unordered set of n points $t_1, t_2, \cdots, t_n$ of H, the roots of the polynomial $f(t)$. If f is on $S_n - \Delta$, these n points are distinct, and conversely. A closed path in $S_n - \Delta$ corresponds to a motion g of the points t_i on H, in the course of which these n points remain always distinct from each other and which carries a given unordered set $(t_1^0, t_2^0, \cdots, t_n^0)$ into its initial position. A slight deformation of g over $S_n - \Delta$ corresponds to a slight deformation of the given motion g, consisting both in a deformation of the paths and, so to speak, of the instantaneous velocities of the individual points t_i, the variable set $(t_1, t_2, \cdots, t_n)$ consisting always of n distinct points in the course of the deformation. If $g = 1$, the given motion can be deformed in the above manner into rest, the initial set $(t_1^0, \cdots, t_n^0)$ remaining fixed during the deformation.

The consideration of the group G_n goes back to Hurwitz [2] who has applied it toward the classification of Riemann surfaces with assigned branch points. In this paper Hurwitz determines the generators of G_n. The group G_n,

interpreted as the group of automorphism classes of the sphere H with n holes (the holes being at the points t_i^0 of the initial set), has been studied by Magnus,[3] who derived the generating relations of G_n and who pointed out the connection between the groups G_n and the "Zopfgruppen" of Artin.[1] In view of the importance of this class of groups, we give here a new and simpler treatment of the group G_n. We point out that the $n-1$ generating relations given in the quoted paper of Magnus (Magnus,[3] relations (19)) are all consequences of the one relation (6) given below.

3. *The generators of G_n.* We fix on H a set of n distinct points $P_1, P_2, \cdots, P_n$ as an initial set and we denote by $X_1, X_2, \cdots, X_n$ the points of a variable set. We join the points $P_1, P_2, \cdots, P_n$, in the order written, by a simple oriented arc, and we denote by s_i the oriented arc joining the points P_i, P_{i+1}. Let g_i denote the motion in which the points P_j, $j \neq i, i+1$, are fixed, while the points P_i and P_{i+1} are interchanged, X_i moving from P_i to P_{i+1} along the right-hand edge of the oriented arc s_i and X_{i+1} moving from P_{i+1} toward P_i along the opposite edge of s_i (Fig. 1). We prove that *the $n-1$ elements $g_1, g_2, \cdots, g_{n-1}$ are generators of G_n.*

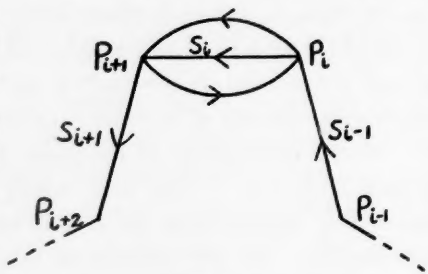

FIG. 1.

Proof. The elements g_i, considered as transpositions $(P_i\, P_{i+1})$, generate the symmetric group of permutations of the n points P_i. Hence, every element of G_n can be written as a product of g_i's multiplied by an element S, representing a motion in which each point X_i comes back to its original position P_i. Let S also denote the corresponding singular 1-sphere in the residual space $S_n - \Delta$; its initial point is $P \equiv (P_1, P_2, \cdots, P_n)$, while a variable point of S_n representing a variable set of n points $(X_1, X_2, \cdots, X_n)$ shall be denoted by X. If Q is a fixed point on the sphere H, we denote by V_Q the $(n-1)$-dimensional variety in S_n representing the sets of n points of H containing Q. We first deform S in such a manner that the closed path described by the point X_1, starting from and returning to P_1, does not pass through the points $P_2, P_3, \cdots, P_n$. We join then each point $X \equiv (X_1, X_2, \cdots, X_n)$ of S to the point $X' \equiv (X_1, P_2, \cdots, P_n)$ by a simple arc l contained in V_{X_1} but not meeting Δ. As X varies on S, we vary the arc l continuously, assuming that when X is very near its initial position P (and hence X' is very near X), the arc l is very small and reduces to a point at the initial position P of X.

The final position of l, as X describes the entire 1-sphere S, is a singular 1-sphere S' on V_{P_1}, in the residual space of Δ, while the point X' describes a singular 1-sphere S_1. The locus of the arc l is a singular 2-cell bounded by $S(S_1S')^{-1}$. Hence we can deform S into the product S_1S', where S_1 represents a motion in which all the points P_i, except P_1, are fixed, and S' is a motion in which the point P_1 is fixed. The same procedure can now be applied to S' and yields a deformation of S' into a product S_2S'', where the motion S_2 leaves all the points P_i, except P_2, fixed and where in S'' the points P_1 and P_2 are fixed. Continuing in this manner we finally express S as a product $S_1S_2\cdots S_n$, where S_a is a motion in which all the points P_i, except P_a, are fixed. Now S_a is a singular 1-sphere in $H - P_1 - P_2 - \cdots - P_{a-1} - P_{a+1} - \cdots - P_n$ and can be deformed into a product of loops issued from the point P_a and surrounding the points $P_1, \cdots, P_{a-1}, P_{a+1}, \cdots, P_n$. It is easily seen that such loops are supplied by products of the elements g_i. For instance, if $\alpha = 1$, then the products $g_1^2, g_1^{-1}g_2^2g_1, \cdots, (g_{n-2}\cdots g_1)^{-1}g^2_{n-1}(g_{n-2}\cdots g_1)$ represent the required loops,[1] q. e. d.

4. *The generating relations of G_n.* We have in the first place the following relations between the generators g_i:

$$g_ig_j = g_jg_i, \qquad |i-j| \neq 1. \tag{3}$$

[1] As originally defined, g_1^2 is a motion in which the variable point X_1, starting from P_1, turns about P_2 in the positive (counterclockwise) sense, while *at the same time* the point X_2, starting from P_2, turns about P_1 in the same sense. However, as can be seen from the accompanying figure (Fig. 2), the loop described by either one of these points, say by X_2, can be pulled over the point P_1 and then deformed into the point P_2, while the path of X_1 is unaltered.

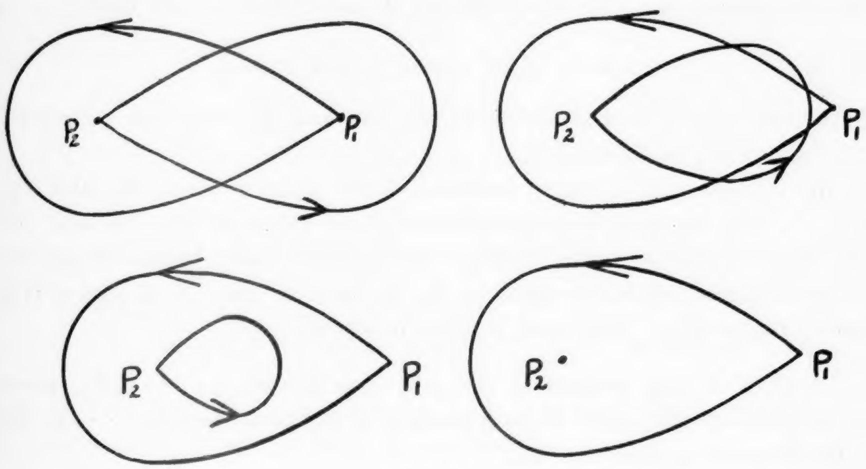

FIG. 2.

These relations are trivial, because if $|i-j| \neq 1$, the arcs s_i and s_j have no end-points in common and it is therefore indifferent which one of the two motions g_i and g_j takes place first.

We now consider an oriented arc s'_i joining the points P_i and P_{i+2}, very near but not meeting the arc $s_i + s_{i+1}$ outside the end points. We assume that s'_i is on the left-hand side of the oriented arc $s_i + s_{i+1}$ and that P_{i+2} is its initial point. Let g'_i be the motion in which the points P_i and P_{i+2} are interchanged along the arc s'_i, the point X_{i+2} moving from P_{i+2} toward P_i on the right-hand edge of s'_i. It is then easily seen that the motion in which the three points P_i, P_{i+1}, P_{i+2} are permuted cyclically, their paths being the oriented arcs s_i, s_{i+1}, s'_i respectively (Fig. 3) is expressable in terms of the elements g_i, g'_i, g_{i+1} in each of the three following manners: $g_{i+1}g_i$, g'_ig_{i+1}, $g_ig'_i$. Hence $g_{i+1}g_i = g_i'g_{i+1} = g_ig'_i$, and eliminating g'_i, we find the following relation:

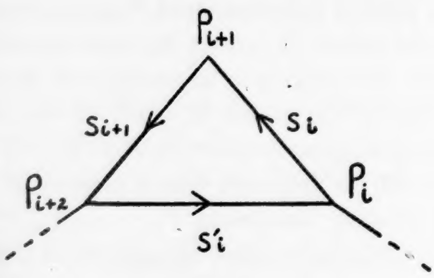

Fig. 3.

(4) $$g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, \qquad (i = 1, 2, \cdots, n-2).$$

We have pointed out in the preceding section that loops, which we shall denote by $a_2, a_3, \cdots, a_n$, which issue from the point P_1 and surround the points $P_2, P_3, \cdots, P_n$ respectively, are given by the following products:

(5) $$a_i = (g_{i-2} \cdots g_1)^{-1}g^2_{i-1}(g_{i-2} \cdots g_1).$$

Since the product $a_2a_3 \cdots a_n$ is obviously the identity in G_n, we have

(6) $$g_1g_2 \cdots g_{n-2}g^2_{n-1}g_{n-2} \cdots g_2g_1 = 1.$$

We proceed to prove that *the relations* (3), (4) *and* (6) *constitute a complete set of generating relations of* G_n.

We introduce the following notation. If two power products Πg_i and $\Pi' g_i$ of $g_1, \cdots, g_n$ are equal *as a consequence of the relations* (3), (4) *and* (6) *only*, then we shall write $\Pi g_i \equiv \Pi' g_i$. Using the ordinary symbol of equality $=$ for elements which are equal in G_n, we have to prove that $\Pi g_i = \Pi' g_i$ implies $\Pi g_i \equiv \Pi' g_i$. The proof is made in several steps.

a. If W is any product of the g_i's, then $W \equiv g_kg_{k-1} \cdots g_1W_1$, where $0 \leqq k \leqq n-1$ and where W_1 is a product of the generators $g_2, \cdots, g_n$ and of the elements a_i given by (5).

We prove this by induction with respect to the number m of factors $g_i^{\pm 1}$ in W, since the statement is trivial in the case $m=1$ $(g_1^{-1}=g_1a_2^{-1})$. Let g $(=g_i^{\pm 1})$ be the first factor of W and let W' be the product of the remaining $m-1$ factors. By our induction we can write $W \equiv gg_kg_{k-1}\cdots g_1W'_1$, where W'_1 is a product of $g_2,\cdots,g_{n-1},a_2,\cdots,a_n$. If $g=g_i^{\pm 1}$, $i>k+1$, then $gg_kg_{k-1}\cdots g_1 \equiv g_kg_{k-1}\cdots g_1g$ and hence $W \equiv g_kg_{k-1}\cdots g_1W_1$, where $W_1=gW'_1$. If $g=g_{k+1}$, then the product $gg_kg_{k-1}\cdots g_1W'_1$ is already of the required form. If $g=g_{k+1}^{-1}$, we have identically:

$$gg_kg_{k-1}\cdots g_1=g_{k+1}g_{k+1}^{-2}g_kg_{k-1}\cdots g_1=g_{k+1}g_k\cdots g_1a_{k+2}^{-1}$$

and hence $W \equiv g_{k+1}g_k\cdots g_1W_1$, $W_1=a_{k+2}^{-1}W'_1$. Finally, if $g=g_i^{\pm 1}$, $i \leqq k$, we observe that, by (4), $(g_{i+1}g_i)^{-1}g_i(g_{i+1}g_i) \equiv g_{i+1}$, and hence

$$(g_kg_{k-1}\cdots g_1)^{-1}g_i^{\pm 1}(g_kg_{k-1}\cdots g_1) \equiv (g_{i+1}g_i\cdots g_1)^{-1}g_i^{\pm 1}(g_{i+1}g_i\cdots g_1) \equiv$$

$$(g_{i-1}\cdots g_1)^{-1}g_{i+1}^{\pm 1}(g_{i-1}\cdots g_1) \equiv g_{i+1}^{\pm 1}. \tag{7}$$

Consequently $W \equiv g_kg_{k-1}\cdots g_1g_{i+1}^{\pm 1}W'_1 \equiv g_kg_{k-1}\cdots g_1W_1$, q. e. d.

b. The product $g_kg_{k-1}\cdots g_1W_1$ represents a motion which carries the point P_1 into the point P_2, provided $k \geqq 1$. Hence, as a corollary of *a*, it follows that *if W is a motion in which P_1 comes back to its original position, and if W is expressed as a product of the generators g_i, then $W \equiv W_1$, where W_1 is a product of the elements $g_2,\cdots,g_{n-1},a_2,\cdots,a_n$.*

c. We observe that *as a consequence of the relations* (3), (4) *and* (6) *the group generated by the elements $a_2,\cdots,a_n$ is an invariant subgroup of the group generated by the elements $g_2,\cdots,g_{n-1}$.* The proof is contained in the following relations:

$$g_ja_i \equiv a_ig_j, \qquad (j \neq 1, i-1, i), \tag{8}$$

trivial if $j>i$, and immediate consequences of the relations (7), if $j \leqq i-2$;

$$\begin{aligned} g_ia_ig_i^{-1} &\equiv (g_{i-2}\cdots g_1)^{-1}g_ig^2_{i-1}g_i^{-1}(g_{i-2}\cdots g_1)\\ &\equiv (g_{i-2}\cdots g_1)^{-1}g_{i-1}^{-1}g_i^2g_{i-1}(g_{i-2}\cdots g_1) \equiv a_{i+1}; \end{aligned} \tag{8'}$$

$$\begin{aligned} g_i^{-1}a_ig_1 &\equiv (g_{i-2}\cdots g_1)^{-1}g_i^{-1}g^2_{i-1}g_i(g_{i-2}\cdots g_1)\\ &\equiv (g_{i-2}\cdots g_1)^{-1}g_{i-1}g_i^2g_{i-1}^{-1}(g_{i-2}\cdots g_1) \equiv a_ia_{i+1}a_i^{-1}. \end{aligned} \tag{8''}$$

From (8′) and (8″) it follows that also $g_i^{\pm 1}a_{i+1}g_i^{\mp 1}$ can be expressed in terms of a_i and a_{i+1}, and hence the proof is complete.

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Corollary. *If a product W of the g_i's represents a motion in which the point P_1 comes back to its original position, and hence in particular if $W = 1$ in G_n, then $W \equiv W_g W_a$, where W_g is a product involving only the generators $g_2, \cdots, g_{n-1}$ and W_a is a product involving only the elements $a_2, \cdots, a_n$.*

d. We are now in position to prove the completeness of the relations (3), (4) and (6). We use an induction with respect to n, since in the case $n = 1$ the group G_n contains only the element 1. Let W be an element of G_n, expressed in terms of the generators $g_1, \cdots, g_{n-1}$ and let $W = 1$ be a true relation in G_n. By *c.*, corollary, we have $W \equiv W_g W_a$ and hence $W_g W_a = 1$. Since W_g is a motion of the points P_i in which the point P_1 is fixed, while in the motion W_a all the points P_i, except P_1, are fixed, it is clear that $W_g = 1$ must be a true relation in G_{n-1}, the generators of G_{n-1} being $g_2, \cdots, g_{n-1}$. By our induction, this relation must be a consequence of the relations (3), (4) and (6) relative to the case $n - 1$. Of these, the only relation which is not included among the relations for the group G_n, is the relation (6), which for G_{n-1} is as follows: $g_2 \cdots g^2_{n-1} \cdots g_2 = 1$. Since

$$g_2 \cdots g^2_{n-1} \cdots g_2 \equiv g_1^{-2} \equiv a_2^{-1},$$

it follows that by using the relations (3), (4), (6) of the group G_n it is possible to express W_g as a product of transforms of $a_2^{\pm 1}$ by elements of G_{n-1}. By *c.*, it follows then that $W = 1$ *implies a relation of the type: $W \equiv W'_a$, where W'_a is a product of the elements $a_2, \cdots, a_n$.*

We now make the following remark. The elements $a_2, \cdots, a_n$ are also generators of the Poincaré group Γ of $H - P_2 - \cdots - P_n$. Given any element V of G_n, and if V carries the point P_1 into itself, then this motion V of the n points $P_1, \cdots, P_n$ defines a deformation of the loops a_i into loops a'_{j_i}, issued from the point P_1, and *the correspondence $a_i \leftrightarrow a'_{j_i}$ defines an automorphism of the free group Γ.* It is clear that $a'_{j_i} = V^{-1} a_i V$, where now the elements a_i and a'_{j_i} are considered as elements of G_n. If the motion V is deformed continuously, while the loops a_i are fixed, then the loops a'_{j_i} are deformed continuously over $H - P_2 - \cdots - P_n$, the point P_1 remaining fixed. Hence *if $V_1 = V_2$ in G_n then $V_1^{-1} a_i V_1 = V_2^{-1} a_i V_2$ in Γ.*

Let now $V_1 = W'_a$. Since $V_1 = 1$ in G_n, it follows that $W'^{-1}_a a_i W'_a = a_i$ in Γ, i. e. W'_a is commutative with each element a_i. Since W'_a is itself an element of Γ and since the elements a_i are generators of Γ, W'_a belongs to the center of Γ. Since Γ is a free group, we must have necessarily $W'_a = 1$ in Γ, i. e. W'_a, given as a product of the elements $a_2, \cdots, a_n$, is a product of transforms of $(a_2 \cdots a_n)^{\pm 1}$. The product $a_2 \cdots a_n$, expressed in terms of the

elements g_i, is nothing but the left-hand member of (6). Hence $W'_a \equiv 1$, and consequently also $W \equiv 1$, q. e. d.

5. *The generating relations of G_n and the associated singularities of the curve C_{2n-2}.* The group G_n is the Poincaré group of the maximal cuspidal curve C_{2n-2} of order $2n-2$. It is interesting to see how the individual generating relations of G_n correspond to the singularities of C_{2n-2}. The curve C_{2n-2} possesses $3(n-2)$ cusps and $2(n-2)(n-3)$ nodes. The $(n-2)(n-3)/2$ commutativity relations (3) are the typical relations at nodes, while the $n-2$ relations (4) are the typical cusp relations (see Zariski [8]). The fact that there are 4 times as many nodes as there are relations (3) and three times as many cusps as there are relations (4), must be due partly to repetitions (two or more singularities giving one and the same relation) and partly to the fact that $g_1, g_2, \cdots, g_{n-1}$ is a reduced set of generators. The curve C_{2n-2} is of order $2n-2$, and originally a set of generators of its Poincaré group would consist of $2n-2$ loops $g_1, g_2, \cdots, g_{2n-2}$, lying in a fixed line and each surrounding a point of intersection of that line with C_{2n-2}. The curve C_{2n-2} is of class n and the n tangents of C_{2n-2} in a pencil of lines supply essentially $n-1$ equalities: $g_i = g_{2n-1-i}$ $(i = 1, 2, \cdots, n-1)$. The trivial relation $g_1 g_2 \cdots g_{2n-2} = 1$ yields the relation (6).

6. *Rational cuspidal curves of even order.* To consider a cusp of C_{2n-2} as a virtual node means to consider that cusp as a limiting case in which two critical points, or two singular lines, coincide, one corresponding to a node and the other being a simple tangent. Since the rational curves of a given order and with a given number of cusps form an irreducible system, it is immaterial which cusps of C_{2n-2} are considered as virtual nodes. We may assume therefore that any one of the relations (4), say the relation $g_1 g_2 g_1 = g_2 g_1 g_2$, is the relation at one of the cusps which are considered as virtual nodes. The above relation must then be replaced by two relations, one $g_1 g_2 = g_2 g_1$, relative to the node, and the second, $g_1 = g_2$, relative to the simple tangent. The new relation $g_1 = g_2$, combined with the relations $g_1 g_3 = g_3 g_1$ and $g_2 g_3 g_2 = g_3 g_2 g_3$, yields the relation $g_2 = g_3$. Combining this relation with the relation $g_2 g_4 = g_4 g_2$ and $g_3 g_4 g_3 = g_4 g_3 g_4$, we find $g_3 = g_4$. Continuing in this manner we get $g_1 = g_2 = \cdots = g_{n-1}$, while relation (6) becomes: $g_1^{2n-2} = 1$. We may therefore state the following result:

Any curve of even order, admitting the maximal cuspidal rational curve C of the same order as a limiting case, but possessing less cusps than C (in particular, any rational curve of even order which possesses only nodes and cusps and is not a maximal cuspidal curve) has a cyclic Poincaré group.

Let us now consider one of the nodes of C_{2n-2} as virtually non-existent, and let the relation at the node be, for instance, $g_1g_3 = g_3g_1$. The node must be then replaced by two simple tangents, very near each other, each yielding the relation $g_1 = g_3$. We must assume $n \geqq 4$, because if $n = 3$, then C_{2n-2} has no nodes. If $n \geqq 5$, the new relation $g_1 = g_3$, combined with the relations $g_1g_4 = g_4g_1$ and $g_3g_4g_3 = g_4g_3g_4$ yields the relation $g_3 = g_4$, and from this equality of two consecutive generators g_i, g_{i+1} we derive, as before, that the group is cyclic, of order $2n - 2$.

If $n = 4$, we are left with the relations:

$$g_1g_2g_1 = g_2g_1g_2, \qquad g_2g_3g_2 = g_3g_2g_3, \qquad g_1 = g_3, \qquad g_1g_2g_3{}^2g_2g_3 = 1,$$

or,

$$g_1g_2g_1 = g_2g_1g_2, \qquad g_1g_2g_1{}^2g_2g_1 = 1.$$

These relations define a group $\bar{G}$, generated by two elements: $u = g_1{}^2g_2$, $v = g_1g_2$, satisfying the relations $u^2 = v^3 = 1$. In the present case the curve C_{2n-2} possesses 6 cusps (lying on a conic) and 4 nodes, and the group G_4 of C_6 reduces to the group $\bar{G}$ if one of these 4 nodes is considered as virtually non-existent. However, there is no further reduction of the group of the curve, if any or all of the 4 nodes are considered as virtually non-existent. In fact, we have shown in another paper (Zariski [8]) that $\bar{G}$ is the Poincaré group of the sextic curve with 6 cusps on a conic, and this curve is obtained from the rational sextic curve C_6 by considering its 4 nodes as virtually non-existent.[2]

Hence, we have the following results:

If a curve C of even order $2n - 2$, $n \neq 4$, and of genus > 0, admits the rational maximal cuspidal curve C_{2n-2} of the same order as a limiting case, then the Poincaré group of C is cyclic of order $2n - 2$.

In the exceptional case $n = 4$, we have the non-rational sextic curves with 6 cusps on a conic, whose Poincaré group is generated by two elements u and v, satisfying the relations $u^2 = v^3 = 1$.

7. *Rational cuspidal curves of odd order.* It has been pointed out above (section 1) that a rational maximal curve C_{2n-1}, of odd order $2n - 1$, is the dual of a rational curve Γ_{n+1}, of order $n + 1$, possessing one cusp. C_{2n-1} is a

[2] If $g_1, g_2, \cdots, g_6$ is a non-reduced set of generators for the rational sextic C_6 (see section 5), such that $g_1 = g_6$, $g_2 = g_5$, $g_3 = g_4$ are relations supplied by tangent lines of C_6, the relations at the 4 nodes of C_6 are likely to be the following: $g_1g_3 = g_3g_1$, $g_1g_4 = g_4g_1$, $g_6g_3 = g_3g_6$, $g_6g_4 = g_4g_6$. In terms of the reduced set of generators g_1, g_2, g_3, we have here only one relation repeated four times, and this explains the stablity of the Poincaré group after one node has been removed.

curve $(2n-1, 3n-5)$ and possesses $2(n-2)^2$ nodes. Let Γ_{n+1} degenerate into a rational curve Γ_n, of order n, possessing only nodes, and into a tangent line t of Γ_n, the point of contact of t with Γ_n (a tacnode of the composite curve $\Gamma_n + t$) being the limit of the cusp of the irreducible curve Γ_{n+1}. The dual curve C_{2n-1} degenerates then into the maximal cuspidal curve C_{2n-2}, the dual of the curve Γ_n, and into a tangent line p of C_{2n-2}. The composite curve $C_{2n-2} + p$ possesses as many nodes as the irreducible curve C_{2n-1} (the $2(n-2)(n-3)$ nodes of C_{2n-2} and the $2n-4$ nodes at the intersections of the line p with C_{2n-2}, outside the point of contact, hence in all $2(n-2)(n-3) + 2(n-2) = 2(n-2)^2$ nodes). Corresponding to the $3n-5$ cusps of the irreducible curve C_{2n-1}, we have on the composite curve $C_{2n-2} + p$ the $3n-6$ cusps of C_{2n-2} and the tacnode at the point of contact of p with C_{2n-2}. Inasmuch as the composite curve $C_{2n-2} + p$ is a limiting case of the curve C_{2n-1}, this tacnode must be considered as a virtual cusp. This enables us to determine the Poincaré group of the irreducible curve C_{2n-1} by investigating first the Poincaré group of the composite curve $C_{2n-2} + p$.

We consider again the space $S_n(a_0, a_1, \cdots, a_n)$ of the polynomials $a_0t^n + \cdots + a_n$, and we consider in S_n the hyperplane $S_{n-1}: a_0\xi^n + \cdots + a_n = 0$, where ξ is a fixed value of t (representative space of all the n-tuples of points on the sphere H of the complex variable t which contain the fixed point ξ). It is immediately seen that S_{n-1} touches the discriminant hypersurface Δ at every common point. This is a consequence of the elementary fact, that if $f(t)$ and $\phi(t)$ are two polynomials both divisible by $t-\xi$ and if $f(t) + \lambda_0\phi(t)$ is divisible by $(t-\xi)^2$, then the discriminant of $f(t) + \lambda\phi(t)$ is divisible by $(\lambda-\lambda_0)^2$ (in geometric language: a fixed point of a linear series $g_n{}^1$ counts for two double points of the series.) It follows that a generic plane section of the composite hypersurface $\Delta + S_{n-1}$ is exactly our composite curve $C_{2n-2} + p$, and hence the Poincaré group of $C_{2n-2} + p$ coincides with the Poincaré group of the residual space of this composite hypersurface. Let G'_n be this group. The group G'_n can be interpreted as the group of motions of n points on a sphere H with one hole, the hole being at the fixed point ξ (or also, as the group of automorphism classes of a sphere with $n+1$ holes, leaving one hole fixed). From this interpretation of G'_n, it is seen that G'_n is obtained from the group G_n by the following modifications: (a) adding to the set of generators $g_1, \cdots, g_{n-1}$ of G_n another generator γ, representing a motion of one of the points P_i, say of P_1, along a loop surrounding the point ξ and not meeting the arcs $s_2, \cdots, s_{n-1}$; (b) adding to the set of relations (3), (4) the following relations:

$$\gamma g_i = g_i\gamma, \qquad (i = 2, 3, \cdots, n-1); \tag{9}$$

$$(\gamma g_1)^2 = (g_1\gamma)^2; \tag{10}$$

and finally, replacing the relation (6) by the following:

$$\gamma g_1 \cdots g_{n-2} g^2_{n-1} g_{n-2} \cdots g_1 = 1. \tag{11}$$

The relations (9) and (11) are obvious. The relation (10) is obtained by observing that $g_1\gamma g_1$ can be deformed into a motion in which the point P_2 turns around both point P_1 and ξ, while the remaining points P_i remain fixed. The path of this motion does not cross the loop γ, and hence γ and $g_1\gamma g_1$ are commutative.

Having proved the completeness of the generating relations (3), (4) and (6) for the group G_n, we could prove without difficulty the completeness of the generating relations (3), (4), (9), (10), (11) for the group G'_n. However, we do not insist on the proof, since we are not immediately concerned with the group G'_n: all we need to know is that the above relations effectively hold true in G'_n.

The relation (10) is a typical relation at a tacnode, in the present case at the point of contact of the line p with C_{2n-2}. We observe incidentally that the commutativity relations (9) arise from the simple intersections of the line p with C_{2n-2}. If we regard the tacnode of the composite curve $C_{2n-2} + p$ as a virtual cusp, i. e. as the limit of a cusp and of a simple tangent line, we must replace the relation (10) by the two relations: $\gamma g_1 \gamma = g_1 \gamma g_1$ and $g_1 = \gamma$, of which the first is a consequence of the second. The new relation $g_1 = \gamma$, combined with the relations $g_1 g_2 g_1 = g_2 g_1 g_2$ and $\gamma g_2 = g_2 \gamma$, yields the relation $g_1 = g_2$. In a similar manner we find $g_2 = g_3 = \cdots g_{n-1}$. Hence *the Poincaré group of C_{2n-1} is cyclic of order $2n - 1$. A fortiori,* the group remains cyclic for any curve of order $2n - 1$ admitting C_{2n-1} as a limiting case. Hence we have the following result:

The maximal cuspidal curve C_{2n-1} of odd order $2n - 1$, and any curve of order $2n - 1$ admitting C_{2n-1} as a limiting case (in particular any rational curve of odd order possessing only nodes and cusps), possesses a cyclic Poincaré group.

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ON THE PRINCIPAL JOIN OF TWO CURVES ON A SURFACE.

By M. L. MacQueen.

1. *Introduction.* Bompiani has made [1] some important contributions to the projective differential geometry of curves in ordinary space by introducing certain lines and points, called principal lines and principal points, which are associated with the intersection of two skew curves. In connection with the investigation of the invariants of intersection of two curves, several different cases present themselves. We shall confine our attention to the case in which the two curves C, $\bar{C}$ pass through a point P with distinct tangents t, $\bar{t}$ at P and also with distinct osculating planes at P, whose line of intersection is different from t and $\bar{t}$. Bompiani shows the existence through P, in the plane determined by t, t, of two lines, called *principal lines*, characterized by the property that the two cones projecting C, $\bar{C}$ from any point on either line have contact of the second or higher order along their common generator through P, instead of contact of the first order as would ordinarily be the case if the center of projection were chosen elsewhere in the plane of t, $\bar{t}$. On each principal line there is a point, called a *principal point*, which has the property that if the projecting cones have their vertices at this point, the cones have contact of the third order.

On considering the case in which the two curves belong respectively to the two families of a conjugate net on a surface, Lane has deduced [2] some interesting results. Among other things, he shows that the principal lines of the parametric curves at a point of a surface are precisely the *associate conjugate tangents* at the point. Moreover, he determines the principal points of the two parametric curves at a point of a surface, and calls the line joining these points the *principal join* of the fundamental parametric curves.

In this note we propose to supplement the investigations of Lane, by presenting other geometric characterizations of the principal join of the fundamental parametric curves at a point of a surface referred to a conjugate net. In connection with our geometric constructions we introduce the neighborhoods of the third and fourth order of the plane curves of section of the surface made

[1] E. Bompiani, "Invarianti d'intersezione di due curve sghembe," *Rendiconti dei Lincei*, ser. 6, vol. 14 (1931), pp. 456-461.

[2] E. P. Lane, "Invariants of intersection of two curves on a surface," *American Journal of Mathematics*, vol. 54 (1932), pp. 699-706.

by variable planes through the tangents of the parametric curves at a point of the surface.

2. *Power series expansions and the principal join.* Let the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x on a surface S referred to a conjugate net N_x in ordinary space be given as analytic functions of two independent variables u, v. The osculating planes of the parametric curves C_u, C_v at the point P_x intersect in the axis of P_x with respect to the net N_x. Let P_y be the point which is the harmonic conjugate of P_x with respect to the two foci of the axis regarded as generating a congruence when the point P_x varies over the surface. Then x and y are solutions of a completely integrable system of differential equations[3] of the form

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \end{aligned} \qquad (LN \neq 0). \tag{1}$$

From the equations

$$(x_{vv})_u = (x_{uv})_v, \qquad (x_{uu})_v = (x_{uv})_u$$

we obtain

$$y_u = fx - nx_u + sx_v + Ay, \qquad y_v = gx + tx_u + nx_v + By, \tag{2}$$

where we have placed

$$\begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ -nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p, \\ A &= b - (\log N)_u, & B &= a - (\log L)_v. \end{aligned} \tag{3}$$

The ray-points, or Laplace transformed points, x_1, x_{-1} of the curves C_u, C_v respectively at the point P_x are defined by the formulas

$$x_{-1} = x_u - bx, \qquad x_1 = x_v - ax. \tag{4}$$

The following formulas give some of the invariants of the parametric conjugate net N_x:

$$\begin{aligned} H &= c + ab - a_u, & K &= c + ab - b_v, \\ \mathfrak{H} &= sN, & \mathfrak{K} &= tL, \\ 8\mathfrak{B}' &= 4a - 2\delta + (\log r)_v, & 8\mathfrak{C}' &= 4b - 2\alpha - (\log r)_u, \\ \mathfrak{D} &= -2nL, & r &= N/L. \end{aligned} \tag{5}$$

[3] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, University of Chicago Press, 1932, p. 138.

Incidentally, it is not difficult to show that the Laplace-Darboux invariants H, K, the tangential invariants $\mathcal{H}$, $\mathcal{K}$, and the invariants $\mathfrak{B}'$, $\mathfrak{C}'$ of Green are connected by the relations

$$(6) \qquad \mathcal{H} = H + 3\mathfrak{B}'_u + \mathfrak{C}'_v, \qquad \mathcal{K} = K + \mathfrak{B}'_u + 3\mathfrak{C}'_v.$$

We shall employ the covariant tetrahedron whose vertices are the points x, x_{-1}, x_1, y as a local tetrahedron of reference with a unit point chosen so that a point

$$X = y_1 x + y_2 x_{-1} + y_3 x_1 + y_4 y$$

has local coördinates $y_1, \cdots, y_4$. In this coördinate system the equations of the osculating planes of the curves C_u, C_v at the point P_x are respectively $y_3 = 0$ and $y_2 = 0$. If we introduce non-homogeneous projective coördinates by the definitions

$$(7) \qquad x = y_2/y_1, \qquad y = y_3/y_1, \qquad z = y_4/y_1,$$

then a power series expansion [4] for one non-homogeneous coördinate z of a point on the surface in terms of the other two coördinates x, y is given, to terms of the fourth order, by

$$(8) \quad z = (1/2)(Lx^2 + Ny^2) + (4/3)(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3) + c_0x^4 + 4c_1x^3y + 4c_3xy^3 + c_4y^4 + \cdots,$$

where

$$c_0 = (1/3)L\mathfrak{C}'[12\mathfrak{C}' + (\log \mathfrak{C}'r^{\frac{1}{2}})_u], \qquad 4c_1 = (1/6)L(H - \mathcal{H}),$$
$$c_4 = (1/3)N\mathfrak{B}'[12\mathfrak{B}' + (\log \mathfrak{B}'r^{-\frac{1}{2}})_v], \qquad 4c_3 = (1/6)N(K - \mathcal{K}).$$

The equation

$$(10) \qquad y = \rho z \qquad (\rho \neq 0)$$

represents a plane passing through the u-tangent, $y = z = 0$, at the point P_x of the surface. This plane cuts the surface in a curve whose projection from the point $(0, 0, 0, 1)$ onto the tangent plane, $z = 0$, is represented by the equation obtained by eliminating z between equations (8) and (10). If this equation is solved for y as a power series in x, the result to terms of the fourth degree is

$$(11) \qquad y = \rho Lx^2/2 + 4\rho L\mathfrak{C}'x^3/3 + \rho(L^2N\rho^2/8 + c_0)x^4 + \cdots.$$

Imposing on the general equation of a conic the conditions that it be satisfied by the series (11) for y identically in x as far as the term in x^3, we obtain

[4] E. P. Lane, "A canonical power series expansion for a surface," *Transactions of the American Mathematical Society*, vol. 37 (1935), p. 481.

the equation of the conics having contact of the third order with a plane section made by a variable plane through the tangent $y = z = 0$, namely,

$$y - \rho L x^2/2 - 8\mathfrak{C}' xy/3 + hy^2 = 0, \tag{12}$$

where h is a parameter. The pole of the v-tangent, $x = z = 0$, with respect to any one of the conics (12) has the coördinates

$$(3/8\mathfrak{C}', 0, 0). \tag{13}$$

Similarly, if we consider a plane

$$x = \sigma z \qquad (\sigma \neq 0) \tag{14}$$

through the v-tangent, $x = z = 0$, we find the equation of the conics having third order contact with the curve of section of the surface to be

$$x - \sigma N y^2/2 - 8\mathfrak{B}' xy/3 + kx^2 = 0, \tag{15}$$

where k is a parameter. The pole of the u-tangent, $y = z = 0$, with respect to any one of the conics (15) is found to be the point

$$(0, 3/8\mathfrak{B}', 0). \tag{16}$$

The join of the points (13) and (16) is a line whose equation is

$$8(\mathfrak{C}' x + \mathfrak{B}' y) = 3, \tag{17}$$

which is precisely the principal join of the fundamental parametric curves at the point P_x. Thus the following theorem is proved.

At each point of a surface referred to a conjugate net, the principal join of the parametric curves at the point crosses each of the parametric tangents in the pole of the other with respect to any conic having contact of the third order with the curves of intersection of the surface and the planes of a pencil with the first parametric tangent as axis.

Another geometric characterization of the principal join can be described briefly in the following way. Let us project from the ray-point $(0, 0, 1, 0)$ onto the osculating plane, $y = 0$, the curves of section of the surface (8) made by the plane (10). Eliminating y between equations (8), (10) and solving the result for z as a power series in x, the equation, in the osculating plane, $y = 0$, of the projection of the curve of section is found to be

$$z = Lx^2/2 + 4L\mathfrak{C}' x^3/3 + \cdots. \tag{18}$$

Similarly, projecting onto the osculating plane, $x = 0$, the curves of section made by the plane (14), we obtain

$$z = Ny^2/2 + 4N\mathfrak{B}'y^3/3 + \cdots. \tag{19}$$

The conics having contact of the third order with the projections (18), (19) are respectively

$$\begin{aligned} z - Lx^2/2 - 8\mathfrak{C}'xz/3 + hz^2 &= 0, \\ z - Ny^2/2 - 8\mathfrak{B}'yz/3 + kz^2 &= 0. \end{aligned} \tag{20}$$

where h, k are parameters. On finding the pole of the axis $x = y = 0$, with respect to each of the conics (20), we again obtain the points (13), (16) which determine the principal join. Thus we have proved the following theorem:

At a point P_x of a surface referred to a conjugate net let the plane curves of section of the surface made by planes passing through the tangent of the u-curve (v-curve) be projected from the ray-point of this curve onto the osculating plane of this curve. The conics in the osculating plane of the u-curve (v-curve) which have four-point contact at P_x with the projected sections determine a point on the u-tangent (v-tangent) which is the pole of the axis with respect to any one of the four-point conics of the pencil. The line which crosses the parametric tangents at the point P_x in the points thus defined is the principal join of the parametric curves at the point P_x.

3. *The quadrics of Moutard for the parametric tangents.* The osculating conic of the curve of section of the surface made by the plane (10) is contained in the pencil (12), and for this conic the parameter is found to have the value

$$h = (256L\mathfrak{C}'^2 - 72c_0 - 9\rho^2L^2N)/9\rho L^2. \tag{21}$$

With this value of h in equation (12), the equation of the quadric of Moutard for the tangent $y = z = 0$ is found, by eliminating ρ between equations (10) and (12), to be

$$z = (1/2)(Lx^2 + Ny^2) + (8/3)\mathfrak{C}'xz - \left(\frac{128\mathfrak{C}'^2}{9L} - \frac{4c_0}{L^2}\right)z^2. \tag{22}$$

Similarly, we find the equation of the quadric of Moutard for the tangent $x = z = 0$ to be

$$z = (1/2)(Lx^2 + Ny^2) + (8/3)\mathfrak{B}'yz - \left(\frac{128\mathfrak{B}'^2}{9N} - \frac{4c_4}{N^2}\right)z^2. \tag{23}$$

The pole of the osculating plane, $x = 0$, of the v-curve with respect to the quadric (22) is found to be the point (13). Moreover, the point (16) is the

pole of the osculating plane, $y=0$, of the u-curve with respect to the quadric (23). Thus one arrives at the following conclusion:

The principal join of the parametric curves at a point of a surface referred to a conjugate net intersects the tangent of each curve in the pole of the osculating plane of the other curve with respect to the quadric of Moutard for the tangent of the first curve.

We interpolate here a few remarks concerning the intersection of the quadrics of Moutard for the parametric tangents. It is well known, as may be verified by inspecting equations (22), (23), that these quadrics are intersected by the tangent plane of the surface in the asymptotic tangents. Furthermore, the cone projecting the curve of intersection of the two quadrics of Moutard from the point P_x consists of two planes, one of which is the tangent plane $z=0$. The other plane, which contains the conic of intersection of the two quadrics, has the equation

$$\mathfrak{C}'x-\mathfrak{B}'y-\left[\frac{16}{3}\left(\frac{\mathfrak{C}'^2}{L}-\frac{\mathfrak{B}'^2}{N}\right)-\frac{3}{2}\left(\frac{c_0}{L^2}-\frac{c_4}{N^2}\right)\right]z=0. \tag{24}$$

The plane (24) intersects the tangent plane of the surface in the line

$$\mathfrak{C}'x-\mathfrak{B}'y=0. \tag{25}$$

Moreover, the line

$$\mathfrak{C}'x+\mathfrak{B}'y=0, \tag{26}$$

which joins the point P_x to the point of intersection of the ray and the associate ray, has been called [5] by Davis the *second canonical tangent* of the conjugate net and its associate conjugate net. We thus obtain the following result.

The plane containing the conic of intersection of the quadrics of Moutard for the two parametric tangents at a point of a surface intersects the tangent plane of the surface in the line which is the harmonic conjugate of Davis's second canonical tangent with respect to the tangents of the net.

4. *Developables of the principal join congruence.* It is of some interest to consider the developables of the congruence of principal joins. By the usual method we find *the differential equation of the curves on the surface corresponding to the developables of the congruence of principal joins to be*

$$Pdu^2-(Rr-S)\,du\,dv-Qr\,dv^2=0, \tag{27}$$

[5] W. M. Davis, *Contributions to the theory of conjugate nets*, Chicago doctoral dissertation (1932), p. 19.

where the functions P, Q, R, S are defined by

$$
(28) \qquad \begin{aligned}
P &= H + 8\mathfrak{B}'_u/3 - 64\mathfrak{B}'\mathfrak{C}'/9,\\
Q &= K + 8\mathfrak{C}'_v/3 - 64\mathfrak{B}'\mathfrak{C}'/9,\\
R &= 8\mathfrak{C}'[4\mathfrak{C}'/3 + (\log \mathfrak{C}'r^{\frac{1}{2}})_u]/3 + \mathfrak{D}/2,\\
S &= 8\mathfrak{B}'[4\mathfrak{B}'/3 + (\log \mathfrak{B}'r^{-\frac{1}{2}})_v]/3 - \mathfrak{D}r/2.
\end{aligned}
$$

We propose to call the curves defined by (27) *the principal join curves* of the net N_x. Calculation of the harmonic invariant of (27) and the asymptotic curves

$$
(29) \qquad L\,du^2 + N\,dv^2 = 0
$$

on the surface shows, with the aid of (6), that *the principal join curves form a conjugate net if, and only if,*

$$
(30) \qquad (H - K) = 4(\mathfrak{H} - \mathfrak{K}).
$$

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DISCONTINUOUS GROUPS ASSOCIATED WITH THE CREMONA GROUPS.

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Introduction. In the theory of *complete* and *regular* linear systems of plane curves there are two closely related fundamental questions that have not been answered. First,[6] there is no simple criterion for determining when a solution $x = \{x_0; x_1, x_2, \cdots, x_\rho\}$ of the Cremona equations,

$$(1) \qquad \begin{aligned} x_1 + x_2 + \cdots + x_\rho - 3x_0 &= -d - (p-1) \\ x_1^2 + x_2^2 + \cdots + x_\rho^2 - x_0^2 &= -d + (p-1), \end{aligned}$$

actually determines a linear system $\Sigma_{p,d}$. The essential difficulty would be dissipated if the structure of the arithmetic group $g_{\rho,2}$ ([1], p. 318) were known for all ρ. $g_{\rho,2}$ is generated by

$$(2) \qquad A_{123}: \quad \begin{aligned} x'_0 &= x_0 + L, \\ x'_1 &= x_1 + L, \\ x'_2 &= x_2 + L, \\ x'_3 &= x_3 + L, \\ x'_{j+3} &= x_{j+3}, \end{aligned} \qquad \begin{aligned} &L = (x_0 - x_1 - x_2 - x_3) \\ \\ &(j = 1, 2, \cdots, \rho - 3) \end{aligned}$$

and the interchanges of $x_1, x_2, \cdots, x_\rho$. $g_{\rho,2}$ leaves invariant the linear and quadratic forms

$$(3) \qquad \begin{aligned} L &\equiv x_1 + x_2 + \cdots + x_\rho - 3x_0 \\ Q &\equiv x_1^2 + x_2^2 + \cdots + x_\rho^2 - x_0^2 \end{aligned}$$

but this property is not sufficient to characterize $g_{\rho,2}$.[11]

The problem considered in this paper arises from two observations made by Coble in connection with these well-known questions. For $\rho = 9$ he found that the nature of a C-characteristic was readily determined by considering the numbers $x_0, x_1, x_2, \cdots, x_\rho$ reduced, mod 3 ([6], p. 475). We will investigate the possibility of a similar criterion for larger values of ρ. This study is made more interesting by the fact that the elements of $g_{\rho,2}$ which reduce to the identity mod 3 must constitute an invariant subgroup.[1] We will denote this invariant subgroup by $I^{(3)}_{\rho,2}$. The factor group of $g_{\rho,2}$ with respect to $I^{(3)}_{\rho,2}$ is simply isomorphic to the finite group $g^{(3)}_{\rho,2}$ obtained by reducing the coefficients in all the elements of $g_{\rho,2}$ with respect to the modulus 3.

We determine the order and nature of $g^{(3)}_{\rho,2}$ and exhibit a complete set of invariants which characterizes the group. In doing so we find that the criterion used by Coble is useful only in that single case.

I. *Definitions, conventions, and preliminary formulae.* In all the work to follow we consider the numbers $x_0, x_1, \cdots, x_\rho$ reduced modulo 3. It will be sufficient then to consider *characteristics* $x = \{x_0; x_1, x_2, \cdots, x_\rho\}$ which consist of numbers of the set 0, 1, 2. In particular,

DEFINITION. *All characteristics $\{x_0, x_1, \cdots, x_\rho\}$ which have $x_0 = i_0$ and contain l twos, m ones, and n zeros, are said to be of the type*

$$i_0; lmn, \qquad l + m + n = \rho.$$

A characteristic $\{x_0; x_1, x_2, \cdots, x_\rho\}$, which satisfies

$$\text{(4)} \qquad \begin{aligned} x_1 + x_2 + \cdots + x_\rho - 3x_0 &\equiv a_1 \\ x_1^2 + x_2^2 + \cdots + x_\rho^2 - x_0^2 &\equiv a_2, \end{aligned} \qquad \text{mod } 3,$$

is said to be of sort (a_1, a_2). By direct substitution in the equations (4) we obtain the theorem:

(5) *All the characteristics of sort (a_1, a_2) are included in the types*

$$0;\ (3b+\beta)(3c+\gamma)q, \quad i_0;\ (3b'+\beta')(3c'+\gamma')q', \qquad (i_0 = 1, 2)$$

where $\beta, \gamma, \beta', \gamma' = 0, 1, 2$ and satisfy the congruences

$$\begin{aligned} \beta &\equiv \quad a_1 - a_2 \equiv \beta' + 1 \\ \gamma &\equiv -(a_1 + a_2) \equiv \gamma' + 1 \end{aligned} \qquad \text{mod } 3,$$

and b, c, q, b', c', q' are any non-negative integers such that

$$3b + \beta + 3c + \gamma + q = 3b' + \beta' + 3c' + \gamma' + q' = \rho.$$

Moreover, all characteristics of these types are of sort (a_1, a_2).

Later it will be necessary to know the number of characteristics of a given sort. It is clear that there are just* $\binom{\rho}{lmn}$ distinct characteristics of type

* $\binom{\rho}{lmn}$ is the usual symbol for the coefficient of $y_1^l y_2^m y_3^n$ in the expansion of $(y_1 + y_2 + y_3)^\rho$. For l, m, n non-negative and $l + m + n = \rho$,

$$\binom{\rho}{lmn} = \frac{\rho!}{l!\, m!\, n!}$$

and is the number of distinct arrangements of ρ things of which l, m, and n are alike.

i_0; lmn. Then by (5) it follows that the number of characteristics of sort (a_1, a_2) is given by

$$(6) \qquad \sum_{b,c,q} \binom{\rho}{3b+\beta\ 3c+\gamma\ q} + 2\sum \binom{\rho}{3b'+\beta'\ 3c'+\gamma'\ q'},$$

where β, γ, β', γ' satisfy the restrictions of (5) and the sums are taken over all non-negative values such that the trinomial coefficients exist. To be able to evaluate sums such as appear in (6) we make the convention:

DEFINITION. *The nine numbers $\rho_{\beta\gamma}$ are defined for $\beta, \gamma = 0, 1, 2$ by the sum*

$$\sum \binom{\rho}{3b+\beta\ 3c+\gamma\ q},$$

where the sum is taken over all non-negative values of b, c, q such that $3b+\beta+3c+\gamma+q=\rho$.

Many interesting properties of these numbers may be derived. We include here those we must use. Since the trinomial coefficients satisfy the recursion relation

$$\binom{\rho}{l\ m\ n} = \binom{\rho-1}{l-1\ m\ n} + \binom{\rho-1}{l\ m-1\ n} + \binom{\rho-1}{l\ m\ n-1},$$

we are able to conclude immediately that the numbers $\rho_{\beta\gamma}$ satisfy the relation

$$(7) \qquad \rho_{\beta\gamma} = (\rho-1)_{\beta\gamma} + (\rho-1)_{\beta-1,\gamma} + (\rho-1)_{\beta,\gamma-1}.$$

By experiment, other algebraic forms of these numbers were found. That the results, which are tabulated below, are correct may be verified by seeing that they verify the initial conditions and the recursion relations (7).

(8) *The numbers $\rho_{\beta\gamma}$ are given by the table below:*

$$\rho \equiv 0 \quad \begin{array}{ll} \rho_{12} = \rho_{21} = 3^{\rho-2}, & \rho_{00} = 3^{\rho-2} - (-3)^{[\rho-1]} + (-1)^{\rho+1}(-3)^{[\rho-1]} \\ \rho_{01} = \rho_{10} = 3^{\rho-2}, & \rho_{11} = 3^{\rho-2} \qquad\qquad\qquad\ \ + (-1)^{\rho}\ (-3)^{[\rho-1]} \\ \rho_{02} = \rho_{20} = 3^{\rho-2}, & \rho_{22} = 3^{\rho-2} + (-3)^{[\rho-1]} \end{array}$$

$$\rho \equiv 1 \quad \begin{array}{l} \rho_{10} = \rho_{01} = \rho_{00} = 3^{\rho-2} - (-3)^{[\rho-2]} + (-1)^{\rho}\ (-3)^{[\rho-2]} \\ \rho_{12} = \rho_{21} = \rho_{11} = 3^{\rho-2} \qquad\qquad\qquad\ + (-1)^{\rho+1}(-3)^{[\rho-2]} \\ \rho_{02} = \rho_{20} = \rho_{22} = 3^{\rho-2} + (-3)^{[\rho-2]} \end{array}$$

$$\rho \equiv 2 \quad \begin{array}{l} \rho_{20} = \rho_{02} = \rho_{00} = 3^{\rho-2} + (-3)^{[\rho-2]} + (-1)^{\rho+1}(-3)^{[\rho-2]} \\ \rho_{10} = \rho_{01} = \rho_{11} = 3^{\rho-2} \qquad\qquad\qquad\ + (-1)^{\rho}\ (-3)^{[\rho-2]} \\ \rho_{21} = \rho_{12} = \rho_{22} = 3^{\rho-2} - (-3)^{[\rho-2]} \end{array}$$

where $[k]$ is the largest integer in $k/2$.

Hence we consider the numbers $\rho_{\beta\gamma}$ as numbers which are readily computed and state the useful result:

(9) *The number of characteristics of sort* (a_1, a_2) *is given by* $\rho_{\beta\gamma} + 2\rho_{\beta-1,\gamma-1}$ *where* β, γ *are determined by* (5).

Having determined the types of all the characteristics of a given sort, we now seek to discover how these divide into conjugate sets under $g^{(3)}_{\rho,2}$. Since $g^{(3)}_{\rho,2}$ by definition includes the permutation group $\Pi(x_1, x_2, \cdots, x_\rho)$ and since any two characteristics of the same type are obviously conjugate under this subgroup, we seek the conditions under which two characteristics of different types are conjugate under A_{123}. In the proper sense of the word, A_{123} transforms characteristics but not types. Hence we make the convention: *we will say that the type* i_0; lmn *is conjugate to the type* i'_0; $l'm'n'$ *if any characteristic of the first type is conjugate to any characteristic of the second type under* A_{123}.* In light of this convention we readily verify the statement below.

(10) *Suppose we choose* λ *twos,* μ *ones, and* ν *zeros from the* l *twos,* m *ones, and* n *zeros of* i_0; lmn

$$\lambda + \mu + \nu = 3, \quad 0 \lesseqgtr \lambda \lesseqgtr l, \quad 0 \lesseqgtr \mu \lesseqgtr m, \quad 0 \lesseqgtr \nu \lesseqgtr m,$$

and consider a characteristic of $x_0 = i_0$ *and with the values chosen at* x_1, x_2, x_3 *in any order. Then the image of this characteristic is a characteristic of type* i'_0; $l'm'n'$ *where*

$$\begin{aligned} i'_0 &= i_0 + (i_0 - 2\lambda - \mu) \\ l' &= l + \lambda' - \lambda \\ m' &= m + \mu' - \mu \\ n' &= n + \nu' - \nu \end{aligned}$$

and λ', μ', ν' *is determined by:*

$$\lambda', \mu', \nu' \quad \textit{is equal to} \quad \lambda, \mu, \nu; \quad \mu, \nu, \lambda; \quad \textit{or} \quad \nu, \lambda, \mu$$

according as $(i_0 - 2\lambda - \mu)$ *is* 0, 1 *or* 2.

The effect of the transformation given is to subtract out the changed numbers and add in the new ones obtained. Since this is accomplished by changing i_0 and adding numbers to l, m, n we put (10) in the following form.

* This means, of course, that a given type may have several conjugates under A_{123}.

(11) *The type conjugate to* i_0; lmn *with* i_0; $\lambda\mu\nu$ *isolated is*

$$\begin{aligned} i'_0 &= i_0 + (i_0 - 2\lambda - \mu) \\ l' &= l + L \\ m' &= m + M \\ n' &= n + N \end{aligned}$$

where $L = \lambda' - \lambda$, $M = \mu' - \mu$, $N = \nu' - \nu$ *with* λ', μ', ν' *determined as in* (10).

In doing the necessary computation it was found convenient to tabulate the values of i_0; LMN which arise from a chosen i_0; $\lambda\mu\nu$ as follows:

(12)

i_0	i'_0	$\lambda\mu\nu \to$	L	M	N	$\lambda\mu\nu \to$	L	M	N	$\lambda\mu\nu \to$	L	M	N
0	1	210	−1	−1	2	021	2	−1	−1	102	−1	2	−1
0	2	120	−1	−1	2	012	2	−1	−1	201	−1	2	−1
1	2	111	0	0	0								
1	2	300	−3	0	3	003	0	3	−3	030	3	−3	0
1	0	102	1	1	−2	210	−2	1	1	021	1	−2	1
2	1	111	0	0	0								
2	1	003	3	0	−3	300	−3	3	0	030	0	3	−3
2	0	012	1	1	−2	201	−2	1	1	120	1	−2	1

II. *The group* $g^{(3)}_{\rho,2}$ *for* $\rho \leq 9$. For $\rho \lesssim 8$ the problem is greatly simplified by the fact that in these cases the order of $g_{\rho,2}$ is finite. Its order and essential properties are known.[7] Indeed, we can see that here $g^{(3)}_{\rho,2}$ and $g_{\rho,2}$ are simply isomorphic as a consequence of a theorem due to Minkowski.[8, 9]

(13) *An integer linear homogeneous substitution which is of finite period and which is different from the identity cannot reduce to the identity with respect to a modulus* $l \gtrsim 3$.

Thus $g^{(3)}_{\rho,2}$ and $g_{\rho,2}$ are of the same order and are obviously simply isomorphic.

For $\rho = 9$ a similar procedure will suffice, even though $g_{9,2}$ is of infinite order. Dr. Taylor[3] obtained a complete determination of all the elements of $g_{9,2}$ in 1932. These results were later put in a more usable form by Dr. Barber.[2] By applying these results it is not difficult to see what happens when the coefficients in the elements of $g_{9,2}$ are reduced modulo 3. We will merely state the results obtained by direct application of the properties of $g_{9,2}$.

$g^{(3)}_{9,2}$ is of order $3^8 \cdot 2 \cdot 8640 \cdot 8!$. It possesses an invariant abelian subgroup of order 3^8 and type $(1, 1, 1, \text{etc.})$. The factor group of $g^{(3)}_{9,2}$ with respect to this invariant subgroup is simply isomorphic to $g^{(3)}_{8,2} = g_{8,2}$. $I^{(3)}_{9,2}$, the invariant subgroup of $g_{9,2}$ characterized by the fact that all its elements reduce

to the identity, mod 3, is comprised of all those elements of Dr. Taylor's[3] Type I which are defined by a $\gamma, \nu, \delta_1, \delta_2, \cdots, \delta_9$ with the property

$$\gamma \equiv \nu \equiv \delta_1 \equiv \delta_2 \equiv \cdots \equiv \delta_9, \quad \text{mod } 3.$$

This could also be stated by saying that $I^{(3)}_{9,2}$ consists of those elements[3] of a_9 which are themselves cubes of some element in a_9.

III. *The group* $g^{(3)}_{\rho,2}$ *for* $\rho > 9$. The essential difficulty in this paper comes in determining the nature of $g^{(3)}_{\rho,2}$ in those cases where the information concerning $g_{\rho,2}$ is fragmentary; i. e. for values of ρ greater than 9. We obtain a grip on the problem by asking how the aggregates of characteristics of the same sort divide into conjugate sets under $g^{(3)}_{\rho,2}$.

In the case $\rho = 10$ there are three self-conjugate characteristics

$$\{0;\ 2, 2, \cdots, 2\}, \quad \{0;\ 1, 1, \cdots, 1\} \quad \text{and} \quad \{0;\ 0, 0, \cdots, 0\}.$$

Excluding these, there remain 195 types of characteristics. Starting with one of these, we can find all the types conjugate to it by use of table (12). Taking the new ones we can find all their conjugates and continue the process until we have a complete conjugate set. Having done this computation, we find that for $\rho = 10$ we have exactly 9 complete conjugate sets. That is, except for the three self-conjugate characteristics, all characteristics of the same sort are actually conjugate under $g^{(3)}_{\rho,2}$. We anticipate then that this is true for all $\rho > 9$. Indeed, it may be shown that when this situation exists for $\rho - 1$, it must also exist for ρ. Suppose all the characteristics of a given sort for $\rho - 1$ are actually conjugate under $g^{(3)}_{\rho-1,2}$. The characteristics obtained by adding a zero to each of these are characteristics of that same sort for ρ and are clearly conjugate under $g^{(3)}_{\rho,2}$. It remains only to show that those characteristics of that sort which contain only twos and ones are conjugate under $g^{(3)}_{\rho,2}$ to one which contains a zero. This is readily accomplished by using table (12).

(13) *The three characteristics of types* $0;\ \rho\, 0\, 0,\ 0;\ 0\, \rho\, 0$, *and* $0, 0\, 0\, \rho$ *are always self-conjugate. For* $\rho > 9$, *the remaining characteristics divide into just nine conjugate sets determined by the nine pairs of congruences* (4).

The significance of this is that any * two characteristics of the same sort are conjugate under $g^{(3)}_{\rho,2}$. We remark that since (5) enables us to write down all the characteristics of a given sort, we have a very complete determination

* Of course, the self-conjugate characteristics are excluded and $\rho > 9$.

of the conjugate sets for $\rho > 9$. The number of characteristics in a given conjugate set is then immediately determined by (9).

We complete the investigation of $g^{(3)}_{\rho,2}$ by comparing it with the group G_ρ. G_ρ is defined to be the group of all linear substitutions with coefficients in the $GF[3]$ which leave Q, L absolutely invariant. The immediate difficulty in comparing these two groups is that $g^{(3)}_{\rho,2}$ is defined by its generators and G_ρ is defined by the invariant forms. The difficulty is resolved by showing that:

(a) for $\rho > 9$, $g^{(3)}_{\rho,2}$ is characterized as a subgroup of G_ρ by the fact that it permutes characteristics of sort $(0,1)$ evenly; and

(b) a simple set of generators exists for G_ρ.

We make use of the notion of involutions in D-conditions introduced by Coble ([4], p. 17) and studied later by Barber.[2] Using the notation of the latter article, if $d = \{d_0; d_1, d_2, \cdots, d_\rho\}$ is a characteristic of sort $(0,2)$, then the linear substitution

$$I_\rho(d): \quad x' = (d_i x) d_i + x, \tag{14}$$

is an involutorial element of G_ρ. By direct substitution we may verify that $I_\rho(d)$ has this useful property.*

(15) *If y is a characteristic of sort $(2,1)$ such that $y_\rho = 0$, then*

$$d = \{y_0; y_1, y_2, \cdots, y_{\rho-1}, 1\}$$

is of sort $(0,2)$ and the image of y under $I_\rho(d)$ is $\{0; 0, 0, \cdots, 0, 2\}$.

Since all characteristics of sort $(0,2)$ are conjugate under $g^{(3)}_{\rho,2}$, and since A_{123} is an involution in a D-condition, all involutions $I_\rho(d)$ are conjugate and are in $g^{(3)}_{\rho,2}$ for $\rho > 9$. By using (9) it is readily seen that $I_\rho(d)$ for $d = \{1; 0, 0, \cdots, 0\}$ always yields an even permutation of characteristics of sort $\{0,1\}$. Hence,

(16) *An involution in a D-condition always provides an even permutation of characteristics of sort $(0,1)$.*

Since $g^{(3)}_{\rho,2}$ is generated by involutions $I_\rho(d)$, this means that every element of the group has that property. However, there are elements of G_ρ which give an odd permutation of the characteristics in question. The involution C_{12} in the non-rational D-condition $\{\sqrt{2}; 2\sqrt{2}, \sqrt{2}, 0, \cdots, 0\}$ has the equations

---

* This statement is also valid in the usual theory. Any P-characteristic [6] defines an involution in a D-condition which sends it into the fundamental P-characteristic $\{0; 0, 0, \cdots, -1\}$.

$$
(17) \qquad C_{12}: \begin{array}{l} x'_0 = \phantom{2x_0 + {}} x_1 + 2x_2 \\ x'_1 = 2x_0 + 2x_1 + 2x_2 \\ x'_2 = x_0 + 2x_1 + 2x_2 \\ x'_{j+2} = \phantom{2x_0 + 2x_1 + {}} x_{j+2}, \qquad (j = 1, 2, \cdots, \rho - 2). \end{array}
$$

Since we can count the characteristics invariant under this by use of (9), and since C_{12} interchanges the remaining ones in pairs, we show that C_{12} gives an odd permutation of characteristics of sort $(0, 1)$.

(18) *The involution C_{12} always provides an odd permutation of characteristics of sort* $(0, 1)$.

By simple direct computation it is verified that G_2 is of order 8 and is generated by involutions in D-conditions and C_{12}. As a consequence of this it may be seen that $\bar{G}_\rho$, the subgroup of G_ρ generated by involutions $I_\rho(d)$, is an invariant subgroup of index 2 under G_ρ. For by (15) an element of G_ρ which has a zero in one of its P-characteristics can be reduced to an element of $G_{\rho-1}$. The number of elements that do not have zeros is restricted and these cases may be considered in detail. This leads to the result:

(19) *G_ρ is generated by the involutions in D-conditions and C_{12}. The subgroup $\bar{G}_\rho$ generated by involutions in D-conditions is an invariant subgroup of index 2 and is completely characterized by the fact that it permutes characteristics of sort* $(0, 1)$ *evenly.*

The order of G_ρ is obtained by using the fact that if h is the total subgroup of H leaving an object θ invariant, then the index of h under H is the number of conjugates of θ under H ([7], p. 356; [5], p. 77). It is readily verified that $G_{\rho-1}$ is simply isomorphic to the subgroup of G_ρ which leaves $\{0; 0, 0, \cdots, 0, 2\}$ unaltered. Thus if $\theta(\rho)$ is the order of $G(\rho)$ and $C(\rho)$ is the number of conjugates of $\{0; 0, 0, \cdots, 0, 2\}$ under $G(\rho)$, then

$$
(20) \qquad \theta(\rho) = C(\rho)\theta(\rho - 1).
$$

By (13) all characteristics of sort $(2, 1)$ are conjugate for $\rho > 9$ and it is readily verified that this is true in the early cases. Hence by (9) we have:

(21) *$C(\rho)$, the number of conjugates of $\{0; 0, 0, \cdots, 0, 2\}$ under G_ρ, is given by $C(\rho) = \rho_{10} + 2\rho_{02} - \epsilon$ where $\epsilon = 0$ if $\rho \equiv 0, 2$, mod 3, and $\epsilon = 1$ if $\rho \equiv 1$, mod 3.*

We remark that the algebraic form of $C(\rho)$ is readily obtained from table (8). $\theta(\rho)$ is completely determined by the initial value $\theta(1)=2$, the recursion relation (20), and the fact that $C(\rho)$ is always known. From (19) it follows that $\frac{1}{2}\theta(\rho)$ is the order of $\bar{G}_\rho$ for $\rho>2$. But for $\rho>9$, $\underline{g}^{(3)}_{\rho,2}$ contains all the generators of $\bar{G}_\rho$ and must then be simply isomorphic to $\bar{G}_\rho$.

(22) *For* $\rho>9$, *the order of* $g^{(3)}_{\rho,2}$ *is* $\frac{1}{2}\theta(\rho)$, *where* $\theta(\rho)$ *is defined as above.* $g^{(3)}_{\rho,2}$ *is characterized as a subgroup of* G_ρ *by the fact that it permutes characteristics of sort* $(0,1)$ *evenly.*

Conclusion. The question concerning the possibility of distinguishing between proper and improper characteristics by reduction with respect to the modulus 3 is completely settled. Since for $\rho>9$ all characteristics of the same sort are actually conjugate under $g^{(3)}_{\rho,2}$, it is clear that the case observed by Coble is the only one in which such procedure is useful.

The nature of $g^{(3)}_{\rho,2}$ has been determined for all $\rho>9$. An explicit formula for the order of the group is given and it is shown that $g^{(3)}_{\rho,2}$ is completely characterized by

(a) the invariance of Q, L, and

(b) the fact that it permutes characteristics of sort $(0,1)$ evenly.*

Since G_ρ is simply isomorphic to the groups studied by Coble [12] and Dickson,[5] and since $g^{(3)}_{\rho,2}$ is an invariant subgroup of index 2 under G_ρ for $\rho>9$, the problem of determining in detail the structure of $g^{(3)}_{\rho,2}$ has been reduced to the application of known results.

These results would be immediately applicable to the study of $g_{\rho,2}$ if the nature of $I^{(3)}_{\rho,2}$ were known for all ρ. For $\rho \lesseqgtr 9$ this was accomplished, but in the later cases only fragmentary information was exhibited. A complete characterization of $I^{(3)}_{\rho,2}$ would be necessary to give this paper a general significance.

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NOTE ON ASTATIC ELEMENTS.

By F. Morley and J. R. Musselman.

Let forces in one plane have points of application a_i, magnitudes μ_i, and directions τ_i. The sum of the areas or moments about any point x is 0 when

$$\Sigma \begin{vmatrix} x & \bar{x} & 1 \\ a & \bar{a} & 1 \\ \mu\tau & \mu/\tau & 0 \end{vmatrix} = 0$$

and therefore for equilibrium

$$(1) \qquad \Sigma\mu\tau = 0$$

and

$$(2) \qquad \Sigma\mu a/\tau = \Sigma\mu\bar{a}\tau.$$

If now each force be given a turn t about its point of application, then τ_i becomes $t\tau_i$, the first equation remains, but the second becomes $\Sigma\mu a/\tau = t^2\Sigma\mu\bar{a}\tau$ and if this be true for one value of t^2 other than 1, it is always true, for $\Sigma\mu a/\tau = 0$. The condition for astatic equilibrium is then

$$(3) \qquad \Sigma\mu a/\tau = 0.$$

We take the simple case of equal forces, that is let $\mu_1 = \mu_2 = \cdots = 1$. And we speak of elements instead of forces with given points of application. We have astatic elements a_i, τ_i when

$$\Sigma\tau = 0 \quad \text{and} \quad \Sigma a_i/\tau_i = 0.$$

For instance three elements are astatic when $\tau_1 : \tau_2 : \tau_3 = 1 : \omega : \omega^2$

$$(\text{where } \omega^2 + \omega + 1 = 0)$$

and

$$a_1 + \omega^2 a_2 + \omega a_3 = 0.$$

If now $\Sigma a/\tau$ is not 0, but say k, we replace a_1 by b_1 where

$$b_1/\tau_1 + a_2/\tau_2 + \cdots = 0.$$

We have then

$$a_1 - b_1 = k\tau_1$$

and similarly replacing each a_i by a b_i

$$a_i - b_i = k\tau_i$$

whence $$\Sigma a = \Sigma b.$$

It suggests itself that the vectors $a_i - b_i$ are forces in equilibrium. This is so if

$$\Sigma \begin{vmatrix} ab \\ \bar{a}\bar{b} \end{vmatrix} = 0$$

that is if

$$\Sigma \begin{vmatrix} a, & k\tau \\ \bar{a}, & \bar{k}/\tau \end{vmatrix} = 0$$

which is true since $\Sigma a/\tau = k$. Thus if we have n elements a_i, τ_i for which

$$\Sigma\tau = 0$$

and if we replace each a_i, τ_i in turn by b_i, τ_i so as to get an astatic set, then $a_i - b_i$ form a set of equal forces in equilibrium.

If we replace $\tau_1, \tau_2, \cdots$ by $1, \epsilon, \epsilon^2 \cdots$ where ϵ is a root of $\epsilon^n = 1$, then $\Sigma\tau = 0$. The expression $\Sigma a/\tau$ becomes a Lagrange resolvent, v. The theorem becomes: If we take n elements $a_1, a_2\epsilon, a_3\epsilon^2 \cdots$ and replace in turn each a_i by b_i so that the Lagrange resolvent v vanishes for b_i and the other a's, then $a_i - b_i$ are a system of equal forces in equilibrium.

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PROPERTIES OF THE VENERONI TRANSFORMATION IN S_4.

By Gertrude K. Blanch.

In a paper which appeared in 1901, Veneroni[1] cited that in a space S_n of n dimensions, the primals V^n_{n-1} of order n, which pass through $(n+1)$ general linear spaces S_{n-2} lying in S_n form a homaloidal system. Such a transformation is referred to as the *Veneroni* transformation. Some properties for S_n are given by Veneroni and Eiesland.[2] More specific detail about the transformation in S_4 is given by J. A. Todd[3] and by Virgil Snyder.[4] The bilinear equations defining the transformation have not heretofore been published, however. They are derived in this paper and further properties are investigated with their aid. Emphasis is laid on the study of involutorial Veneroni transformations. In S_3 any Veneroni transformations can be made involutorial by a proper choice of the frame of reference—an elegant derivation is given by H. F. Baker.[5] We show in this paper that in S_4 this is no longer true; one condition among the coefficients of the equations becomes necessary for an involution. It is found that quite generally, but not always, the involutorial case can be represented as a polarity with respect to four composite quadric primals, and the fundamental elements are considerably more specialized than in the more general involutorial transformations studied by Schoute[6] and by Alderton.[7] Furthermore, *there exist Veneroni transformations in S_4 which are involutorial; but the bilinear forms cannot be represented as polarities with respect to quadric primals by any linear transformations.* Properties which have already been found by other investigators will be included for the sake of clarity and completeness, when necessary.

Notation. Let

$$(ax) \equiv a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5;$$
$$(bx) \equiv b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5.$$

The binomial $(a_ib_k - b_ia_k)$ will be denoted by (ik). Thus $(a_1b_2 - b_1a_2) \equiv (12)$. S_n: linear space of n dimensions. V_r^n: a variety of dimension r and order n.

[1] *Lombardo Rendiconti II*, vol. 34, pp. 640-654.

[2] *Rendiconti Circolo Matematico di Palermo*, vol. 54, pp. 335-365.

[3] *Proceedings of the Cambridge Philological Society*, vol. 26 (1930), pp. 323-333.

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[6] *Archives du Musée Teyler*, ser. 2, vol. 7.

[7] *California Publications in Mathematics*, no. 15, vol. 1 (1923), pp. 345-358.

A V_r^n of S_{r+1} will be called a *primal*; if $n=1$, it will be called a *prime* and denoted by S_r. The transformation involves two spaces S_4, S'_4. The system in S_4 will be referred to as the (x)-system; that in S'_4 will be called the (x')-system. The symbol $\mathfrak{Z}$ means "transform into."

1. *The derivation of the bilinear equations.* The general homaloid of this $(4-4)$ transformation in S_4 is a quartic primal V_3^4 passing through five general planes of S_4, such that any two of them intersect in one point only. It is easy to show that the system $|V_3^4|$ is ∞^4. Let the five planes have the equations:

$$\pi_1:\ x_1=x_2=0;\qquad \pi_2:\ x_3=x_4=0;\qquad \pi_3:\ x_2-x_5=x_4-x_5=0.$$
$$\pi_4:\ x_1-x_5=x_3-x_5=0;\qquad \pi_5:\ (ax)=(bx)=0.$$

The first four planes intersect by twos in the vertices of the simplex of reference and the unit point. These are entirely general for S_4. Any other six points in equally general position uniquely determine a set of four other planes which can be carried into these by a non-singular linear transformation. The fifth plane being entirely general the homaloidal system defined by these five planes will be projectively equivalent to that defined by any other five equally general planes in S_4. We demand that π_5 be non-incident [8] with the other four planes, and that the ten points of intersection of the planes π_i by twos be distinct. Then $\sum_{i=1}^{5} a_i\equiv A$; $\sum_{i=1}^{5} b_i\equiv B$ cannot both be zero; for then $(1,1,1,1,1)$ would satisfy π_5, and would coincide with the intersection $\pi_3\cdot\pi_4$ of π_3 and π_4. It is therefore no restriction to define π_5 by $(ax)=0$; $(bx)=0$; $A\neq 0$; $\sum_{i=1}^{5} b_i=0$.

Consider the five Segre cubic primals, each determined by four of the five planes.[9] Their equations are given below:

$$\begin{aligned}
W_1:\ &[(ax)b_2-(bx)a_2](x_5-x_2)(x_3-x_4)\\
&\quad+[(ax)b_4-(bx)a_4](x_5-x_4)(x_3-x_4)+A(bx)\cdot x_3\cdot(x_5-x_4)=0\\
W_2:\ &[(ax)b_2-(bx)a_2\cdot](x_2-x_5)(x_2-x_1)\\
&\quad+[(ax)b_4-(bx)a_4]\cdot(x_4-x_5)(x_2-x_1)+A(bx)\cdot x_1\cdot(x_5-x_2)=0\\
W_3:\ &[(ax)b_3-(bx)a_3]\cdot x_3\cdot(x_3-x_1)\\
&\quad+[(ax)b_4-(bx)a_4]\cdot x_4(x_3-x_1)+[(ax)b_5-(bx)a_5]\cdot x_3(x_5-x_1)=0\\
W_4:\ &[(ax)b_3-(bx)a_3]\cdot x_3\cdot(x_4-x_2)\\
&\quad+[(ax)b_4-(bx)a_4]\cdot x_4\cdot(x_4-x_2)+[(ax)b_5-(bx)a_5]\cdot x_4\cdot(x_5-x_2)=\\
W_5:\ &x_1x_3x_4-x_1x_2x_3+x_1x_2x_4-x_1x_4x_5+x_2x_3x_5-x_2x_3x_4=0.
\end{aligned}$$

[8] Two planes are *non-incident* in S_4 when they have only a point in common; if they have a line in common, call them *incident*.

[9] Bertini, *Einführung in die projective Geometrie mehrdimensionaler Räume*, chapter VIII, p. 193.

It will be useful to recall some of the properties of these primals. Each can be generated as the locus of lines which meet the four planes determining it. Each contains fifteen planes; from every one of its points there can be drawn six lines, each such line meeting six of the fifteen planes. We may readily verify the following identity:

$$(x_1 - x_2) W_1 + (x_4 - x_3) W_2 + (bx) A W_5 \equiv 0.$$

Let

$$x'_1 = x_1 W_1; \quad x'_2 = x_2 W_1; \quad x'_3 = x_3 W_2; \quad x'_4 = x_4 W_2; \quad x'_5 = A(ax) W_5. \tag{1. 1}$$

Each x'_i is satisfied by all five planes π_i and is therefore a composite member of $| V_3{}^4 |$; further these five homaloids are linearly independent and hence they define a proper Veneroni transformation. To obtain now the bilinear equations consider the determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ -x_2 & x_1 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & x_3 & 0 \\ (ax) & -(ax) & -(ax) & (ax) & (bx) \\ \begin{matrix}[(a_2 - A)(x_5 - x_2) \\ + a_4(x_5 - x_4)]\end{matrix} & \begin{matrix}[a_2(x_2 - x_5) \\ + a_4(x_4 - x_5)]\end{matrix} & \begin{matrix}[a_2(x_2 - x_5) \\ + (A - a_4)(x_5 - x_4)]\end{matrix} & \begin{matrix}[a_2(x_5 - x_2) \\ + a_4(x_5 - x_4)]\end{matrix} & \begin{matrix}[b_2(x_5 - x_2) \\ + b_4(x_5 - x_4)]\end{matrix} \end{vmatrix} = 0. \tag{1. 2}$$

This expands into

$$\lambda_1 x_1 W_1 + \lambda_2 x_2 W_1 + \lambda_3 x_3 W_2 + \lambda_4 x_4 W_2 + \lambda_5 A(ax) W_5 = 0.$$

The determinant is therefore the complete system $| V_3{}^4 |$. Replace the first row of the determinant successively by the second, third, fourth, and fifth rows; taking account of (1. 1), we have the four bilinear equations:

M_1: $x_1 x'_2 - x_2 x'_1 = 0$

M_2: $x_3 x'_4 - x_4 x'_3 = 0$

M_3: $(ax)(x'_1 - x'_2 - x'_3 + x'_4) + (bx) x'_5 = 0$

M_4: $(x_5 - x_2) \cdot (a_2 C - A x'_1 + b_2 x'_5) + (x_5 - x_4) \cdot (a_4 C + A x'_3 + b_4 x'_5) = 0.$

The linear combination $A(M_1 - M_2) + M_3 + M_4$ gives

$$M_5\colon (x_5 - x_1) \cdot (-a_1 C - A x'_2 - b_1 x'_5) \\ + (x_5 - x_3) \cdot (-a_3 C + A x'_4 - b_3 x'_5) = 0$$

where $C \equiv x'_1 - x'_2 - x'_3 + x'_4$.

The matrix of each bilinear equation has rank 2; the direct and inverse transformation are of the same nature; and the fundamental planes of (x) and (x') can be read off from inspection of these equations. Naming the fundamental planes of (x') T_1, T_2, T_3, T_4, T_5, we may say that each bilinear

equation associates a plane T_i with a plane π_i; and we shall hereafter call two such planes T_i, π_i satisfying the same bilinear equation *associated* planes.

3. *Locus of invariant points.* When the spaces (x) and (x') are superposed, the invariant points satisfy M_1 and M_2 identically. The locus of invariant points therefore consists of the quartic surface defined by M_3 and M_4, with the primed symbols removed. The planes π_3, π_4, π_5 intersect the surface in conics, but π_1, π_2 cut the surface in only four points. Thus the invariant locus is not symmetric with respect to the fundamental planes and depends on the choice of the frame of reference.

Involutorial Transformations.

4. *A special involutorial transformation.* The transformation already developed will be involutorial when and only when $(ax) \equiv x_5$ and $(bx) \equiv -C$. The fundamental system becomes very much specialized—$R_2{}^5$ breaks up into two planes and an $R_2{}^3$, and four of the Segre cubic primals on $J_3{}^{15}$ contain double lines, with only 11 planes on each. The transformation

$$x'_1 = y_2;\ x'_2 = y_1;\ x'_3 = y_4;\ x'_4 = y_3;\ x'_5 = y_5 \tag{2.1}$$

applied to (x') changes these forms to a polarity with respect to four composite quadric primals.

5. *Veneroni transformations in the form of quadric polarities.* Subjecting (x') to a linear transformation is equivalent to taking for the simplex of reference in (x') different linear combinations of the homaloids in (x). When the two spaces are regarded as superposed, $J_2{}^{10}$, $J'_2{}^{10}$ have two planes in common, each of which $\mathfrak{z}$ a W_i. Hence the square of the derived transformation is of order ten. But we may transform any four planes of (x') into any other four planes in equally general position and we may so choose such a transformation that four, three, two, one or no planes T_i of (x') coincide with fundamental planes of (x). Hence the square of the general Veneroni transformation may be of order 4, 7, 10, or 16 respectively, depending merely on the frame of reference. But under what minimum restrictions will the transformation be involutorial? We have studied one involutorial transformation in which the fundamental system was very much specialized. We will now obtain others much more general. In any involutorial transformation, the fundamental planes of (x) and (x') must coincide; but conceivably they may be associated with one another in any one of the 5! ways in the bilinear equations. If the plane T_i coincides with its *associated* plane π_i for every i,

we shall call such a transformation *form I*. We inquire under what conditions form I is possible, with at least four planes T in perfectly general position. In that case there will exist a non-singular linear transformation on (x') carrying the four planes T_i into π_i, which will also transform T_5 into π_5. Then such a transformation must have the form:

$$x'_1 = \alpha_1 y_1 + \alpha_2 y_2; \quad x'_2 = \alpha_3 y_1 + \alpha_4 y_2; \quad x'_3 = \alpha_5 y_3 + \alpha_6 y_4; \quad x'_4 = \alpha_7 y_3 + \alpha_8 y_4;$$
$$x'_5 = \alpha_9 (ay) + \alpha_{10}(by).$$

In addition, it must carry the remaining planes T_i into the associated planes. Completing the algebraic details[10] we find that *one restriction* on the coefficients is necessary; namely

$$(2.2) \qquad (24)\cdot(51)b_3 + (25)\cdot(13)b_4 = 0, \quad \text{when} \quad \sum_{i=1}^{5} b_i = 0.$$

If, in addition

$$(2.3) \quad [(53)\cdot(51)\cdot(42)\cdot(31)b_2b_4]\cdot[(24)b_1 + (13)b_2 + Ab_1b_2] \cdot[(13)b_4 + (42)b_3 - Ab_3b_4] \neq 0$$

the transformation will always be non-singular. If some of the factors in (2.3) are zero, form I may still be possible in some cases, and impossible in others. The special restrictions in individual cases can be derived quite readily. The resulting bilinear equations will show *that when form I is possible, the transformation is always a polarity with respect to four composite quadrics* (and indeed, a fifth also, when M_5 is considered). No essential restriction is imposed on the fundamental elements. Thus the double points are distinct and no three need be collinear; the fifty planes of J_3^{15} can all be distinct. In S_3, every Veneroni transformation can be made involutorial by a suitable linear transformation; and since so much depends on the frame of reference, we will examine more closely the choice used to see whether (2.2) is really an essential restriction for S_4. First, we imposed the restriction that $\Sigma b_i = 0$ in developing the bilinear equations. It may be verified[11] that when this restriction is removed, condition (2.2) takes the form

$$B[(51)\cdot(24)a_3 + a_4(31)\cdot(52)] + A[b_4(13)\cdot(52) + b_3(51)\cdot(42)] = 0,$$

where $B \equiv \Sigma b_i$. This is a relative invariant under the transformations $(ay) = c(ax) + d(bx)$; $(by) = e(ax) + g(bx)$. It remains to be seen whether by imposing a transformation (ω) on (x) and another transformation

[10] The algebra in full detail is given in the author's dissertation, 1935, Library of Cornell University.

[11] Author's dissertation, Appendix I b.

(τ) on (x'), we may obtain form I without any restriction on the coefficients a_i, b_i. Suppose one such had been found. Then the fundamental planes of (x) and (x') now being alike, use (ω^{-1}) on both. Then $(\tau\omega^{-1})$ transforms all five planes of (x') into the associated planes of (x) under the old frame of reference, and if form I is possible at all, we may obtain it by keeping (x) fixed and impositing a linear transformation on (x'). *Hence condition* (2.2) *is a necessary restriction.*

6. *Involutorial Veneroni transformations which cannot assume form I.* The first special involutorial transformation in II is not a polarity with respect to quadric primals, since $a_{ik} \neq a_{ki}$ in the fourth bilinear equation. But condition (2.2) is satisfied and (2.1) carries the transformation into form I. Is it possible to find involutorial Veneroni transformations, not form I, and such that form I is never possible for them? Such transformations exist and we give a method of obtaining them by examining certain anharmonic ratios.

From every double-point $\pi_i\pi_k$ can be drawn a line of R_2^5 to meet the other three fundamental planes. Let A_3, A_4, A_5 be the points in π_3, π_4, π_5 respectively on the line of R_2^5 from $\pi_i\pi_2$. Let $p_{12} = (\pi_i\pi_2, 3, 4, 5)$ be the anharmonic ratio of the points $\pi_i\pi_2$, A_3, A_4, A_5. Obtain the anharmonic ratios p_{ik} on every one of the ten lines through the points $\pi_i\pi_k$. Similarly, obtain the corresponding cross-ratios p'_{ik} on the lines in (x') from the associated points T_iT_k, in the same order. *Then it is readily verified that* $p_{ik} = p'_{ik}$ *for every* i, k.

Now suppose the five planes T_i are projective with the planes π_j but not in the associated order. Suppose, for instance, there exists a linear transformation under which $T_1 \wr \pi_1$; $T_2 \wr \pi_2$; $T_5 \wr \pi_5$; $T_3 \wr \pi_4$: $T_4 \wr \pi_3$. Since cross-ratios are invariant under a linear transformation, we must have $(\pi_1\pi_2, 3, 4, 5) = (\pi_1\pi_2, 4, 3, 5)$. Hence $p_{12} = 1/2$. The other anharmonic ratios impose three more independent conditions, and we exhibit the following equations for π_5 which will satisfy all requirements:

$$(ax): (2b+1)x_1 + b(2b+1)x_2 + 2(b^2+b+1)x_3 - 2(b^2+b+1)x_5 = 0$$

$$(bx): (-2b-1)x_1 + (2b+1)x_2 + 2bx_3 - 2(b+1)x_4 + 2x_5 = 0$$

subject to the condition $(2b+1)(b+1)(b^2+b+1) \neq 0$, but b unrestricted otherwise. The following linear transformation carries $T_i \wr \pi_j$:

$$x'_1 = a_1{}^2(y_1 + by_2)/a_3;\ x'_2 = a_1{}^2(by_1 + y_1 - y_2)/a_3;\ x'_3 = y_3 + 2by_4$$
$$x'_4 = 2(b+1)y_3 - y_4;\ x'_5 = (a_1/2)[a_1y_1 + 2(y_3 - y_5) + 2b(y_4 - y_5)].$$

where $a_1 = 2b + 1$; $a_3 = 2(b^2 + b + 1)$.

The five bilinear equations become:

$$M_1:\ x_1[(b+1)y_1-y_2]-x_2(y_1+by_2)=0$$
$$M_2:\ x_3[(2b+2)y_3-y_4]-x_4(y_3+2by_4)=0$$
$$M_3:\ (ax)[(ay)+(b^2+b+1)\cdot(by)] + (bx)[(ay)-b(by)]\cdot(b^2+b+1)=0$$
$$M_4:\ (x_5-x_2)[(2b+1)\cdot(y_3-y_5)] + (x_5-x_4)[(2b+1)\cdot(y_5-y_1)+(y_5-y_3)]=0$$
$$M_5:\ (x_5-x_1)\cdot[(2b+1)(y_5-y_4)] + (x_5-x_3)\cdot[(2b+1)(y_2-y_5)+(y_5-y_4)]=0.$$

The transformation is non-singular when $(b^2+b+1)(b+1)(2b+1)\neq 0$. It is readily verified that

$$(24)(51)b_3+(25)(13)b_4=-8(2b+1)^3\cdot(b+1)^2(b^2+b+1)$$

and is never zero when the transformation is non-singular. Hence form I is never possible for these transformations.

In existing literature, involutorial transformations in S_4 expressible by means of bilinear equations are given as polarities with respect to quadric primals by Schoute [12] and by Alderton.[13] The results of this paper show that not all involutorial transformations so definable are expressible as quadric polarities. These Veneroni transformations also throw more light on the character of the fundamental system for the case of the quadric polarities. Miss Alderton classifies the transformations according to the characteristics of the quadric primals defining the polarity. For the general case of all four primals being "space-pairs" (composite primals), she cites that J_2^{10} consists of four planes and a sextic surface, with more specialization when the planes common to the space-pairs are incident. The Veneroni transformations show that J_2^{10} can be considerably more specialized even when the four "space-pairs" are perfectly general.

Clearly a transformation may be involutorial even when $a_{ik}\neq a_{ki}$ in all the bilinear equations. A less exacting criterion—quite obvious yet worth noting is this: Denote by the (x') matrix the one from which the ratios $x'_1:x'_2:x'_3:x'_4:x'_5$ are obtained. If the (x) matrix may be obtained from the (x') matrix by elementary transformations on the rows of the latter, then the transformation is involutorial. The cases given in this paper satisfy this criterion.

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[12] Schoute, *loc. cit.*

[13] Alderton, *loc. cit.*

SECOND ORDER DIFFERENTIAL EQUATIONS WITH TWO POINT BOUNDARY CONDITIONS IN GENERAL ANALYSIS.[1]

By A. D. Michal and D. H. Hyers.

Introduction. The abstract differential calculus initiated by M. Fréchet[2] has led several authors to the study of both "ordinary"[3] and "total"[4] differential equations in abstract normed vector spaces. So far the existence theorems given have all been for one point initial conditions.[5]

In this paper we consider the following "ordinary" second order differential equation with *two point* boundary conditions:

$$(1) \qquad d^2\xi(t)/dt^2 = F(t, \xi, d\xi/dt), \qquad \xi(t_0) = A, \quad \xi(t_1) = B,$$

where the values of the functions $\xi(t)$, $F(t, \xi, \eta)$ and the second and third arguments of $F(t, \xi, \eta)$ are elements of a Banach space[5] while t is a real variable.

Section 1 contains essentially an abstraction of Picard's two point existence theorem.[6] In section 2 we consider special systems of type (1) (Cf. systems (2.1) and (2.6)) occurring in abstract differential geometry,[7] and study the continuity and Fréchet differentiability of the solution $x(t, x_0, x_1)$ as a function of t and the boundary values x_0, x_1. The main results of the

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[1] Presented to the American Mathematical Society, September, 1935, and November, 1935.

[2] M. Fréchet, *Comptes Rendus*, vol. 180 (1925), pp. 806-809; *Annales Scientifique de l'École Normale Supérieure*, vol. 42 (1925), pp. 293-323. For the differentials of functionals see V. Volterra, *Theory of Functionals* (London, 1930); G. C. Evans, *American Mathematical Society Colloquium Lectures* (New York, 1918).

[3] L. M. Graves, *Transactions of the American Mathematical Society*, vol. 29 (1927), p. 514; M. Kerner, *Prace Matematyczno-Fizyczne*, vol. 40 (1933), pp. 47-67. For the earlier work on linear differential equations in Moore's general analysis see *E. H. Moore, Atti di IV Congresso* (Rome, 1908), vol. 2, pp. 98-114; T. H. Hildebrandt, *Transactions of the American Mathematical Society*, vol. 18 (1917), p. 73.

[4] A. D. Michal and V. Elconin, *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 534-536; A. D. Michal, *Annali di Matematica* (in press).

[5] S. Banach, *Opérations Linéaires* (Warsaw, 1932). In (1) we have written $\frac{d^2\xi(t)}{dt^2}$ in the place of $\frac{d}{dt}\left(\frac{d\xi(t)}{dt}\right)$. For other definitions of second derivatives see M. Kerner, *Annali di Matematica*, vol. 10 (1932), pp. 145-164.

[6] E. Picard, *Traité d'Analyse*, Tome 3 (Paris, 1928), pp. 90-96.

[7] A. D. Michal, *Annali di Matematica, loc. cit.*; *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 526-529.

paper center around Theorems 2, 3, 4 and 5 of this section. Sections 3 and 4 are concerned with a differential equation in a normed ring and with an integro-differential system arising in functional geometry.[8]

1. *A general existence theorem.* In this section we give an existence theorem for the differential system (1) together with a lemma needed in the proof of this theorem and of Theorem 6.

LEMMA 1. *Let $\Phi(t)$ be a continuous function on the closed real interval $(0, c)$ to a Banach space and let*

$$m = \max_t \| \Phi(t) \|.$$

Then for $0 \leq t \leq c$ the unique solution of the differential equation

$$d^2\xi/dt^2 = \Phi(t) \tag{1.1}$$

taking on the two-point conditions $\xi(0) = 0$, $\xi(c) = B$ is given by

$$\xi(t) = \int_0^c g(s, t)\Phi(s)ds + (t/c)B,$$

where

$$\begin{aligned} g(s,t) &= s(t/c - 1) \qquad s < t \\ &= t(s/c - 1) \qquad s > t. \end{aligned}$$

Furthermore we have

$$\| \xi \| < mc^2/8 + \| B \|, \qquad \| d\xi/dt - B/c \| < mc/2.$$

THEOREM 1. *Let L and L' be any two chosen positive numbers. Denote the real closed interval (a, b) by $\boldsymbol{I}$, let $\boldsymbol{N}_1$ be the set of points ξ of a Banach space*[9] *$\boldsymbol{E}$ such that*

$$\| \xi - A \| \leq L$$

and let N_2 be the set of points η of E such that

$$\| \eta - \eta_0 \| \leq L'.$$

Suppose $F(t, \xi, \eta)$ to be a continuous function in the set t, ξ, η for t in $\boldsymbol{I}$, ξ in $\boldsymbol{N}_1$, η in $\boldsymbol{N}_2$, and with values in $\boldsymbol{E}$. Further suppose that for the same set of values of t, ξ, η, the function F satisfies the Lipschitz condition

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[8] A. D. Michal, *American Journal of Mathematics*, vol. 50 (1928), pp. 473-517; *Proceedings of the National Academy of Sciences*, vol. 16 (1930), (three papers) and vol. 17 (1931).

[9] Banach spaces and their subsets will be denoted throughout the paper by bold face letters.

$$\| F(t, \xi_1, \eta_1) - F(t, \xi_2, \eta_2) \| \leq \alpha \| \xi_1 - \xi_2 \| + \beta \| \eta_1 - \eta_2 \|$$

(and hence $\| F \|$ *is bounded, say* $\| F(t, \xi, \eta) \| \leq U$*). Then subject to the following inequalities*

$$(1.2) \qquad \begin{cases} a \leq t_0 < t_1 \leq b \\ \dfrac{U(t_1 - t_0)^2}{8} + \| B - A \| < L \\ \dfrac{U(t_1 - t_0)}{2} + \left\| \eta_0 - \dfrac{B - A}{t_1 - t_0} \right\| < L' \\ \dfrac{\alpha(t_1 - t_0)^2}{8} + \dfrac{\beta(t_1 - t_0)}{2} < 1, \end{cases}$$

the differential system (1) *has a solution on* (t_0, t_1)*. Furthermore under the same conditions there exists on* (t_0, t_1) *a unique solution* $\xi(t)$ *of the differential system* (1) *and the inequalities*

$$(1.3) \qquad \| \xi(t) - A \| \leq L, \qquad \| d\xi(t)/dt - \eta_0 \| \leq L'.$$

The proof of Lemma 1 and the existence of a solution of the differential system (1) can be obtained by an evident modification of Picard's [10] well known methods and by the application of some known or easily proved results on abstractly valued functions.[11] To prove the uniqueness of the solution of (1) and (1.3) we take $t_0 = 0$ without loss of generality. Assume there are two distinct solutions $\xi(t)$ and $\eta(t)$ in $(0, t_1)$. By the Lipschitz condition we obtain the inequality

$$\| F(t, \xi(t), d\xi(t)/dt) - F(t, \eta(t), d\eta(t)/dt) \| \\ \leq \alpha \| \xi(t) - \eta(t) \| + \beta \| d\xi(t)/dt - d\eta(t)/dt \|$$

for t in $(0, t_1)$. The non-negative continuous function on the right side of this inequality takes on its positive maximum value, say N, at some point τ in $(0, t_1)$. Hence an application of the lemma to the differential equation

$$\frac{d^2(\xi(t) - \eta(t))}{dt^2} = F\left(t, \xi(t), \frac{d\xi(t)}{dt}\right) - F\left(t, \eta(t), \frac{d\eta(t)}{dt}\right)$$

yields

$$N < N(\alpha t_1^2/8 + \beta t_1/2).$$

But this contradicts the last inequality (1.2) for $t_0 = 0$.

2. *A differential equation arising in abstract differential geometry.* One

[10] E. Picard, *loc. cit.*

[11] M. Kerner, *Prace Matematyczno-Fizyczne, loc. cit.*

of us [12] has initiated the study of Riemannian and non-Riemannian differential geometries in abstract vector spaces. In these geometries the geodesics (or paths) satisfy special differential equations of type (1). In this section we shall prove several theorems on geodesics.

LEMMA 2. *Let* $\Gamma(x, \xi, \eta)$, *an affine connection,*[12] *be defined on*[13] $\boldsymbol{E}((\bar{x})_a)\boldsymbol{E}^2$ *to a Banach space* $\boldsymbol{E}$. *If*

(i) $\Gamma(x, \xi, \eta)$ *is additive in* ξ *and in* η;

(ii) $\| \Gamma(x, \xi, \eta) \| \leq M \| \xi \| \ \| \eta \|$ *(M, constant)*;

(iii) *The partial Fréchet differential*[14] $\Gamma(x, \xi, \eta; \delta x)$ *exists and is continuous in* x, *then there exist positive numbers* N *and* $\bar{a} < a$ *such that for any* λ

$$\| \Gamma(x_1, \xi_1, \xi_1) - \Gamma(x_2, \xi_2, \xi_2) \| \leq N\lambda^2 \| x_1 - x_2 \| + 2M\lambda \| \xi_1 - \xi_2 \|$$

for all x *in* $(\bar{x})_{\bar{a}}$ *and all* ξ *in* $(0)_\lambda$.

Proof. Clearly $\Gamma(x, \xi, \eta)$ is bilinear [15] in ξ and η. On using hypothesis (iii) and a theorem on differentials proved by one of us [16] we find that $\Gamma(x, \xi, \eta; z)$ is trilinear in ξ, η, z.

Therefore there exists [17] an $\bar{a}$, $0 < \bar{a} < a$ and a constant N such that

$$\| \Gamma(x, \xi, \eta; z) \| \leq N \| \xi \| \ \| \eta \| \ \| z \|$$

for x in $(\bar{x})_{\bar{a}}$. For any positive λ let x_1 and x_2 be in $(\bar{x})_{\bar{a}}$ and ξ_1, ξ_2 be in $(0)_\lambda$. On using the formula [18] for the difference in terms of the differential and noting that a neighborhood is a convex point set we obtain

$$\| \Gamma(x_1, \xi_1, \xi_1) - \Gamma(x_2, \xi_1, \xi_1) \| \leq N\lambda^2 \| x_1 - x_2 \|.$$

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[12] A. D. Michal, *Annali di Matematica, loc. cit.*; *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 526-529.

[13] $\boldsymbol{E}((\bar{x})_a)$ or simply $(\bar{x})_a$ is the set of points of $\boldsymbol{E}$ for which $\| x - \bar{x} \| < a$. By "$f(x_1, \cdots, x_n)$ on $\boldsymbol{E}_1\boldsymbol{E}_2 \cdots \boldsymbol{E}_n$ to $\boldsymbol{E}$" we mean that the i-th argument x_i ranges over the set $\boldsymbol{E}_i$ while the values of $f(x_1, \cdots, x_n)$ are in $\boldsymbol{E}$.

[14] M. Fréchet, *loc. cit.* See also L. M. Graves and T. H. Hildebrandt, *Transactions of the American Mathematical Society*, vol. 29 (1927).

[15] Additive and continuous in ξ and η separately.

[16] A. D. Michal, *Annali di Matematica, loc. cit.*

[17] M. Kerner, *Annals of Mathematics*, vol. 34 (1933), pp. 546-572.

[18] L. M. Graves, *Transactions of the American Mathematical Society*, vol. 29 (1927), p. 173. See also M. Kerner, *Prace Matematyczno-Fizyczne*, and *Annals of Mathematics, loc. cit.*

From hypotheses (i) and (ii) we have

$$\| \Gamma(x_2, \xi_1, \xi_1) - \Gamma(x_2, \xi_1, \xi_2) \| \leq M\lambda \| \xi_1 - \xi_2 \|$$
$$\| \Gamma(x_2, \xi_1, \xi_2) - \Gamma(x_2, \xi_2, \xi_2) \| \leq M\lambda \| \xi_1 - \xi_2 \|$$

and the lemma follows immediately.

So far we have considered two point boundary problems in which one of the points x_0 is kept fixed while the other point x_1 is any chosen point within a suitable domain. We shall now give some theorems on geodesics in which both end points x_0 and x_1 are any chosen points within a suitable domain.

THEOREM 2. *Suppose the hypotheses of Lemma 2 are satisfied. Using the notations of this lemma choose* $\lambda <$ *the smaller of*

$$\frac{\bar{a}}{2(t_1 - t_0)}, \quad \frac{1}{M(t_1 - t_0)}, \quad \frac{2}{N(t_1 - t_0)} (\sqrt{4M^2 + 2N} - 2M)$$

and choose $\delta < (\lambda/4)(t_1 - t_0)$. *Then for any such choice of* λ *and* δ, *and for any* x_0, x_1 *in* $(\bar{x})_\delta$, *the system*

$$(2.1) \qquad d^2x/dt^2 = \Gamma(x, dx/dt, dx/dt), \qquad x(t_0) = x_0, \quad x(t_1) = x_1$$

has a solution which with its derivative is uniformly continuous in t, x_0, x_1. *Furthermore there exists on* (t_0, t_1) *a unique solution* $x(t)$ *of the system consisting of* (2.1) *and*

$$\| x(t) - \bar{x} \| \leq \bar{a}, \qquad \| dx/dt \| \leq \lambda.$$

Proof. From Lemma 2 and the inequality $2\delta < \bar{a}/4$ we see that Theorem 1 applies here if we take $L = \bar{a}/2$, $L' = \lambda$, $\eta_0 = 0$, $\alpha = N\lambda^2$, $\beta = 2M\lambda$, $U = M\lambda^2$. From the hypotheses on λ it follows that the inequalities (1.2) of Theorem 1 are satisfied. A glance at the uniqueness proof of Theorem 1 shows the truth of the last statement in our theorem. To prove the uniform continuity write $t_0 = 0$, $t_1 = c$ and let

$$(2.2) \qquad \begin{cases} y_0(t, x_0, x_1) = x_0 + (t/c)(x_1 - x_0) \\ y'_0(t, x_0, x_1) = (1/c)(x_1 - x_0). \end{cases}$$

Consider the following system of integral equations which is easily shown to be equivalent to differential system (2.1):

$$(2.21) \qquad \begin{cases} x(t, x_0, x_1) = y_0(t, x_0, x_1) + \int_0^c g(s, t)\Gamma(x(s), x'(s), x'(s))\,ds \\ x'(t, x_0, x_1) = y'_0(t, x_0, x_1) + \int_0^c g_t(s, t)\Gamma(x(s), x'(s), x'(s))\,ds. \end{cases}$$

The functions y_i and y'_i given recurrently below approximate to the solution $x(t, x_0, x_1)$ and its derivative respectively:

$$(2.3) \quad \begin{cases} y_i(t, x_0, x_1) = y_0(t, x_0, x_1) + \int_0^c g(s, t)\Phi_{i-1}(s, x_0, x_1)ds \\ y'_i(t, x_0, x_1) = y'_0(t, x_0, x_1) + \int_0^c g_t(s, t)\Phi_{i-1}(s, x_0, x_1)ds \\ \qquad (i = 1, 2, \cdots) \end{cases}$$

where

$$\Phi_i(s, x_0, x_1) = \Gamma(y_i(s, x_0, x_1), y'_i(s, x_0, x_1), y'_i(s, x_0, x_1)).$$

By means of the inequailty

$$(2.31) \quad \begin{cases} \| \Phi_i(s, x_0 + \delta x_0, x_1 + \delta x_1) - \Phi_i(s, x_0, x_1) \| \\ \quad \leq N\lambda^2 \| y_i(s, x_0 + \delta x_0, x_1 + \delta x_1) - y_i(s, x_0, x_1) \| \\ \quad + 2M\lambda \| y'_i(s, x_0 + \delta x_0, x_1 + \delta x_1) - y'_i(s, x_0, x_1) \| \end{cases}$$

and an induction we see that y_i and y'_i are uniformly continuous in t, x_0, x_1 for t in $(0, c)$ and x_0, x_1 in $(\bar{x})_\delta$. The uniform continuity of the limit functions follows by the usual method.

To investigate the differentiability of the solution $x(t, x_0, x_1)$ as a function of the set of boundary values (x_0, x_1) we could proceed directly as in the proof of continuity of Theorem 2. An alternative method,[19] which we employ below, is to make use of the general implicit function theorems of Hildebrandt and Graves.

Before studying the differentiability of $x(t, x_0, x_1)$ we prove the following lemma concerning functions [20] of class $C^{(n)}$.

LEMMA 3. *Let* $\mathbf{E}$ *and* $\mathbf{E}_1$ *be Banach spaces and* $\mathbf{X}$ *be a bounded convex region of* $\mathbf{E}$. *The necessary and sufficient conditions that a function* $F(x)$ *on* $\mathbf{E}$ *to* $\mathbf{E}_1$ *be of class* $C^{(n)}$ *uniformly on* $\mathbf{X}$ *is that the following conditions* (A) *and* (B) *be satisfied for* $k = n$.

(A) $F(x; \delta_1 x; \cdots; \delta_k x)$ *exists and is continuous in* x *uniformly* (*in the ordinary sense*) *with respect to its entire set of arguments for* $\| \delta_i x \| \leq 1$, $i = 1, \cdots, k$ *and* x *in* $\mathbf{X}$.

(B) *There exists a constant* M_k *with*

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[19] We are indebted to the referee for helpful suggestions in this connection. Originally we employed the direct method, without using the results of Hildebrandt and Graves. For the general implicit function theorems see *Transactions of the American Mathematical Society*, vol. 29 (1927), pp. 127-153. From now on this paper will be referred to as HG.

[20] For the denition of functions of class $C^{(n)}$ see HG, p. 137 and 140.

$$\| F(x;\delta_1 x;\cdots;\delta_k x)\| \leq M_k \|\delta_1 x\| \cdots \|\delta_k x\|$$

for all x in $\mathbf{X}$.

Proof. The necessity of the conditions of the lemma is obvious from the definition of functions of class $C^{(n)}$ uniformly on $\mathbf{X}$ (uniformly $(\mathbf{X};1)$ in the notation of HG). To prove the sufficiency we note that condition (3), p. 137 of HG is automatically fulfilled since clearly

$$(2.4)\qquad F(x)-F(x_0)=\int_0^1 F(x_0+\sigma(x-x_0);x-x_0)\,d\sigma$$

for any two points x_0, x of $\mathbf{X}$. We now make use of the similar formula

$$(2.5)\qquad \begin{aligned} &F(x;\delta_1 x;\cdots;\delta_r x)-F(x_0;\delta_1 x;\cdots;\delta_r x)\\ &\quad=\int_0^1 F(x_0+\sigma(x-x_0);\delta_1 x;\cdots;\delta_r x;x-x_0)\,d\sigma\\ &\quad(x_0, x \text{ in } \mathbf{X})\end{aligned}$$

to prove that conditions (A) and (B) are satisfied for $k=1,2,\cdots,n-1$, and therefore that the lemma holds. In fact (2.5) shows that (B) for $k=r+1$ implies (A) for $k=r$. For each x_0 in $\mathbf{X}$, $F(x_0;\delta_1 x;\cdots;\delta_r x)$ is a multilinear function [21] in $\delta_1 x,\cdots,\delta_r x$ so that

$$\| F(x_0;\delta_1 x;\cdots;\delta_r x)\| \leq M_r(x_0)\|\delta_1 x\|\cdots\|\delta_r x\|.$$

Thus by (2.5) we see that (B) for $k=r+1$ implies (B) for $k=r$. The lemma now follows by repeated applications of this argument.

The following theorem on existence and differentiability is independent of Theorems 1 and 2.

THEOREM 3. *Let $\mathbf{X}$, $\mathbf{\Xi}$ stand for the neighborhoods $(\bar{x})_b$ and $(0)_\nu$ of $\mathbf{E}$ and let the function $H(x,\xi)$ on $\mathbf{XE}$ to $\mathbf{E}$ have the following properties:*

(i) *$H(x,\xi)$ is of class $C^{(n)}$ in (x,ξ) uniformly on $\mathbf{X\Xi}$ $(n\geq 1)$;*

(ii) *$H(x,\xi)$ is homogeneous in ξ of degree $r>1$. Then having chosen two real numbers t_0, t_1, there exist positive numbers $\beta \leq b$ and $\lambda \leq \nu$ such that*

(I) *for every x_0, x_1 in $(\bar{x})_\beta$ there is a solution $x=\phi(t,x_0,x_1)$ of the differential system*

$$(2.6)\qquad d^2x/dt^2=H(x,dx/dt),\qquad x(t_0)=x_0,\quad x(t_1)=x_1$$

and this is the only solution with these end points satisfying

$$\|x(t)-\bar{x}\|<\lambda,\qquad \|dx/dt\|<\lambda$$

[21] A. D. Michal, *Annali di Matematica, loc. cit.*

(II) *the function* $\phi(t, x_0, x_1)$ *is of class* $C^{(n)}$ *in* (x_0, x_1) *uniformly on* $\boldsymbol{E}^2((\bar{x})_\beta)$.

Proof. As in the proof of Theorem 2 we deal with the following system of integral equations which is equivalent to the differential system (2.6):

$$(2.7) \qquad \begin{cases} x(t) = \int_0^c g(s, t)H(x(s), \xi(s))ds + x_0 + (t/c)(x_1 - x_0) \\ \xi(t) = \int_0^c g_t(s, t)H(x(s), \xi(s))ds + (1/c)(x_1 - x_0) \end{cases}$$

where we have chosen $t_0 = 0$, $t_1 = c$ for simplicity.

Let $\boldsymbol{E}_1$ be the space of pairs $y \equiv (x, \xi)$ where $x, y \in \boldsymbol{E}$, with the norm $\| y \| =$ greater of $\| x \|$, $\| \xi \|$, and let $\boldsymbol{E}_2$ be the space of continuous functions $Y \equiv y(t) \equiv (x(t), \xi(t))$ on $(0, c)$ to $\boldsymbol{E}_1$ with the norm $\| Y \| = \max\limits_{0 \leq t \leq c} \| y(t) \|$. With the usual definitions of equality, addition and multiplication by reals, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are Banach spaces. Put $\omega \equiv (x_0, x_1)$. Next let $F_1(\omega, Y)$, $F_2(\omega, Y)$ respectively stand for the right members of the integral equations (2.7) and write $F(\omega, Y) = (F_1(\omega, Y), F_2(\omega, Y))$, so that $F(\omega, Y)$ is a function with values in $\boldsymbol{E}_2$ for $\omega \in \boldsymbol{E}_1$, $Y \in \boldsymbol{E}_2$. Then the system (2.7) may be replaced by the single equation

$$(2.8) \qquad G(\omega, Y) \equiv Y - F(\omega, Y) = 0.$$

We shall now show that hypotheses (H_1), (H_2), (H_3) of Theorem 4 of HG, p. 150 are satisfied by the functional equation (2.8) with a proper choice of regions. In fact let $\boldsymbol{Y}_1$ be the region $\boldsymbol{X\Xi}$ of the composite space $\boldsymbol{E}_1$ and let $\boldsymbol{Y}_2$ be that subset of $\boldsymbol{E}_2$ consisting of all continuous functions $Y \equiv y(t)$ for which $y(t) \in \boldsymbol{Y}_1$, when $0 \leq t \leq c$. It is clear from its definition that $\boldsymbol{Y}_2$ is a bounded convex region. Take ω_0 as the point $x = \bar{x}$, $\xi = \bar{x}$ of $\boldsymbol{E}_1$ and Y_0 as the point $x(t) = \bar{x}$, $\xi(t) = 0$ of $\boldsymbol{E}_2$. If $\boldsymbol{\Omega}_1$ is any chosen bounded convex region of $\boldsymbol{E}_1$ containing ω_0, take the region $\boldsymbol{W}_0$ of the composite space $\boldsymbol{E}_1\boldsymbol{E}_2$ as $\boldsymbol{\Omega}_1\boldsymbol{Y}_2$, so that $\boldsymbol{W}_0$ is bounded, convex and contains the point (ω_0, Y_0).

(H_1) Since $H(x, \xi)$ is homogeneous in ξ it is clear that $F_1(\omega_0, Y_0) = \bar{x}$, $F_2(\omega_0, Y_0) = 0$, so that (ω_0, Y_0) is a solution of (2.8).

(H_2) It is sufficient to show that $F_1(\omega, Y)$, $F_2(\omega, Y)$ are of class $C^{(n)}$ uniformly on $\boldsymbol{W}_0$. We give the details of proof for F_1 only, since those for F_2 are similar. Now $F_1(\omega, Y)$ can be written in the form

$$\Phi(Y) + \Lambda(\omega),$$

where Λ is a uniformly continuous linear function of ω on the bounded

region $\mathbf{\Omega}_1$, so that $F_1(\omega, Y)$ is of class $C^{(n)}$ uniformly on $\boldsymbol{W}_0$ if $\Phi(Y)$ has this property on $\boldsymbol{Y}_2$. Evidently

$$\Phi(Y) \equiv \int_0^c g(s,t)H(x(s),\xi(s))ds$$

is linear in H, while H may be considered as a function on $\boldsymbol{E}_2$ to $\boldsymbol{E}_2$ and written $H(Y)$. As a consequence of the uniformity conditions in (i) of our theorem it is easy to demonstrate that the differentials of $H(Y)$ up to and including the n-th exist for $Y \in \boldsymbol{Y}_2$. An application of Lemma 3 then shows that $H(Y)$ is actually of class $C^{(n)}$ uniformly on $\boldsymbol{Y}_2$. The function Φ being linear in $H(Y)$, is also of class $C^{(n)}$ uniformly on $\boldsymbol{Y}_2$. Consequently $F_1(\omega, Y)$, $F_2(\omega, Y)$ and $F(\omega, Y)$ are of class $C^{(n)}$ uniformly on $\boldsymbol{W}_0 = \mathbf{\Omega}_1\boldsymbol{Y}_2$.

$$(H_3) \qquad G(\omega_0, Y_0;\, \delta Y) = \delta Y - F(\omega_0, Y_0;\, \delta Y).$$

It is evident from the above discusssion that

$$F_1(\omega, Y; \delta Y) = \int_0^c g(s,t)H(x(s),\xi(s);\delta x(s),\delta\xi(s))ds$$

$$F_2(\omega, Y; \delta Y) = \int_0^c g_t(s,t)H(x(s),\xi(s);\delta x(s),\delta\xi(s))ds.$$

Since $H(x,\xi)$ is homogeneous of degree > 1 in ξ, the differential

$$H(x(s),\xi(s);\delta x(s),\delta\xi(s)) = H(x(s),\xi(s);\delta x(s)) + H(x(s),\xi(s);\delta\xi(s))$$

vanishes for $\xi(s) \equiv 0$ as is easy to see by calculating $H(x,\xi;\delta x)$ and $H(x,\xi;\delta\xi)$ by means of Gateaux's limit method. Hence for all δY, $G(\omega_0, Y_0; \delta Y) = \delta Y$ and (H_3) of Theorem 4 of HG is satisfied.

The hypotheses in Theorem 4 of HG hold. Hence by an application of this theorem the reader can readily complete the proof of our Theorem 3.

On differentiating the first equation of (2. 7), we obtain with the aid of a theorem on total differentials proved by one of us [22] the following corollary.

COROLLARY 1. *The differentials* $\phi(t, w; \delta_1\omega; \cdots; \delta_k\omega)$ $(k = 1, 2, \cdots, n)$ *are uniformly modular (Cf. Lemma 3, property* (B)) *and continuous in* (t, ω) *uniformly with respect to their arguments for* $t \in \boldsymbol{T} : t_1 < t < t_2$, $\omega \in \boldsymbol{E}^2((\bar{x})_\beta)$ *and* $\| \delta_i\omega \| \leq 1$ $(i = 1, \cdots, k)$.

COROLLARY 2. *If in Theorem 3,* $n = 1$, *then* $\phi(t, \omega)$ *is of class* $C^{(1)}$ *in* (t, ω) *uniformly on* $\boldsymbol{TE}^2((\bar{x})_\beta)$.

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[22] A. D. Michal, *Annali di Matematica, loc. cit.*

Proof. Since the pair of functions ϕ, $\partial\phi/\partial t$ satisfies (2.7) it can be shown by a simple calculation that $\partial\phi(t,\omega)/\partial t$ and $\phi(t,\omega;\delta\omega)$ are continuous in (t,ω) uniformly in $(t,\omega,\delta\omega)$ for $t \epsilon \boldsymbol{T}$, $\omega \epsilon \boldsymbol{E}^2((\bar{x})_\beta)$, $\| \delta\omega \| = 1$, and have bounded norms for the same ranges of the variables. With the aid of Corollary 1 we obtain for a preassigned $\epsilon > 0$ the inequality

$$\| \phi(t+\delta t, \omega+\delta\omega) - \phi(t,\omega) - \frac{\partial\phi(t,\omega)}{\partial t}\delta t - \phi(t,\omega;\delta\omega) \|$$
$$\leq | \delta t | \left\| \int_0^1 \left\{ \frac{\partial\phi(t+\sigma\delta t, \omega+\delta\omega)}{\partial t} - \frac{\partial\phi(t,\omega)}{\partial t} \right\} d\sigma \right\| + \epsilon \| \delta\omega \|$$

for t, $t+\delta t \epsilon \boldsymbol{T}$, $\omega \epsilon \boldsymbol{E}^2((\bar{x})_\beta)$ and $\| \delta\omega \| < \delta(\epsilon)$ ($\delta(\epsilon)$ independent of t and ω).

We return now to our study of the differential system (2.1).

THEOREM 4. *Suppose that besides satisfying hypotheses* (i), (ii) *of Lemma* 2, *the affine connection* $\Gamma(x,\xi,\eta)$ *has an n-th partial Fréchet differential* $\Gamma(x,\xi,\eta;\delta_1 x;\cdots;\delta_n x)$ $n \geq 2$, *which is continuous in* x *for* $x \epsilon (\bar{x})_{\bar{a}}$ *and for all* ξ, η. *Then there exist positive numbers* λ *and* d *such that the unique solution* $x(t,x_0,x_1)$ *of the system consisting of* (2.1) *and*

$$\| x(t,x_0,x_1) - \bar{x} \| \leq \bar{a}, \qquad \left\| \frac{\partial x(t,x_0,x_1)}{\partial t} \right\| \leq \lambda$$

has Fréchet differentials of orders $1, 2, \cdots, n-1$ *in the set* (x_0, x_1) *which are uniformly continuous in* (t, x_0, x_1) *on* $\boldsymbol{TE}^2((\bar{x})_d)$.

Proof. As in Lemma 2 one can deduce that $\Gamma(x,\xi,\eta;\delta_1 x;\cdots;\delta_k x)$ $(k = 1, 2, \cdots, n)$ is multilinear in $(\xi,\eta,\delta_1 x,\cdots,\delta_k x)$, so that constants $b \leq \bar{a}$ and P_k exist such that

$$\| \Gamma(x,\xi,\eta;\delta_1 x;\cdots;\delta_k x) \| \leq P_k \| \xi \| \, \| \eta \| \, \| \delta_1 x \| \cdots \| \delta_k x \|$$

for $x \epsilon (\bar{x})_b$ and hence, on using the integral formula for the difference repeatedly, that $\Gamma(x,\xi,\eta;\delta_1 x;\cdots;\delta_l x)$ $(l = 1, 2, \cdots, n-1)$ is continuous in (x,ξ,η) uniformly with respect to its entire set of arguments for $x \epsilon (\bar{x})_b$ and $\xi, \eta, \delta_1 x, \cdots, \delta_l x$ each in norm ≤ 1. A formula proved by one of us [23] for the total differential of a bilinear function depending on a parameter gives

$$(2.9) \qquad \Gamma(x,\xi,\xi;\delta x,\delta\xi) = \Gamma(x,\xi,\xi;\delta x) + \Gamma(x,\xi,\delta\xi) + \Gamma(x,\delta\xi,\xi).$$

Repeated applications of this formula show that the $n-1$-st total differential of $\Gamma(x,\xi,\xi)$ exists. Each term of this total differential has properties of continuity and modularity similar to those of $\Gamma(x,\xi,\eta;\delta_1 x;\cdots;\delta_l x)$, and there-

[23] A. D. Michal, *Annali di Matematica*, *loc. cit.*

fore by Lemma 3 $\Gamma(x, \xi, \xi)$ is of class $C^{(n-1)}$ uniformly on $E((\bar{x})_b)E^2((0)_\nu)$ where ν is any number ≤ 1. Hence by Theorem 3 there exist positive numbers $\beta(\nu)$ and $\lambda(\nu)$ such that $x(t, x_0, x_1)$ is of class $C^{(n-1)}$ in (x_0, x_1) uniformly on $E^2((\bar{x})_{\beta(\nu)})$ and is unique among the class of functions for which

$$\| x(t, x_0, x_1) - \bar{x} \| < \lambda(\nu), \quad \left\| \frac{\partial x(t, x_0, x_1)}{\partial t} \right\| < \lambda(\nu).$$

Theorem 4 now follows by choosing ν small enough so that $\lambda(\nu)$ satisfies the hypotheses of Theorem 2, and by applying Corollary 1 of Theorem 3.

On making special use of Theorem 4 and Corollary 2 of Theorem 3 we obtain the following additional result on geodesics.

THEOREM 5. *Suppose that besides satisfying the hypotheses of Lemma 2, the affine connection* $\Gamma(x, \xi, \eta)$ *has a second partial Fréchet differential*[24] $\Gamma(x, \xi, \eta; \delta_1 x; \delta_2 x)$ *which is continuous in* x *for* $x \in (\bar{x})_{\bar{a}}$ *and for all* ξ, η. *Then there exist positive numbers* λ *and* d *such that the solution* $x(t, x_0, x_1)$ *of the system consisting of* (2.1) *and*

$$\| x(t, x_0, x_1) - \bar{x} \| \leq \bar{a}, \qquad \| dx/dt \| \leq \lambda$$

is of class $C^{(1)}$ *uniformly on* $TE^2((\bar{x})_d)$.

If the real Banach space E is replaced by a complex Banach space,[25] i. e., a complete normed vector space closed under multiplication by complex numbers, Theorems 3, 4 and 5 are still valid. Thus, by a result on abstract analytic functions given independently by L. M. Graves and A. E. Taylor,[26] the solution $x(t, x_0, x_1)$ is analytic, and hence indefinitely Fréchet differentiable.

The attention of the reader is drawn to the fact that the validity of Theorems 2 and 5 does not depend on a particular parameterization of the geodesic $x(t, x_0, x_1)$. More precisely, the region $(\bar{x})_\delta$ is the same for all affine parameterizations.

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[24] The conclusions of Theorem 5 will still be valid if we merely require that the first partial differential $\Gamma(x, \xi, \eta; \delta x)$ be continuous in (x, ξ, η) uniformly for $x \in (\bar{x})_b$, $\xi, \eta \in (0)_\nu$ and $\| \delta x \| \leq 1$. A similar remark of course holds for Theorem 4.

[25] By a linear function in a complex Banach space we shall mean a homogeneous function of degree one which is additive and continuous. It is not possible to prove homogeneity from additivity and continuity alone as is the case for real Banach spaces.

[26] A. E. Taylor, *Bulletin of the American Mathematical Society*, November (1935); L. M. Graves, *Bulletin of the American Mathematical Society*, October (1935). For the earlier work on abstract analytic function theory see R. S. Martin, *California Institute thesis* (1932); A. D. Michal and R. S. Martin, *Journal de Mathématiques Pures et Appliquées*, vol. 13 (1934), pp. 69-91; also N. Wiener, *Fundamenta Mathematicae* (1923), for abstractly valid analytic functions of a complex variable.

The uniqueness of the path passing through two points has been shown only for the class of paths such that $\| dx/dt \| < \lambda$. It is however possible to prove a modified uniqueness theorem where no restriction is placed on dx/dt.

THEOREM 6. *There exist two neighborhoods $(\bar{x})_\sigma$, $(\bar{x})_\tau$, $(\sigma < \tau)$ such that a unique path joins any two points of $(\bar{x})_\sigma$ and lies wholly within $(\bar{x})_\tau$.*

Proof. The proof depends on the following lemma, given by Whitehead [27] for the classical case.

LEMMA 4. *There is a neighborhood $(\bar{x})_\tau$ $(\tau < \delta)$ which is simple,* i. e., *not more than one path joins any two points of $(\bar{x})_\tau$ and lies wholly within the neighborhood.*

Let $\mu(t) = \lambda(t)(t - t_0)$ where

$$\lambda(t) = \max_{t_0 \leq s \leq t} \| dx/ds \|,$$

and $\mu_0 = \lambda(t_1 - t_0)$ where λ is defined in Theorem 2; let a μ-path be one for which $\mu(t) < \mu_0$, $t_0 \leq t \leq t_1$. Then Theorem 2 states that there is a unique μ_0-path joining any two points of $(\bar{x})_\delta$. Having shown that

$$\| x(s, x_0, x_1) - x_0 \| \geq \mu(s) - \tfrac{1}{2} M \mu^2(s)$$

one can proceed exactly as in the last part of Whitehead's proof. But on taking $t_0 = 0$ for simplicity and applying Lemma 1 with

$$\Phi(t) = \Gamma[x(t, x_0, x_1), dx/dt, dx/dt], \quad c = s, \quad m = M\lambda^2(s)$$

we find that

$$\left\| \frac{dx(t, x_0, x_1}{dt} \right\| \leq \frac{sM\lambda^2(s)}{2} + \frac{\| x(s, x_0, x_1) - x_0 \|}{s}$$
$$\text{for} \quad 0 \leq t \leq s \leq t_1.$$

Now $\| dx/dt \|$ is a real continuous function of t on $0 \leq t \leq s$ and attains its maximum $\lambda(s)$ on this interval so that the required inequality and the lemma follow.

Since $\tau < \bar{a}$, Theorem 2 is valid with $\bar{a}$ replaced by τ, and the proof of Theorem 6 is complete.

3. *A differential equation in a normed ring.* Let $\boldsymbol{E}$ be a Banach space in which there exists on $\boldsymbol{E}^2$ to $\boldsymbol{E}$ an associative bilinear function denoted by xy.

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[27] J. H. C. Whitehead, *Quarterly Journal of Mathematics*, vol. 3 (1932), pp. 38-39.

For simplicity, let the modulus of this bilinear function be unity, i. e., $\| xy \| \leq \| x \| \; \| y \|$. We also require that there exist a unit element I in $\boldsymbol{E}$. Clearly the functions $x + y$ and xy satisfy the postulates for an abstract ring.[28]

If the affine connection is taken to be

$$\Gamma(x, \xi, \eta) = \xi\eta,$$

the corresponding differential system for the paths, passing through two points is

$$(3.1) \qquad \begin{cases} d^2x/dt^2 = (dx/dt)^2 \\ x(0) = x_0, \qquad x(t_1) = x_1. \end{cases}$$

It is perhaps worthy of note that this extremely simple affinely connected space is not in general "flat," since the curvature tensor[29]

$$B(x, \xi, \delta_1 x, \delta_2 x) = \xi\delta_1 x\delta_2 x - \xi\delta_2 x\delta_1 x$$

vanishes identically if and only if the ring is commutative.

The system (3.1) can be integrated by means of elementary functions. In fact define $ln(I + x)$ and e^x as the respective power series

$$\sum_{n=1}^{\infty} ((-1)^{(n-1)}/n)x^n \quad \text{when} \quad \| x \| < 1$$

and

$$I + \sum_{n=1}^{\infty} (x^n/n!).$$

Then we can show that for sufficiently small values of $\| x_0 - x_1 \|$, the system (3.1) has the solution

$$(3.2) \qquad \phi(t, x_0, x_1) = x_0 - ln[I - (t/t_1)(I - e^{x_0 - x_1})]$$

By using a method similar to that of J. von Neumann[30] in the case of a finite matric ring, it may be shown that the relation $ln(e^x) = x$ holds in our normed ring providing that $\| I - e^x \| < 1$. Hence if $\| I - e^{x - x_0} \| < 1$, the boundary conditions are satisfied by the function $\phi(t, x_0, x_1)$. The ordinary rules for differentiation of a power series can be extended to the case where the coefficients are elements of our normed ring. Moreover Cauchy's rule for the multiplication of absolutely convergent series holds here. Hence, with the aid of Lemmas 1 and 2 and the uniqueness proof of Theorem 1, we have

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[28] Van der Waerden, *Moderne Algebra*, vol. 1 (Berlin, 1930), pp. 36-37.

[29] A. D. Michal, *Annali di Matematica*, *loc. cit.*

[30] *Mathematische Zeitschrift*, vol. 30 (1929), p. 9.

THEOREM 7. *The function* $\phi(t, x_0, x_1)$ *defined in* (3.2) *satisfies the differential system* (3.1) *for* $0 \leq t \leq t_1$ *and for* x_0, x_1 *such that*

$$\| I - e^{x_1 - x_0} \| < 1.$$

If moreover

$$\| x_0 - x_1 \| < \lambda t_1/3 < 1/3,$$

then $\phi(t, x_0, x_1)$ *is the unique solution among the class of functions for which*

$$\| dx/dt \| < \lambda.$$

Given the power series

$$f(x) = \sum_n a_n x^n,$$

where a_n, x are elements of our ring, define

$$h_n(x_1, x_2, \cdots, x_n) = \sum (a_n/n!) x_{i_1} x_{i_2} \cdots x_{i_n},$$

the summation extending over all permutations of $x_1, x_2, \cdots, x_n$. It may be proved that the r-th differential of $f(x)$ exists for x in the region of convergence of $\sum a_n x^n$ and is given by

$$f(x; \delta_1 x; \delta_2 x; \cdots; \delta_r x) = \sum_{n=r}^{\infty} (n!/r!) h_n(x, x, \cdots, x, \delta_1 x, \delta_2 x, \cdots, \delta_r x).$$

Consequently the total differentials of all orders of $\phi(t, x_0, x_1)$ exist for $\| I - e^{x_0 - x_1} \| < 1$.

THEOREM 8. *The solution* $\phi(t, x_0, x_1)$ *of the differential system* (3.1) *has total Fréchet differentials of all orders for* $\| I - e^{x_0 - x_1} \| < 1$.

Besides the well known finite matric examples of a normed ring, we may take the class of ordered pairs of real functions $(x(r, s), x(u))$ with suitably defined operations,[31] where $x(r, s)$ and $x(r)$ are continuous for r, s in (0.1). Then the following integro-differential system is an instance of system (3.1).

$$\begin{cases} \dfrac{\partial^2 x(r, s, t)}{\partial t^2} = \displaystyle\int_0^1 \dfrac{\partial x(r, p, t)}{\partial t} \dfrac{\partial x(p, s, t)}{\partial t} dp \\ \qquad\qquad + \dfrac{\partial x(r, s, t)}{\partial t} \dfrac{\partial x(r, t)}{\partial t} + \dfrac{\partial x(r, s, t)}{\partial t} \dfrac{\partial x(s, t)}{\partial t} \\ \dfrac{\partial^2 x(r, t)}{\partial t^2} = \left(\dfrac{\partial x(r, t)}{\partial t} \right)^2 \\ x(r, s, 0) = x_0(r, s), \qquad x(r, 0) = x_0(r) \\ x(r, s, t_1) = x_1(r, s), \qquad x(r, t_1) = x_1(r). \end{cases}$$

[31] Compare with Evans' and Volterra's functional algebra. See Evans, *loc. cit.*, and Volterra, *loc. cit.* See also A. D. Michal and R. S. Martin, *loc. cit.*

4. *An integro-differential system in a functional geometry.* In certain functional differential geometries[32] the following integro-differential system is the analogue of the equations for the paths in a finite dimensional affine geometry.

$$(4.1)\quad \begin{cases} \dfrac{\partial^2 x^i(s)}{\partial s^2} + \Gamma_{\alpha\beta}{}^i \dfrac{\partial x^\alpha}{\partial s}\dfrac{\partial x^\beta}{\partial s} + M_\alpha{}^i\left(\dfrac{\partial x^\alpha}{\partial s}\right)^2 + 2N_\alpha{}^i \dfrac{\partial x^i}{\partial s}\dfrac{\partial x^\alpha}{\partial s} + P^i\left(\dfrac{\partial x^i}{\partial s}\right)^2 = 0, \\ \Gamma_{\alpha\beta}{}^i = \Gamma_{\beta\alpha}{}^i, \qquad x^i(s_0) = x_0{}^i, \quad x^i(s_1) = x_1{}^i. \end{cases}$$

Take the Banach space $\boldsymbol{E}$ of Theorem 2 to be the space of real functions x^i continuous over the interval (σ, τ) with $\| x^i \| = \max | x^i |$.

To satisfy the hypotheses (i) and (ii) of Lemma 2, we require that the functionals Γ, M, N, P be bounded and have continuous values when i, α, β are in (σ, τ) and x^i satisfies $\max\limits_i | x^i - \bar{x}^i | < a$. Let $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ be the spaces of continuous functions of two and three variables respectively with each variable ranging over (σ, τ) and the norm taken as the maximum of the absolute values. Then $\Gamma_{\alpha\beta}{}^i$, for example, is a function on $\boldsymbol{E}$ to $\boldsymbol{E}_3$. For hypothesis (iii) to be satisfied it is sufficient that $\Gamma_{\alpha\beta}{}^i$, $M_\alpha{}^i$, $N_\alpha{}^i$, P^i have continuous first Fréchet differentials provided we take $\Gamma(x, \xi, \eta)$ to be

$$\Gamma_{\alpha\beta}{}^i\xi^\alpha\eta^\beta + M_\alpha{}^i\xi^\alpha\eta^\alpha + N_\alpha{}^i\xi^i\eta^\alpha + N_\alpha{}^i\xi^\alpha\eta^i + P^i\xi^i\eta^i.$$

Theorem 2 can now be applied immediately to the integro-differential system (4.1).

If in addition to the above hypotheses on the functionals Γ, M, N, P, we require that the second or higher order differentials of these functionals exist and are continuous the hypotheses of Theorems 4 and 5 are also satisfied.

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[32] A. D. Michal, *Proceedings of the National Academy of Sciences*, vol. 16 (January, 1930), pp. 88-94; *American Journal of Mathematics*, *loc. cit.*

In equation (4.1), the indices are continuous real variables on the interval (σ, τ) and a repetition of a subscript and superscript in a term denotes integration over (σ, τ). The values of the functionals $\Gamma_{\alpha\beta}{}^i[x^\lambda]$, $M_\alpha{}^i[x^\lambda]$, $N_\alpha{}^i[x^\lambda]$, $P^i[x^\lambda]$ are real continuous functions of three, two, two and one real variables respectively.

UNIVERSAL HOMOLOGY GROUPS.[1]

By Norman E. Steenrod.

Introduction. The development of algebraic topology of the past ten years has been marked by a constant effort to extend its principal theorems to ever more general spaces. Outstanding in this respect has been the endeavor to extend the Alexander duality theorem, the theory of manifolds, and to bring within the framework the theory of dimensionality. These investigations have compelled the introduction of homology groups based on chains and cycles with coefficients in a general topological abelian group.

There have been many reasons to suspect that one does not need to consider all possible coefficient groups to obtain a complete set of homology relations in a space. Thus, in the case of a complex, it has been proved that it is sufficient to know the homology groups based on integer coefficients. These groups and an abelian group $\mathfrak{G}$ determine completely the homology groups based on chains with coefficients in $\mathfrak{G}$. We say therefore that the integers form a universal coefficient group for the homology theory of a complex.

There are examples to show that this theorem does not hold if "complex" is replaced by "compact metric space." Alexandroff [5][2] has raised the problem of finding a universal coefficient group for a compact metric space. He surmised that the group $\mathfrak{X}$ of real numbers reduced modulo 1 is such a group. A proof of this is contained in the present paper. In fact, using Čech's definition [8] of the homology groups, we shall prove that $\mathfrak{X}$ is universal for the homology theory of a general topological space. To express the result differently: *the use of other coefficient groups cannot lead to any new topological invariants.* It will also be shown that $\mathfrak{X}$ is universal for the homology theory of the infinite cycles of an infinite complex. All of these results hold for relative homology groups in the sense of Lefschetz [15].

There is a dual result. Using the procedure of Čech, one may define, in the manner of Alexander [1], the dual homology groups of a topological space. Then the group of integers (the character group of $\mathfrak{X}$) is a universal coefficient group for this homology theory. This generalizes in a sense a theorem of Čech [9] which asserts that the integers are universal for the homology theory of the finite cycles of an infinite complex.

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There is a dual result. Using the procedure of Čech, one may define, in the manner of Alexander [1], the dual homology groups of a topological space. Then the group of integers (the character group of $\mathfrak{X}$) is a universal coefficient group for this homology theory. This generalizes in a sense a theorem of Čech [9] which asserts that the integers are universal for the homology theory of the finite cycles of an infinite complex.

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These results have the implication that, in any investigation into the relations between a space and its homology groups, one obtains a theory with a maximum content by using either dual homology groups with integer coefficients or ordinary homology groups with coefficients in $\mathfrak{X}$. The two procedures are entirely equivalent, for the two types of groups of the same dimension are character groups of one another (Pontrjagin [17]). Since the dual homology groups with integer coefficients are somewhat easier to handle, this result constitutes a strong argument for the future exclusive use of these groups.

In part I we formalize a well-known tool of the topologist: mapping systems and homomorphism systems. These notions are found first in the projection spectrum of Alexandroff [4], and they have since been used by Pontrjagin [16], Lefschetz [15], and others. In keeping with the ideas of Čech [8], we do not restrict ourselves to sequences but admit partially ordered systems. This requires a complete treatment of the subject. By using the van Kampen [13] extension of the character group theory of Pontrjagin [17], we establish two theorems on the representations of bicompact abelian groups as limit groups of homomorphism systems of finite dimensional, locally-connected groups.

In part II we consider first the homology groups of a finite complex. We give the structure of these groups in terms of the coefficient group and known invariants of the complex. A proof of this decomposition, omitting all continuity considerations, can be found in the book of Alexandroff and Hopf [7; pp. 228-240]. Then following Čech [8], we define the homology groups of a general topological space. We establish the decomposition of these groups into direct sums of "torsion groups" and "reduced homology groups." In this is contained a corresponding direct sum theorem for the dual homology groups.

Part III contains the proof proper that $\mathfrak{X}$ is a universal coefficient group for a topological space. In part IV we indicate the modifications necessary to obtain the same theorem for an infinite complex.

The first appendix contains a proof that a connected, bicompact, abelian group is continuously isomorphic with its 1-dimensional homology group over $\mathfrak{X}$. This indicates a probability that almost any bicompact abelian group can appear as a modulo 1 homology group of some space. In the second appendix we give a number of examples to support earlier statements of the paper.

This investigation was made under the guidance of Professor S. Lefschetz. I am indebted to him and to Professor A. W. Tucker for many helpful suggestions. I am likewise indebted to Dr. Leo Zippin for general criticisms

and for constant assistance with the group theoretic considerations which, it is clear, occur throughout.

Certain results of this paper were obtained independently by Čech [10]. During the past year the author has had the benefit of many conversations with Professor Čech. These have contributed materially to the generality of our results.

I. Homomorphism Systems.

1. Definitions and notation. We shall have occasion to use only commutative groups; we may therefore, without confusion, omit the word commutative. All groups are written additively. Groups are denoted by capital German letters and their elements by small German letters.

By *topological space* we mean a set in which certain subsets are called open so that the axioms 1 to 4 of Hausdorff [12, p. 228] are satisfied. In particular we use only the separation axiom which asserts that a point is closed. Spaces and their elements are denoted by Latin letters.

A *bicompact space* is a topological space with the property that every covering by open sets contains a finite covering. A closed subset of a bicompact space is bicompact;[3] and a continuous image of a bicompact space is bicompact. By interpreting the definition of bicompactness in terms of the complementary closed sets, one obtains the following characterization:

LEMMA 1.1. *For a space to be bicompact it is necessary and sufficient that, for any collection* $\{F^\alpha\}$ *of closed subsets,* $\prod F^\alpha \neq 0$ *if this relation holds for each finite subcollection of* $\{F^\alpha\}$.

A *topological group* is a group whose elements are the points of a topological space whose neighborhoods satisfy the additional axiom:

α) if U is a neighborhood of $\mathfrak{x} - \mathfrak{y}$ there are neighborhoods V and W of $\mathfrak{x}$ and $\mathfrak{y}$ respecitvely such that $\mathfrak{x}' - \mathfrak{y}' \in U$ whenever $\mathfrak{x}' \in V$ and $\mathfrak{y}' \in W$.

Since the zero point is closed, it follows from α) that a topological group satisfies the separation axiom 5 of Hausdorff.

By setting $\mathfrak{x} = 0$ in α), we find that $f(\mathfrak{y}) = -\mathfrak{y}$ is a continuous function of $\mathfrak{y}$. Combining this with α) we obtain that $f(\mathfrak{x}, \mathfrak{y}) = \mathfrak{x} + \mathfrak{y}$ is continuous simultaneously in $\mathfrak{x}$ and $\mathfrak{y}$. In general $\sum_{i=1}^{n} a_i \mathfrak{x}_i$ ($a_i =$ integer) is continuous simultaneously in all its variables.

If $\mathfrak{A}$ and $\mathfrak{B}$ are topological groups and H is a homomorphic mapping of

[3] For proofs of these statements and the following lemma see Alexandroff and Urysohn [6]. They assume that a bicompact space has the separation axiom 5 of Hausdorff, but this is not essential.

$\mathfrak{A}$ into $\mathfrak{B}$ which is continuous at the zero of $\mathfrak{A}$, then, by a simple application of α), H is continuous at each point of $\mathfrak{A}$. Thus to prove the continuity of a homomorphism it suffices to prove its continuity at the zero.

If $\mathfrak{B}$ is a subgroup of $\mathfrak{A}$, the residue (factor) group $\mathfrak{A}$ mod $\mathfrak{B}$ is denoted $\mathfrak{A} - \mathfrak{B}$. If $\mathfrak{A}$ is a topological group and $\mathfrak{B}$ is closed in $\mathfrak{A}$, we may designate as neighborhoods in $\mathfrak{A} - \mathfrak{B}$ the images of neighborhoods of $\mathfrak{A}$. Then, since $\mathfrak{B}$ is closed, it can be proved that $\mathfrak{A} - \mathfrak{B}$ is a topological group.

A topological group in which each point is a neighborhood of itself is called a *discrete group.*

Let $\{\mathfrak{A}^\alpha\}$ be a well-ordered collection of topological groups. By the *direct sum* $\sum \mathfrak{A}^\alpha$ we mean the topological group whose elements are collections $\{\mathfrak{a}^\alpha\}$, $\mathfrak{a}^\alpha \epsilon \mathfrak{A}^\alpha$, in which difference is defined by $\{\mathfrak{a}^\alpha\} - \{\mathfrak{a}'^\alpha\} = \{\mathfrak{a}^\alpha - \mathfrak{a}'^\alpha\}$, and a neighborhood of $\{\mathfrak{a}^\alpha\}$ is defined by a finite number of indices α_i ($i = 1, \cdots, k$) and neighborhoods V^{α_i} of the coördinates $\mathfrak{a}^{\alpha_i}$ and consists of all points $\{\mathfrak{a}'^\alpha\}$ such that $\mathfrak{a}'^{\alpha_i} \epsilon V^{\alpha_i}$ ($i = 1, \cdots, k$). It is not difficult to prove

LEMMA 1.2. *If $\mathfrak{A} = \sum \mathfrak{A}^\alpha$ and the subgroup $\mathfrak{B}$ of $\mathfrak{A}$ is the direct sum $\sum \mathfrak{B}^\alpha$ where $\mathfrak{B}^\alpha$ is a closed subgroup of $\mathfrak{A}^\alpha$, then $\mathfrak{A} - \mathfrak{B}$ and $\sum (\mathfrak{A}^\alpha - \mathfrak{B}^\alpha)$ are bicontinuously isomorphic.*

2. Inverse mapping systems. Let $\{A^\alpha\}$ be a collection of topological spaces (the index α runs over the set of ordinals less than a fixed but arbitrary ordinal κ). Suppose that to some pairs (α, β) of ordinals $< \kappa$ there corresponds a continuous mapping $M_\beta{}^\alpha$ of A^β into A^α, so that the following conditions are satisfied:

a) $M_\alpha{}^\alpha$, for each α, is defined and is the identity,

b) if $M_\gamma{}^\beta$ and $M_\beta{}^\alpha$ are defined, then $M_\gamma{}^\alpha$ is defined and is the product of the two transformations: $M_\gamma{}^\alpha = M_\beta{}^\alpha M_\gamma{}^\beta$,

c) to each pair (α, β) of ordinals $< \kappa$ corresponds a γ such that $M_\gamma{}^\alpha$ and $M_\gamma{}^\beta$ are defined.

Such a collection of spaces and continuous mappings we shall call an *inverse system.*

The existence of $M_\beta{}^\alpha$ we shall indicate by $A^\beta \to A^\alpha$; $M_\beta{}^\alpha$ is called a *projection* of the system. The space A^γ postulated in c) is called a *common refinement* of A^α and A^β. Iteration of c) shows that any finite number of spaces of the system have a common refinement.

We define the limit space A of an inverse system S. The points of A shall be all collections $\{a^\alpha\}$ consisting of one point a^α from each A^α such that, if $A^\beta \to A^\alpha$, then $M_\beta{}^\alpha(a^\beta) = a^\alpha$. The point a^α is said to be the coördinate in

A^α of the point $a = \{a^\alpha\}$ of A. If V^β is a neighborhood of the fixed coördinate a^β of a, the set V in A of all points $\tilde{a} = \{\tilde{a}^\alpha\}$ satisfying $\tilde{a}^\beta \epsilon V^\beta$ is said to be a neighborhood of a in A. Allowing V^β to range over a complete set of neighborhoods of a^β and β to range over all ordinals $< \kappa$, we define in this way a complete set of neighborhood of a in A. We must verify that these neighborhoods make of A a topological space.

It is trivial that $a \epsilon V$. Suppose $\tilde{a} \epsilon V$; then let $\tilde{V}^\beta$ be a neighborhood of $\tilde{a}^\beta$ lying in V^β. It follows that $\tilde{V}^\beta$ determines a neighborhood $\tilde{V}$ of $\tilde{a}$ contained in V. Suppose V and V' are two neighborhoods of a determined by V^α and V^β respectively. Let A^γ be a common refinement of A^α and A^β. As $M_\gamma{}^\alpha$ and $M_\gamma{}^\beta$ are continuous, there is a neighborhood V^γ of a^γ such that $M_\gamma{}^\alpha(V^\gamma) \subset V^\alpha$ and $M_\gamma{}^\beta(V^\gamma) \subset V^\beta$. Then V^γ determines a neighborhood V'' of a contained in V and V'. Suppose a and b are distinct points of A. Then, for some β, their coördinates a^β and b^β are distinct. Then a neighbor- V^β of a^β not containing b^β determines a neighborhood V of a not containing b. This proves that A is a topological space. We remark that, if in each A^α, two points can be separated by neighborhoods, the same holds true in A.

The mapping $M^\alpha(a) = a^\alpha$ of A into A^α is continuous and is called the projection of A into A^α. If A and the projections M^α are added to S, we obtain a new inverse system whose limit space is A.

If each A^α is regular, A is regular. If the system S is countable, then the first (second) countability axiom in each A^α implies the first (second) countability axiom in A. If, in each A^α, we substitute an equivalent system of neighborhoods, we obtain an equivalent system of neighborhoods in A.

As we shall be frequently concerned with inverse systems of bicompact spaces, we prove the following important lemma for such systems.

LEMMA 2.1. *If S is an inverse system of bicompact spaces, and, for a fixed δ, a^δ is a point of A^δ such that $A^\beta \to A^\delta$ implies $M_\beta{}^\delta(A^\beta) \supset a^\delta$, there is a point a of the limit space A such that $M^\delta(a) = a^\delta$.*

We suppose the spaces of S are well-ordered so that A^δ occurs as the first element (i. e. we suppose $\delta = 1$). The coördinates of the point a are chosen inductively. The first coördinate is a^1. Suppose we have chosen a^α for each $\alpha < \beta$ so that, if A^γ is a common refinement of $A^{\alpha_1}, \cdots, A^{\alpha_k}$ $(\alpha_i < \beta)$, the sets $F^{\alpha_1}, \cdots, F^{\alpha_k}$ of inverse images in A^γ of $a^{\alpha_1}, \cdots, a^{\alpha_k}$, respectively, have a non-vacuous intersection. Clearly the choice of a^1 satisfies this condition. Suppose $\alpha < \beta$ and $A^{\gamma(\alpha)}$ is a refinement of A^α and A^β. Denote by $F^{\gamma(\alpha)}$ the image in A^β of the inverse image in $A^{\gamma(\alpha)}$ of a^α. We assert that

$\prod_{\substack{\gamma(a)\\ a<\beta}} F^{\gamma(a)}$ is non-vacuous. As $F^{\gamma(a)}$ is closed, by Lemma 1. 1, it is sufficient to prove that $\prod_{i=1}^{k} F^{\gamma(a_i)}$ is non-vacuous for any finite set $\gamma(\alpha_i)$ $(i=1,\cdots,k)$. Let A^γ be a refinement of $A^{\gamma(a_i)}$ $(i=1,\cdots,k)$. By the hypothesis of the induction, the inverse images in A^γ of $a^{a_1},\cdots,a^{a_k}$ have a non-vacuous intersection F^γ whose image in A^β lies in each $F^{\gamma(a_i)}$. Thus $\prod F^{\gamma(a)}$ is non-vacuous, and we may select the coördinate a^β of a in this set. The hypothesis of the induction is still satisfied. For, if A^γ is a refinement of $A^\beta, A^{a_1},\cdots, A^{a_k}$, we may find a common refinement A^ϵ of A^γ and $A^{\gamma(a_i)}$ $(i=1,\cdots,k)$. Since A^ϵ is a refinement of $A^{\gamma(a_i)}$, the projection on A^β of the inverse image of a^{a_i} in A^ϵ equals $F^{\gamma(a_i)}$. As $a^\beta \epsilon F^{\gamma(a_i)}$, the inverse images of a^β and a^{a_i} $(i=1,\cdots,k)$ in A^ϵ have a non-vacuous intersection. The projection on A^γ of this intersection is common to the inverse images in A^γ of a^β and a^{a_i} $(i=1,\cdots,k)$.

The procedure outlined determines a collection $\{a^a\}$ of one point from each space. If $A^\beta \to A^a$, since A^β is a common refinement of A^a and A^β, the inverse images in A^β of a^a and a^β have a non-vacuous intersection. Thus a^β lies in the inverse image of a^a. This means $M_\beta{}^a(a^\beta)=a^a$; and $\{a^a\}$ is the desired point of A.

THEOREM 2. 1. *The limit space of an inverse system of bicompact spaces is non-vacuous and bicompact.*

If $A^\beta \to A^1$, let $A_1{}^\beta$ be the image of A^β in A^1. Now each $A_1{}^\beta$ is closed and any finite number of sets $A_1{}^{\beta_i}$ $(i=1,\cdots,k)$ have a non-vacuous intersection; for, if A^γ is a refinement of A^{β_i} $(i=1,\cdots,k)$, $A_1{}^\gamma$ lies in this intersection. Thus, by Lemma 1. 1, $\prod A_1{}^\beta$ is non-vacuous; and there is a point a^1 of A^1 contained in the image of each A^β such that $A^\beta \to A^1$. By Lemma 2. 1, there is a point a of A such that $M^1(a)=a^1$. Thus A is non-vacuous.

Let F be a closed set in A. We shall prove that $F^a = M^a(F)$ is closed in A^a. If $A^\beta \to A^a$, the statement $M_\beta{}^a(F^\beta)=F^a$ implies $M_\beta{}^a(\bar{F}^\beta)=\bar{F}^a$ (the bar denoting the closure). Suppose $\bar{F}^a - F^a$ contains a point a^a. By Lemma 2. 1, the limit space of the inverse system $\{\bar{F}^\beta\}$ of bicompact spaces contains a point a whose projection in $\bar{F}^a$ is a^a. Clearly $a \epsilon A$. In each neighborhood of this point are points of F; hence $a \epsilon \bar{F}=F$. This implies $a^a \epsilon F^a$ which is impossible. Therefore $F^a=\bar{F}^a$.

To prove that A is bicompact we shall use the sufficiency of the condition of Lemma 1. 1. Let $\{F_\lambda\}$ be any collection of closed sets in A such that any finite subcollection has a non-vacuous intersection. Let $F_\lambda{}^a$ be the projection of F_λ in A^a. As proved above, $F_\lambda{}^a$ is closed. Any finite number of the sets

$F_\lambda{}^\alpha$ (α fixed) have a non-vacuous intersection; for this intersection contains the projection in A^α of the intersection of the corresponding sets in A. Therefore $\prod_\lambda F_\lambda{}^\alpha = F^\alpha$ is non-vacuous. If $A^\beta \rightarrow A^\alpha$, it is obvious that $M_\beta{}^\alpha(F^\beta) \subset F^\alpha$. By the first part of the theorem, the inverse system consisting of the projections $\{M_\beta{}^\alpha\}$ and the bicompact spaces $\{F^\alpha\}$ has a non-vacuous limit set $F \subset A$. In each neighborhood of a point of F there are points of F_λ; hence $F \subset \bar{F}_\lambda = F_\lambda$. As λ is arbitrary, $F \subset \prod F_\lambda$; and the theorem is proved.

We shall need later the following lemma whose proof properly belongs here.

LEMMA 2.2. *If $S = \{A^\alpha\}$ is an inverse system of bicompact spaces and U is an open set of A^α containing the image $M^\alpha(A)$ of the limit space A of S, then there is a refinement A^β of A^α such that $M_\beta{}^\alpha(A^\beta) \subset U$.*

If, to the contrary, the image in A^α of each refinement A^β of A^α meets $A^\alpha - U$, the collection of closed sets consisting of $A^\alpha - U$ and these images will have a non-vacuous intersection (use Lemma 1.1). By Lemma 2.1, this intersection which is a subset of $A^\alpha - U$ contains the coördinate of a point of A. The contradiction proves the lemma.

3. Equivalence of inverse systems. If S_1 and S_2 are two inverse systems, and the spaces and mappings of S_2 are included among those of S_1, then S_2 is called a *subsystem* of S_1. A subsystem S_2 of S_1 is said to be *complete* if each space of S_1 has a refinement in S_2.

LEMMA 3.1. *If S_2 is a complete subsystem of S_1, the limit spaces of S_1 and S_2 are homeomorphic.*

The coördinates of a point of the limit space A_1 of S_1 in the spaces of S_2 are the coördinates of a point of the limit space A_2 of S_2. The mapping of A_1 into A_2 so defined is $1:1$ and bicontinuous. First, the image of A_1 in A_2 covers A_2; for, any point a_2 of A_2 has a projection in each space of S_2 and therefore, since S_2 is complete in S_1, a unique projection in each space of S_1. The set of these projections are the coördinates of a point a_1 of A_1 whose image in A_2 is a_2. Next, if a_1 and a'_1 are distinct in A_1, their images in A_2 are distinct. The distinctness of a_1 and a'_1 implies that of their projections $a_1{}^\alpha$ and $a'_1{}^\alpha$ for some α. Let A^β be a refinement of A^α in S_2. Then $a_1{}^\beta$ and $a'_1{}^\beta$ are distinct; but this implies that a_2 and a'_2 are distinct. Finally, the correspondence is bicontinuous. For, if V_2 is a neighborhood of a_2 determined by V^β in A^β (A^β in S_2), the neighborhood V_1 of a_1 determined by V^β images into V_2. If V_1 is a neighborhood of a_1 determined by V^α in A^α, let V^β be a neighborhood of $a_1{}^\beta$ in the refinement A^β of A^α (A^β in S_2) whose image in A^α

lies in V^α. Then V^β determines a neighborhood V_2 of a_2 covered by the image of V_1.

Two inverse systems are said to be *immediately equivalent* if both are complete subsystems of a third. Now it is not yet clear that this relation is a transitive one; so we define the following clearly transitive relation. Two inverse systems S and S' are said to be *equivalent* if there exists a finite ordered set of inverse systems $S, S_1, \cdots, S_k, S'$ such that neighboring pairs are immediately equivalent. As an immediate consequence of this definition and Lemma 3. 1, we have

THEOREM 3. 1. *Equivalent inverse systems have homeomorphic limit spaces.*

A countable inverse system is called an *inverse sequence* if, for any pair of its spaces, at least one is a refinement of the other. A simple induction proves that any countable inverse system contains a complete subsequence.

4. Inverse homomorphism systems. An inverse system is said to be an *inverse homomorphism system* if each space of the system is a topological group and each mapping of the system is a continuous homomorphism.[4] The homomorphisms of the system are denoted by $H_\beta{}^\alpha$. In the expression "inverse homomorphism system" we shall omit the word "homomorphism" when it is clear that we are dealing with groups.

The limit space $\mathfrak{A}$ of the inverse system $S = \{\mathfrak{A}^\alpha\}$ is converted into a topological group by the following definition of addition. If $\mathfrak{a}_1 = \{\mathfrak{a}_1{}^\alpha\}$, $\mathfrak{a}_2 = \{\mathfrak{a}_2{}^\alpha\}$ are two points of $\mathfrak{A}$, then $\mathfrak{a}_1 + \mathfrak{a}_2$ is $\{\mathfrak{a}_1{}^\alpha + \mathfrak{a}_2{}^\alpha\}$ which is clearly an element of $\mathfrak{A}$. The zero of $\mathfrak{A}$ is $\{0\}$, and $-\{\mathfrak{a}^\alpha\} = \{-\mathfrak{a}^\alpha\}$. It is a simple matter to prove that $\mathfrak{A}$ is a group. We consider now the axiom α) of No. 1. Let U be a neighborhood of $\mathfrak{a}_1 - \mathfrak{a}_2$ determined by U^α in $\mathfrak{A}^\alpha$. As $\mathfrak{A}^\alpha$ is a topological group, there are neighborhoods V^α and W^α of $\mathfrak{a}_1{}^\alpha$ and $\mathfrak{a}_2{}^\alpha$, respectively, as postulated in α). Then V^α and W^α determine neighborhoods V and W in $\mathfrak{A}$ satisfying α). Thus $\mathfrak{A}$ is a topological group. Let us remark that $\mathfrak{a} \to \mathfrak{a}^\alpha$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}^\alpha$.

As in No. 3, we define for inverse homomorphism systems the notions of subsystem, complete subsystems, and equivalence. Noting that, for an inverse homomorphism system, the correspondence set up in Lemma 3. 1 is an isomorphism, we have

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[4] It is clear that a similar definition can be made for non-commutative groups, and the limit group described below exists.

THEOREM 4. 1. *Equivalent inverse homomorphism systems have bicontinuously isomorphic limit groups.*

5. Direct homomorphism systems. Let $\{\mathfrak{B}^\alpha\}$ be a well-ordered collection of groups and, for some pairs $\mathfrak{B}^\alpha$, $\mathfrak{B}^\beta$ let there be defined a homomorphism $H^*{}_\beta{}^\alpha$ of $\mathfrak{B}^\beta$ into $\mathfrak{B}^\alpha$ so that axioms a) and b) of No. 2 are satisfied and, in place of c), the axiom

c') to each pair (α, β) corresponds a γ such that $H^*{}_\alpha{}^\gamma$ and $H^*{}_\beta{}^\gamma$ are defined.

Such a system S^* of groups and homomorphisms we shall call a *direct homomorphism system.*

We define the limit group $\mathfrak{B}$ of the direct system S^*. An element of $\mathfrak{B}$ will be a collection $\{\mathfrak{b}^\alpha\}$ of one element from each of a subcollection of the groups of S^* satisfying the condition that, if $\mathfrak{B}^\alpha \to \mathfrak{B}^\beta$ and $\mathfrak{B}^\alpha$ is in the subcollection, so also is $\mathfrak{B}^\beta$ and $H^*{}_\alpha{}^\beta(\mathfrak{b}^\alpha) = \mathfrak{b}^\beta$. Two elements of $\mathfrak{B}$ are equal if in some group of S^* their coördinates are defined and are equal. Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two elements of $\mathfrak{B}$. Consider the collection of groups of S^* in each of which the coördinates of both $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are defined. In each group of this collection let us add the coördinates of $\mathfrak{b}_1$ and $\mathfrak{b}_2$. The sums so obtained determine an element of $\mathfrak{B}$ called the sum $\mathfrak{b}_1 + \mathfrak{b}_2$ of $\mathfrak{b}_1$ and $\mathfrak{b}_2$. It is verified without difficulty that $\mathfrak{B}$ constitutes a group with addition so defined.

If $\mathfrak{b}^\alpha$ is an element of $\mathfrak{B}^\alpha$ the collection of its images in each $\mathfrak{B}^\beta$ such that $\mathfrak{B}^\alpha \to \mathfrak{B}^\beta$ determines an element $\mathfrak{b}$ of $\mathfrak{B}$. The correspondence $H^*{}_\alpha(\mathfrak{b}^\alpha) = \mathfrak{b}$ so defined is a homomorphism.

We assign to $\mathfrak{B}$ the discrete topology. The justification will be found in the theorem of the next section. We might remark that, in case each $\mathfrak{B}^\alpha$ is a topological group and each $H^*{}_\beta{}^\alpha$ is continuous, the method of No. 2 defines neighborhoods in $\mathfrak{B}$ which in general fail to satisfy the axiom that two neighborhoods of a point contain a third in their common part. Other obvious methods fail to make of $\mathfrak{B}$ a topological group.

6. Dual systems. Let $\mathfrak{X}$ be the group of real numbers reduced modulo 1. Let S be an inverse system of bicompact groups $\{\mathfrak{A}^\alpha\}$. Let $\mathfrak{B}^\alpha$ be the discrete group of continuous characters [5] of $\mathfrak{A}^\alpha$. The element of $\mathfrak{X}$ determined by $\mathfrak{b}^\alpha \in \mathfrak{B}^\alpha$ and $\mathfrak{a}^\alpha \in \mathfrak{A}^\alpha$ we denote by $(\mathfrak{a}^\alpha, \mathfrak{b}^\alpha)$. Suppose $\mathfrak{A}^\beta \to \mathfrak{A}^\alpha$. The result of the homomorphisms $\mathfrak{A}^\beta \to \mathfrak{A}^\alpha$ then $\mathfrak{A}^\alpha \to \mathfrak{X}$ as determined by the element $\mathfrak{b}^\alpha \in \mathfrak{B}^\alpha$ is a continuous character $\mathfrak{b}^\beta$ of $\mathfrak{A}^\beta$. The correspondence $H^*{}_\alpha{}^\beta(\mathfrak{b}^\alpha) = \mathfrak{b}^\beta$

[5] A continuous character of $\mathfrak{A}$ is a continuous homomorphic mapping of $\mathfrak{A}$ into the group $\mathfrak{X}$ (see [17; p. 362]).

so obtained is a homomorphism of $\mathfrak{B}^{\alpha}$ into a subgroup of $\mathfrak{B}^{\beta}$. This homomorphism is defined to satisfy

$$(6.1) \qquad (\mathfrak{a}^{\beta}, H^*{}_{\alpha}{}^{\beta}(\mathfrak{b}^{\alpha})) = (H_{\beta}{}^{\alpha}(\mathfrak{a}^{\beta}), \mathfrak{b}^{\alpha})$$

for all $\mathfrak{a}^{\beta} \epsilon \mathfrak{A}^{\beta}$ and $\mathfrak{b}^{\alpha} \epsilon \mathfrak{B}^{\alpha}$. The collection of groups $\{\mathfrak{B}^{\alpha}\}$ and homomorphisms $\{H^*{}_{\beta}{}^{\alpha}\}$ form a direct system S^* called the system dual to S.

Conversely, given a direct system S^* of discrete groups, by passing to character groups we can define an inverse system S of bicompact groups having S^* as its dual system.

THEOREM 6.1. *If S is an inverse system of bicompact groups and S^* is its dual direct system, then the limit group of S^* is the group of all continuous characters of the limit group of S.*

Let $\mathfrak{b}$ be an arbitrary element of the limit group $\mathfrak{B}$ of S^*. The function

$$(6.2) \qquad \mathfrak{b}(\mathfrak{a}) = (H^{\alpha}(\mathfrak{a}), \mathfrak{b}^{\alpha})$$

where $\mathfrak{b}^{\alpha}$ is some fixed coördinate of $\mathfrak{b}$ is a continuous character of the limit group $\mathfrak{A}$ of S. By (6.1), if $\mathfrak{B}^{\alpha} \to \mathfrak{B}^{\beta}$,

$$(H^{\alpha}(\mathfrak{a}), \mathfrak{b}^{\alpha}) = (H_{\beta}{}^{\alpha}H^{\beta}(\mathfrak{a}), \mathfrak{b}^{\alpha}) = (H^{\beta}(\mathfrak{a}), H^*{}_{\alpha}{}^{\beta}(\mathfrak{b}^{\alpha})) = (H^{\beta}(\mathfrak{a}), \mathfrak{b}^{\beta}).$$

Therefore the character defined by (6.2) is independent of the particular coördinate $\mathfrak{b}^{\alpha}$ chosen; so the character $\mathfrak{b}(\mathfrak{a})$ may be uniquely associated with the element $\mathfrak{b}$ of $\mathfrak{B}$. In this way $\mathfrak{B}$ is a group of continuous characters of $\mathfrak{A}$.

Let $\mathfrak{a}$ be a non-zero element of $\mathfrak{A}$. Then, for some α, the coördinate $\mathfrak{a}^{\alpha}$ of $\mathfrak{a}$ is not equal to zero. Let $\mathfrak{b}^{\alpha}$ be a character of $\mathfrak{A}^{\alpha}$ such that $(\mathfrak{a}^{\alpha}, \mathfrak{b}^{\alpha}) \neq 0$. The image $\mathfrak{b}$ of $\mathfrak{b}^{\alpha}$ in $\mathfrak{B}$ is then a character of $\mathfrak{A}$ such that $\mathfrak{b}(\mathfrak{a}) \neq 0$. Hence the annihilator [6] of $\mathfrak{B}$ in $\mathfrak{A}$ is the zero.

Let $\mathfrak{b}$ be a non-zero element of $\mathfrak{B}$. Then each coördinate $\mathfrak{b}^{\alpha}$ of $\mathfrak{b}$ is not equal to zero. Let $\tilde{\mathfrak{A}}^{\alpha}$ be the annihilator of $\mathfrak{b}^{\alpha}$ in $\mathfrak{A}^{\alpha}$. Let U be a neighborhood of zero in $\mathfrak{X}$ containing no subgroup of $\mathfrak{X}$ other than the zero.[7] As $\mathfrak{b}^{\alpha}$ maps $\tilde{\mathfrak{A}}^{\alpha}$ into zero, there is a neighborhood V of $\tilde{\mathfrak{A}}^{\alpha}$ which is mapped by $\mathfrak{b}^{\alpha}$ into U. Suppose, as is impossible, that $H^{\alpha}(\mathfrak{A}) \subset \tilde{\mathfrak{A}}^{\alpha}$ for a fixed $\mathfrak{a}$. By Lemma 2.2, there is a refinement $\mathfrak{A}^{\beta}$ of $\mathfrak{A}^{\alpha}$ such that $H_{\beta}{}^{\alpha}(\mathfrak{A}^{\beta}) \subset V$. Then the coördinate $\mathfrak{b}^{\beta}$ of $\mathfrak{b}$ is defined, and, as $\mathfrak{b}^{\beta} \neq 0$, there is an element $\mathfrak{a}^{\beta}$ of $\mathfrak{A}^{\beta}$ such that $(\mathfrak{a}^{\beta}, \mathfrak{b}^{\beta}) \neq 0$. However

$$(\mathfrak{a}^{\beta}, \mathfrak{b}^{\beta}) = (\mathfrak{a}^{\beta}, H^*{}_{\alpha}{}^{\beta}(b^{\alpha})) = (H_{\beta}{}^{\alpha}(\mathfrak{a}^{\beta}), \mathfrak{b}^{\alpha}) \neq 0;$$

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[6] That is, the set of elements of $\mathfrak{A}$ mapping each element of $\mathfrak{B}$ into zero.

[7] The complement of the closed interval [1/3, 2/3] is such a neighborhood.

so there is an element $\mathfrak{a}^\alpha$ of $H_\beta{}^\alpha(\mathfrak{A}^\beta)$ such that $(\mathfrak{a}^\alpha, \mathfrak{b}^\alpha) \neq 0$. But $H_\beta{}^\alpha(\mathfrak{A}^\beta)$ is a closed subgroup of V; and, as $\mathfrak{b}^\alpha$ maps V into U, $\mathfrak{b}^\alpha$ maps $H_\beta{}^\alpha(\mathfrak{A}^\beta)$ into a subgroup of U—therefore into the zero. As this contradicts the preceding statement, the assumption $H^\alpha(\mathfrak{A}) \subset \bar{\mathfrak{A}}^\alpha$ is false. So there is an element $\mathfrak{a}$ of $\mathfrak{A}$ whose coördinate $\mathfrak{a}^\alpha$ is not in $\bar{\mathfrak{A}}^\alpha$. As $\mathfrak{b}(\mathfrak{a}) = (\mathfrak{a}^\alpha, \mathfrak{b}^\alpha) \neq 0$, we have proved that the annihilator of $\mathfrak{A}$ in $\mathfrak{B}$ is the zero. The results of the last two paragraphs are sufficient to prove that each of the groups $\mathfrak{A}$, $\mathfrak{B}$ is the group of continuous characters of the other ([13], Lemma 5).

7. The representations of bicompact groups. We shall establish two theorems on the representations of discrete groups by direct systems of groups having finite bases. The results of the preceding section enable us to state the two dual propositions on bicompact groups.

THEOREM 7. 1. *A discrete group is always the limit group of some direct system of groups each on a finite basis. A system may be chosen so that its homomorphisms are isomorphic transformations into subgroups.*

THEOREM 7. 2. *Two direct systems of discrete groups having finite bases are equivalent if their limit groups are isomorphic.*

Let $\mathfrak{B}$ be a discrete group; and let $\{\mathfrak{B}^\alpha\}$ be the collection of all subgroups of $\mathfrak{B}$ on a finite basis. The system S will consist of these groups. If $\mathfrak{B}^\alpha \subset \mathfrak{B}^\beta$ in $\mathfrak{B}$, we introduce in S the isomorphism $H_\alpha{}^\beta$ transforming the group $\mathfrak{B}^\alpha$ into the subgroup of $\mathfrak{B}^\beta$ with which it is identified in $\mathfrak{B}$. Since any two subgroups of $\mathfrak{B}$, each on a finite basis, are contained in a third subgroup on a finite basis, it follows that S is a direct system. The isomorphism between $\mathfrak{B}$ and the limit group of S is set up in an obvious fashion. This proves Theorem 7. 1.

Let $S_1 = \{\mathfrak{B}_1{}^\alpha\}$ be a direct system of discrete groups each on a finite basis. Let $\mathfrak{B}$ be its limit group. We prove the second theorem by showing that S_1 and the system S just constructed are equivalent.

Let $\mathfrak{B}_0{}^\alpha$ be the subgroup of $\mathfrak{B}_1{}^\alpha$ whose image in $\mathfrak{B}$ is the zero. $\mathfrak{B}_0{}^\alpha$ has a finite basis; to each such basis element there is a refinement of $\mathfrak{B}_1{}^\alpha$ in which its projection is zero. Passing to a common refinement of these groups we find a refinement $\mathfrak{B}_1{}^\beta$ of $\mathfrak{B}_1{}^\alpha$ in which the image of $\mathfrak{B}_0{}^\alpha$ is the zero. Let S'_1 be the subsystem of S_1 consisting of the same groups as S_1 but contains, for each α, only those homomorphisms $\mathfrak{B}_1{}^\alpha \to \mathfrak{B}_1{}^\beta$ of S_1 under which $\mathfrak{B}_0{}^\alpha$ is mapped into zero (we except of course the identity transformations: $\mathfrak{B}_1{}^\alpha \to \mathfrak{B}_1{}^\alpha$). Then S_1 and S'_1 are equivalent systems.

Let $\mathfrak{B}'^\alpha = \mathfrak{B}_1{}^\alpha - \mathfrak{B}_0{}^\alpha$. We shall enlarge S'_1 obtaining an equivalent system S_1'' by adding the groups $\mathfrak{B}'^\alpha$. We must add also certain homomorphisms.

First $\mathfrak{B}'^{\alpha}$ is the homomorphic image of $\mathfrak{B}_1^{\alpha}$, we include this homomorphism and then any homomorphism obtained by a combination $\mathfrak{B}_1^{\gamma} \to \mathfrak{B}_1^{\alpha} \to \mathfrak{B}'^{\alpha}$. If $\mathfrak{B}_1^{\alpha} \to \mathfrak{B}_1^{\beta}$ (in S'_1), then $\mathfrak{B}'^{\alpha}$ is isomorphic with a subgroup of $\mathfrak{B}_1^{\beta}$; we include this isomorphism. Then we include any homomorphism resulting from a combination of any above: $\mathfrak{B}'^{\alpha} \to \mathfrak{B}_1^{\beta} \to \mathfrak{B}'^{\delta}$. Then S_1'' has S_1' as a complete subsystem. The system S' consisting of all groups of the type $\mathfrak{B}'^{\alpha}$ of S_1'' and all homomorphisms of S_1'' between any two such groups, also forms a complete subsystem. The system S' has the advantage that all of its homomorphisms are really isomorphic transformations (i. e. as above, $\mathfrak{B}'^{\alpha}$ is mapped isomorphically into a subgroup of $\mathfrak{B}'^{\delta}$). This follows from the fact that no non-zero element of $\mathfrak{B}'^{\alpha}$ images into the zero of $\mathfrak{B}$. Thus each group $\mathfrak{B}'^{\alpha}$ may be identified with a subgroup of $\mathfrak{B}$; and a projection $\mathfrak{B}'^{\alpha} \to \mathfrak{B}'^{\delta}$ goes over into a relation of inclusion in $\mathfrak{B}$. It is trivial that the system S' is equivalent with the system S constructed above (except for the possible repetition of the same group, it is a complete subsystem of S). We merely use inclusion relations in $\mathfrak{B}$ to define isomorphisms interlocking the two systems.

By using the theorem of the preceding section we can state two dual theorems. The character group of a free discrete group is called a *toral group.* A toral group is always the direct sum of a number of groups each isomorphic with the group $\mathfrak{X}$ (No. 6) [17; p. 370]. The number is called the *dimension* and is equal to the rank of the free group. The character group of a discrete group on a finite basis we shall call an *elementary group.* It is always the direct sum of a finite group and a toral group of finite dimension.

THEOREM 7. 3. *To each bicompact group corresponds at least one inverse system of elementary groups of which it is the limit group. A system can be chosen so that each of its groups is covered by the image of any of its refinements.*

THEOREM 7. 4. *Two inverse systems of elementary groups are equivalent if their limit groups are continuously isomorphic.*[8]

II. Homology groups over a general group of coefficients.

8. The finite complex. K denotes a finite complex (it need not be simplicial) of dimension n. L will denote a closed subcomplex of K. The

[8] In view of recent results of van Kampen: "Almost periodic functions and compact groups," *Annals of Mathematics*, vol. 37 (1936), pp. 78-91, it can be shown that these two theorems hold in the non-abelian case if Lie group replaces elementary group. The proof of the first depends on the existence in any neighborhood of the identity of a closed invariant subgroup such that the factor group is a Lie group. The proof of the second depends on the fact that any decreasing sequence of closed subgroups of a Lie group is finite.

p-cells of $K - L$ are denoted $E_p{}^i$ $(i = 1, \cdots, \alpha_p;\ p = 0, 1, \cdots, n)$. The boundary relations of K mod L are written [9]

$$F(E_p{}^i) = \eta^i{}_{p-1,j} E^j{}_{p-1}$$

where η_p is a finite matrix of integers. We assume that $\mathfrak{G}$ is a topological abelian group.

A p-chain over $\mathfrak{G}$ of K mod L is a linear form $\mathfrak{g}_i E_p{}^i$ in the p-cells of $K - L$ with coefficients $\mathfrak{g}_i \in \mathfrak{G}$. Addition is defined by: $(\mathfrak{g}_i E_p{}^i) + (\mathfrak{g}_i' E_p{}^i) = \mathfrak{g}_i'' E_p{}^i$ where $\mathfrak{g}_i'' = \mathfrak{g}_i + \mathfrak{g}_i'$. In this way the p-chains over $\mathfrak{G}$ of K mod L constitute a group denoted by $\mathfrak{K}_p(K, L, \mathfrak{G})$ (We shall omit parts of the expression $\mathfrak{K}_p(K, L, \mathfrak{G})$ whenever confusion is impossible).

We introduce a topology in $\mathfrak{K}$ as follows. Let $\{U^\alpha\}$ be a complete set of neighborhoods of zero in $\mathfrak{G}$. The set V^α of those p-chains whose coefficients lie in U^α is defined to be a neighborhood of zero in $\mathfrak{K}$. The set $\{V^\alpha\}$ is taken to be complete. The set $V^\alpha(\mathfrak{k})$ obtained by adding the chain $\mathfrak{k}$ to each element of V^α is a neighborhood of $\mathfrak{k}$; and the set $\{V^\alpha(\mathfrak{k})\}$ is taken to be complete. We proceed to verify that $\mathfrak{K}$ is a topological group. Clearly $\mathfrak{k} \in V^\alpha(\mathfrak{k})$.

If $V^\alpha(\mathfrak{k})$ and $V^\beta(\mathfrak{k})$ are two neighborhoods of $\mathfrak{k}$, let U^γ be common to U^α and U^β; then $V^\gamma(\mathfrak{k}) \subset V^\alpha(\mathfrak{k}) V^\beta(\mathfrak{k})$.

Suppose $\mathfrak{k}' \in V^\alpha(\mathfrak{k})$. Let $\mathfrak{g}_1, \cdots, \mathfrak{g}_{\alpha_p}$ be the coefficients of $\mathfrak{k}' - \mathfrak{k}$; then $\mathfrak{g}_i \in U^\alpha$. Let U^β be such that $\mathfrak{g}' \in U^\beta$ implies $\mathfrak{g}' + \mathfrak{g}_i \in U^\alpha$. Then $V^\beta(\mathfrak{k}' - \mathfrak{k}) \subset V^\alpha$ and this implies $V^\beta(\mathfrak{k}') \subset V^\alpha(\mathfrak{k})$.

Suppose now that $\mathfrak{k}' \neq \mathfrak{k}$. Let $\mathfrak{g}_i$ be a non-zero coefficient of $\mathfrak{k}' - \mathfrak{k}$; then let U^α be chosen so that $\mathfrak{g}_i$ is not the difference of any two of its elements. Then $V^\alpha(\mathfrak{k})$ and $V^\alpha(\mathfrak{k}')$ have no common point.

Finally let $V^\alpha(\mathfrak{k}_1 - \mathfrak{k}_2)$ be given. Let U^β be such that the difference of any two of its elements lies in U^α. Then $\mathfrak{k}'_1 \in V^\beta(\mathfrak{k}_1)$ and $\mathfrak{k}'_2 \in V^\beta(\mathfrak{k}_2)$ implies $\mathfrak{k}'_1 - \mathfrak{k}'_2 \in V^\alpha(\mathfrak{k}_1 - \mathfrak{k}_2)$. Thus $\mathfrak{K}$ is a Hausdorff space satisfying the axiom α) of No. 1.

Let $\mathfrak{K}^i$ be the subgroup of $\mathfrak{K}$ of those chains whose coefficients are all zero except possibly the i-th. Then it is easy to prove that $\mathfrak{K}^i$ is bicontinuously isomorphic with $\mathfrak{G}$ and $\mathfrak{K}$ with the direct sum $\sum \mathfrak{K}^i$. Thus

Lemma 8.1. *$\mathfrak{K}_p(K, L, \mathfrak{G})$ is the direct sum of α_p groups each bicontinuously isomorphic with $\mathfrak{G}$.*

The *boundary* mod L of a p-chain is defined by

[9] When a latin index occurs twice in a term, once above and once below, it is to be summed over its range.

$$(8.1) \qquad F(\mathfrak{g}_i E_p{}^i) = \mathfrak{g}_i \eta^i{}_{p-1,j} E^j{}_{p-1}.$$

It is clear that F is a homomorphic mapping of $\mathfrak{K}_p$ into $\mathfrak{K}_{p-1}$. In fact.

LEMMA 8.2. *The boundary operator is a continuous homomorphic mapping of $\mathfrak{K}_p$ into $\mathfrak{K}_{p-1}$.*

Let V^α be a neighborhood of zero in $\mathfrak{K}_{p-1}$. Now $\mathfrak{g}_i \eta^i{}_{p-1,j}$ is a continuous function in the $\mathfrak{g}_i$ simultaneously and is zero for each $\mathfrak{g}_i = 0$. Therefore there is a neighborhood $U_j{}^\beta$ of zero in $\mathfrak{G}$ such that $\mathfrak{g}_i \eta^i{}_{p-1,j} \in U^\alpha$ whenever each $\mathfrak{g}_i \in U_j{}^\beta$. Let $U^\beta \subset \prod_j U_j{}^\beta$. Then the neighborhood V^β of zero in $\mathfrak{K}_p$ determined by U^β is such that $F(V^\beta) \subset V^\alpha$. Thus F is continuous at the zero of $\mathfrak{K}_p$. As F is homomorphic, it is continuous everywhere.

It follows from the lemma that the subgroup of $\mathfrak{K}_p$ mapped by F into the zero of $\mathfrak{K}_{p-1}$ is closed in $\mathfrak{K}_p$. This group, $\mathfrak{C}_p(K, L, \mathfrak{G})$, is called the *group of p-cycles over* $\mathfrak{G}$ of K mod L. The image under F of $\mathfrak{K}_{p+1}$ in $\mathfrak{K}_p$ is a subgroup, $\mathfrak{B}_p(K, L, \mathfrak{G})$, of $\mathfrak{K}_p$ called the *group of bounding p-cycles over* $\mathfrak{G}$ of K mod L. Since $\eta_p \eta_{p-1} = 0$, we have $\mathfrak{B}_p \subset \mathfrak{C}_p$.

DEFINITION 1. The *p-th homology group over* $\mathfrak{G}$ of K mod L is

$$\mathfrak{H}_p(K, L, \mathfrak{G}) = \mathfrak{C}_p(K, L, \mathfrak{G}) - \bar{\mathfrak{B}}_p(K, L, \mathfrak{G})$$

where $\bar{\mathfrak{B}}$ is the closure of $\mathfrak{B}$.

We will see shortly the conditions that $\mathfrak{G}$ must satisfy in order that $\mathfrak{B}$ should be closed. We proceed to give the structure of $\mathfrak{H}$ in terms of $\mathfrak{G}$ and the Betti numbers and torsion coefficients of K mod L.

Let λ_p $(p = 0, 1, \cdots, n)$ be a unimodular matrix of integers of order α_p, and let $\bar{\lambda}_p$ denote its inverse: $\lambda_p \bar{\lambda}_p = 1$. Let $G_p{}^j$ denote the linear form $\lambda_{pi}{}^j E_p{}^i$. We define for each G a boundary

$$(8.2) \qquad F(G_p{}^i) = \zeta^i{}_{p-1,j} G^j{}_{p-1}, \qquad \zeta_{p-1} = \lambda_p \eta_{p-1} \bar{\lambda}_{p-1}.$$

Let $\mathfrak{K}'_p$ denote the group of linear forms in the $G_p{}^i$ with coefficients from $\mathfrak{G}$. We define a topology for $\mathfrak{K}'_p$ as we defined one for $\mathfrak{K}_p$. In analogy with (8.1) the transformation

$$(8.3) \qquad F(\mathfrak{g}_i G_p{}^i) = \mathfrak{g}_i \zeta^i{}_{p-1,j} G^j{}_{p-1}$$

is a continuous homomorphism of $\mathfrak{K}'_p$ into $\mathfrak{K}'_{p-1}$. The transformation

$$f(\mathfrak{g}_i G_p{}^i) = \mathfrak{g}_i \lambda_{pj}{}^i E_p{}^j$$

is a homomorphism of $\mathfrak{K}'_p$ into $\mathfrak{K}_p$ which is continuous (compare with proof of Lemma 8.2). It has the continuous inverse

$$f'(\mathfrak{g}_j E_p{}^j) = \mathfrak{g}_j \bar{\lambda}_{pk}{}^j G_p{}^k,$$

and is therefore a bicontinuous isomorphism between $\mathfrak{K}_p$ and $\mathfrak{K}'_p$. Comparing (8.2) with (8.1) we see that the boundary operator F commutes with this isomorphism. Therefore the groups $\mathfrak{K}'_p$ and the boundary operator (8.3) may be used to define the homology groups over $\mathfrak{G}$ of K mod L.

It is known [15; p. 27] that there exist unimodular matrices λ_p such that the matrices ζ_p of (8.2) are all in quasi-canonical form (i. e. at most one non-zero element in any row or column). In this case we can distinguish five types of G's (which are given the new notation $a, b, \cdots$ respectively) so that the boundary relations (8.2) assume the form

$$\begin{array}{llll}
 & F(e^h{}_{p+1}) = a_p{}^h, & F(a_p{}^h) = 0, & (h = 1, \cdots, \omega_p), \\
 & F(d^i{}_{p+1}) = \theta_p{}^i b_p{}^i, & F(b_p{}^i) = 0, & (i = 1, \cdots, \tau_p), \\
(8.4) & & F(c_p{}^j) = 0, & (j = 1, \cdots, R_p), \\
 & & F(d_p{}^k) = \theta^k{}_{p-1} b^k{}_{p-1}, & (k = 1, \cdots, \tau_{p-1}), \\
 & & F(e_p{}^l) = a^l{}_{p-1}, & (l = 1, \cdots, \omega_{p-1}),
\end{array}$$

where the $\theta_p{}^i$ $(i = 1, \cdots, \tau_p)$ are the invariant factors of η_p different from 1, $R_p = \alpha_p - \rho_p - \rho_{p-1}$ $(\rho_p = \text{rank } \eta_p)$, and $\omega_p = \rho_p - \tau_p$. If $\mathfrak{A}_p{}^h$ denotes the subgroup of $\mathfrak{K}_p(\mathfrak{G})$ of elements of the form $\mathfrak{g} a_p{}^h$, etc., then

$$\mathfrak{K}_p(K, L, \mathfrak{G}) = \sum \mathfrak{A}_p{}^h + \sum \mathfrak{B}_p{}^i + \sum \mathfrak{C}_p{}^j + \sum \mathfrak{D}_p{}^k + \sum \mathfrak{E}_p{}^l.$$

If $\mathfrak{D}'_p{}^k$ is the subgroup of $\mathfrak{D}_p{}^k$ of elements of orders dividing $\theta^k{}_{p-1}$, then

$$\mathfrak{C}_p(K, L, \mathfrak{G}) = \sum \mathfrak{A}_p{}^h + \sum \mathfrak{B}_p{}^i + \sum \mathfrak{C}_p{}^j + \sum \mathfrak{D}'_p{}^k.$$

If $\mathfrak{B}'_p{}^i$ is the subgroup of elements of $\mathfrak{B}_p{}^i$ divisible by $\theta_p{}^i$ and $\bar{\mathfrak{B}}'_p{}^i$ is its closure, then

$$\bar{\mathfrak{B}}_p(K, L, \mathfrak{G}) = \sum \mathfrak{A}_p{}^h + \sum \bar{\mathfrak{B}}'_p{}^i.$$

Referring to Lemma 1.2, we can state

THEOREM 8. *If R_p, $\theta^i{}_{p-1}$ $(i = 1, \cdots, \tau_{p-1};\ p = 0, 1, \cdots, n)$ are the Betti numbers and torsion coefficients of the finite n-complex K mod L, then $\mathfrak{H}_p(K, L, \mathfrak{G})$ can be continuously decomposed into the direct sum*

$$(8.5) \qquad \sum_{i=1}^{\tau_p} \mathfrak{G}^*{}_p{}^i + \sum_{j=1}^{R_p} \mathfrak{G}_p{}^j + \sum_{k=1}^{\tau_{p-1}} \mathfrak{G}'_p{}^k$$

where $\mathfrak{G}^{*}{}_p{}^i$ is the group obtained by reducing $\mathfrak{G}$ modulo the closure of the subgroup of elements divisible by $\theta_p{}^i$, each $\mathfrak{G}_p{}^j$ is bicontinuously isomorphic with $\mathfrak{G}$, and $\mathfrak{G}'{}_p{}^k$ is the subgroup of $\mathfrak{G}$ of elements whose orders divide $\theta^k{}_{p-1}$.

The condition that $\mathfrak{B}_p$ should be closed is clearly that each $\mathfrak{B}'{}_p{}^i$ should be closed. Thus the following requirement on $\mathfrak{G}$ is sufficient for $\mathfrak{B}_p$ to be closed:

Division-closure property. For each positive integer m, the subgroup of $\mathfrak{G}$ of elements divisible by m is closed.

In general this condition is also necessary; for, if $m\mathfrak{G}$ is not closed for some integer m, the complex K composed of a circle bounding a 2-cell where sets of m equally spaced points on the circle are identified has the property that $\mathfrak{B}_1(K, \mathfrak{G})$ is not closed.

If $\mathfrak{G}$ has the division-closure property, then the algebraic structure of $\mathfrak{H}_p(\mathfrak{G})$ depends only on the algebraic structure of $\mathfrak{G}$. Thus, if $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic (though not continuously so) and both have the division-closure property, then $\mathfrak{H}_p(\mathfrak{G})$ and $\mathfrak{H}_p(\mathfrak{G}')$ are isomorphic.

It is easy to prove that discrete groups, bicompact groups, and vector groups have the division-closure property. Likewise the groups of rational numbers. If two groups have the property, so also does their direct sum. A connected, locally-bicompact group has the property. On the other hand, the group of rational numbers of the form $p/2^n$ (p and n are integers) with the topology it has as a subset of the linear continuum does not have the division-closure property. For $1/2$ is not divisible (in the group) by 3, while the elements divisible by 3 are everywhere dense.

We state some simple consequences of Theorem 8.

Corollary 8. 1. *If $\mathfrak{G}$ is bicompact, so also is $\mathfrak{H}_p(\mathfrak{G})$. If $\mathfrak{G}$ is an elementary group (No. 7), so is $\mathfrak{H}_p(\mathfrak{G})$. If $\mathfrak{G}$ is the group $\mathfrak{X}$ of real numbers reduced mod 1, then $\mathfrak{H}_p(\mathfrak{G})$ is the direct sum of a toral group of dimension R_p and of a finite group isomorphic with the torsion group over the integers of one less dimension.*

Thus the Betti numbers and torsion coefficients are invariants of the mod 1 homology groups as well as the integer homology groups; hence

Corollary 8. 2. *The group $\mathfrak{X}$ and the group of integers are each universal coefficient groups for the homology theory of a finite complex.*

Definition 2. $\mathfrak{T}_p(K, L, \mathfrak{G}) = \sum \mathfrak{G}^{*}{}_p{}^i$ is called the *torsion group over* $\mathfrak{G}$. $\mathfrak{S}_p(K, L, \mathfrak{G}) = \mathfrak{H}_p(\mathfrak{G}) - \mathfrak{T}_p(\mathfrak{G})$ is called the *reduced homology group*. $\sum \mathfrak{G}'{}_p{}^k$ is called the *group of impure cycles* (see Čech [9]).

REMARK. The decomposition (8.5) in general cannot be specified in terms of invariants of $\mathfrak{H}_p(\mathfrak{G})$. For example there may exist a bicontinuous isomorphism of $\mathfrak{H}_p(\mathfrak{G})$ into itself which carries the torsion group into another subgroup. The decomposition is imposed by the structure of the complex, not by that of the group.

9. The abstract space. We describe briefly the procedure of Čech [8]. Let A be a set of points. A finite covering ϕ of A is a finite collection of subsets of A whose sum is A. A finite covering is a refinement of another if every set of the first is contained in some set of the second. In particular, any finite covering is a refinement of itself.

Each finite covering may be regarded as a complex K (the nerve of the covering): the sets are the vertices, a collection of $n+1$ vertices are the vertices of an n-simplex if the corresponding sets have a non-vacuous intersection. The intersection is associated with the n-simplex. If B is a subset of A, it determines a closed subcomplex L of K where a simplex is in L if and only if its associated set meets B.

If ϕ^2 is a refinement of ϕ^1 ($\phi^2 \to \phi^1$) and if K^2, L^2 and K^1, L^1 are the corresponding complexes and subcomplexes, we may define a simplicial mapping π of K^2 into K^1. To each set of ϕ^2 we associate a definite one of the sets of ϕ^1 which contains it; then π maps the vertex of K^2 corresponding to the set into the vertex if K^1 corresponding to the associated containing set. It is not difficult to see that π is a simplicial mapping and carries L^2 into L^1. The projection π obtained in this manner is called an allowable projection. In general there are many allowable projections.

If π maps a simplex σ_p of $K^2 - L^2$ into a simplex of L^1 or a simplex of dimension $< p$ of $K^1 - L^1$ we write algebraically $\pi(\sigma_p) = 0$. If σ_p is carried into a simplex τ_p of $K^1 - L^1$ we write $\pi(\sigma_p) = \tau_p$. Then, for any chain $\mathfrak{g}_i\sigma_p{}^i$ of $K^2 - L^2$ we define $\pi(\mathfrak{g}_i\sigma_p{}^i) = \mathfrak{g}_i\pi(\sigma_p{}^i)$. In this way we define a homomorphism π of $\mathfrak{K}_p(K^2, L^2, \mathfrak{G})$ into $\mathfrak{K}_p(K^1, L^1, \mathfrak{G})$. Since π (like F) is determined by a finite matrix of integers, we can prove that π is continuous (just as we proved that F is continuous). Since $\pi F(\sigma_p) = F\pi(\sigma_p)$ we find that π maps cycles (bounding cycles) into cycles (bounding cycles). Thus π induces a continuous homomorphism H of $\mathfrak{H}_p(K^2, L^2, \mathfrak{G})$ into $\mathfrak{H}_p(K^1, L^1, \mathfrak{G})$.

If π' is another allowable projection of K^2 into K^1, and c_p is a cycle of K^2 mod L^2, then, as shown by Čech, $\pi(c_p) \sim \pi'(c_p)$ mod L^1. Thus any allowable projection induces the same homomorphism H.

Now let Φ be a family $\{\phi^\alpha\}$ of finite coverings of A satisfying the condition that any two members of Φ have a common refinement in Φ. If $\phi^\beta \to \phi^\alpha$, let $H_\beta{}^\alpha$ be the induced homomorphism of $\mathfrak{H}_p{}^\beta = \mathfrak{H}_p(K^\beta, L^\beta, \mathfrak{G})$ into $\mathfrak{H}_p{}^\alpha$. The

limit group $\mathfrak{H}_p$ of the inverse system $\{\mathfrak{H}_p{}^a\}$ is called the *p-th homology over* $\mathfrak{G}$ of the family Φ mod B.

If A is a topological space and Φ is the family of all finite coverings of A by closed sets, $\mathfrak{H}_p$ is called the *p-th homology group over* $\mathfrak{G}$ of A mod B, written $\mathfrak{H}_p(A, B, \mathfrak{G})$. By definition $\mathfrak{H}_p$ is a topological invariant of A (i. e. homeomorphic spaces have bicontinuously isomorphic homology groups).

Remark I. We may also choose Φ to be the family of all finite coverings of A by open sets, and $\mathfrak{H}$ is once again a topological invariant. Čech [8; p. 180] has proved that, in a normal space, the two homology theories are equivalent.[10] It must be remarked that the proof of the theorem of the next section uses closed coverings in an essential way. Thus we have no proof of the analogous theorem for the homology group of a non-normal space based on open coverings. This remark applies also to Theorem 12.

II. If $\mathfrak{G}$ is bicompact, by Theorem 2.1, $\mathfrak{H}_p(A, B, \mathfrak{G})$ is bicompact. According to Čech, a cycle of the covering ϕ^a is *essential* if it is homologous to the projection of a cycle of ϕ^β for every refinement ϕ^β of ϕ^a. Čech proved for rational coefficients the fundamental existence theorem that an essential cycle of ϕ^a is a representative of the α-th coördinate of some element of $\mathfrak{H}_p$. In Lemma 2.1, we established this fundamental existence theorem for bicompact coefficient groups. According to Čech, a refinement ϕ^β of ϕ^a is *normal relative to* $\mathfrak{G}$ if the projection in ϕ^a of each p-cycle over $\mathfrak{G}$ of ϕ^β is essential. By Corollary 8.1, $\mathfrak{H}_p(K^a, L^a, \mathfrak{G})$ is an elementary group if $\mathfrak{G}$ is an elementary group. In proving Theorem 7.4, we showed that any elementary group in an inverse system of bicompact groups has a normal refinement. Thus normal refinements always exist relative to coefficient groups which are elementary. If $\mathfrak{G}$ is bicompact but not elementary, then in general normal refinements do not exist. However Lemma 2.2 states that, for a bicompact $\mathfrak{G}$, we may obtain refinements as close to being normal as we please.

III. If K is a finite complex and L a closed subcomplex, we have two definitions of $\mathfrak{H}_p(K, L, \mathfrak{G})$, the first, considering K as a complex, the second, considering K as a topological space. One establishes a bicontinuous isomorphism between these two groups as follows. Let $\{K^m\}$ be a sequence of barycentric subdivisions of K. The set ϕ^m of the stars of vertices of K^m form a finite covering of K by open sets whose associated complex is K^m. The family $\Phi = \{\Phi^m\}$ is a complete family in the sense that every covering of K by open sets has a refinement in Φ (see Remark I, K is normal). It is then proved in the usual way that the projection of K^{m+1} into K^m induces an iso-

[10] Čech's proof assumes that A is completely normal, but this is not necessary.

morphism of $\mathfrak{H}_p(K^{m+1}, L^{m+1}, \mathfrak{G})$ into the whole of $\mathfrak{H}_p(K^m, L^m, \mathfrak{G})$. It is not difficult to see that this isomorphism is bicontinuous. Then the limit group is bicontinuously isomorphic with each group of the sequence.

10. General direct sum theorem. The *torsion group over* $\mathfrak{G}$ of A mod B is the subgroup of $\mathfrak{H}$ of those elements represented in each covering complex by elements of the torsion group of that complex (Def. 8.2). We shall prove [11]

THEOREM 10.1. *$\mathfrak{H}_p(A, B, \mathfrak{G})$ is the direct sum of the torsion group $\mathfrak{T}_p$ and a reduced homology group $\mathfrak{S}_p$.*

In order to exhibit this decomposition of $\mathfrak{H}$ we shall define a canonical inverse system of groups of chains which carries the homology theory of A mod B. This canonical system will also be used in the next section.

To obtain an inverse system of chains we use a type of closed covering introduced by Kurosh [14]. We then consider the direct system of groups of dual chains (in the sense of Alexander [1]) determined by these coverings. To avoid a digression into the theory of dual chains, we define this direct system in terms of the inverse system in an invariant fashion using a type of argument due to Pontrjagin [16]. In the limit group of the direct system, a decomposition into a direct sum is established. Herein is contained a direct sum theorem for the dual homology group. We then argue backwards and define in each group of the inverse system a canonical basis so that the matrices exhibiting the homomorphisms of the system have a simple form. Finally we interpolate certain auxiliary groups of chains into the system. The subsystem of auxiliary groups is the desired canonical system and the direct sum theorem is immediate.

In the manner of Kurosh [14], we consider finite coverings of A by closed sets having the property that each set is the closure of its interior and the interiors of distinct sets are non-overlapping. Let Φ' be the aggregate of such coverings of A. It is easy to prove that any finite covering of A by closed sets has a refinement in Φ'. Thus Φ' is a complete subfamily of Φ, and we may restrict our considerations to Φ'. These coverings of A have the following advantages:

1°. If $\phi^\alpha \to \phi^\beta$, each set of ϕ^β is an exact sum of sets of ϕ^α.

2°. If $\phi^\alpha \to \phi^\beta$, each set of ϕ^α is contained in only one set of ϕ^β.

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[11] For the special case of a compact metric space and homology groups based on integer coefficients, this theorem was proved by Alexander and Cohen [2].

From 1° we deduce that a covering is a refinement of only finitely many others. Thus a covering possesses an immediate refinement in the sense that there are no coverings of Φ' between the two. In general a covering possesses many immediate refinements. From 2° it follows that the projection of the nerves $K^\alpha \to K^\beta$ is uniquely determined. As a consequence the groups of chains $\mathfrak{K}_p{}^\alpha = \mathfrak{K}_p(K^\alpha, L^\alpha, \mathfrak{J})$ ($\mathfrak{J}$ = group of integers) of the covering complexes K^α form an inverse system. From 1° and 2° it is proved that K^β is completely covered by the image of K^α. Then the homomorphism of $\mathfrak{K}_p{}^\alpha$ into $\mathfrak{K}_p{}^\beta$ is determined by a matrix reducible to the form $\left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|$.

Let $\mathfrak{K}^{*\alpha}$ be the group of homomorphic mappings [12] of $\mathfrak{K}^\alpha$ into $\mathfrak{J}$. $\mathfrak{K}^{*\alpha}$ is isomorphic to $\mathfrak{K}^\alpha$, and $\mathfrak{K}^\alpha$ is the group of homomorphic mappings of $\mathfrak{K}^{*\alpha}$ into $\mathfrak{J}$. The element of $\mathfrak{J}$ determined by an element $\mathfrak{k} \in \mathfrak{K}^\alpha$ and $\mathfrak{k}^* \in \mathfrak{K}^{*\alpha}$ is denoted $(\mathfrak{k}, \mathfrak{k}^*)$. Following Alexander [1], we refer to the elements of $\mathfrak{K}^{*\alpha}$ as dual chains. An independent basis $\mathfrak{k}^1, \mathfrak{k}^2, \cdots, \mathfrak{k}^h$ in $\mathfrak{K}^\alpha$ determines a dual basis in $\mathfrak{K}^{*\alpha}$ (and conversely) as follows: $\mathfrak{k}^{*i}$ $(i = 1, \cdots, h)$ is defined by $(\mathfrak{k}^j, \mathfrak{k}^{*i}) = \delta_j{}^i$ $(\delta_i{}^i = 1,\ \delta_j{}^i = 0\ (i \neq j))$.

The boundary homomorphism F of $\mathfrak{K}^\alpha{}_{p+1}$ into $\mathfrak{K}_p{}^\alpha$ determines a homomorphism F^* of $\mathfrak{K}^*{}_p{}^\alpha$ into $\mathfrak{K}^{*\alpha}_{p+1}$ as follows. If $\mathfrak{k}^*{}_p \in \mathfrak{K}^*{}_p{}^\alpha$, the function $\mathfrak{k}^*{}_p(\mathfrak{k}_{p+1}) = (F(\mathfrak{k}_{p+1}), \mathfrak{k}^*{}_p)$ is a homomorphism of $\mathfrak{K}^\alpha{}_{p+1}$ into $\mathfrak{J}$; it can therefore be identified with an element $\mathfrak{k}^*{}_{p+1} \in \mathfrak{K}^{*\alpha}_{p+1}$. Then $F^*(\mathfrak{k}^*{}_p)$ is the element $\mathfrak{k}^*{}_{p+1}$ so obtained. By definition

$$(10.1) \qquad (F(\mathfrak{k}_{p+1}), \mathfrak{k}^*{}_p) = (\mathfrak{k}_{p+1}, F^*(\mathfrak{k}^*{}_p))$$

for arbitrary $\mathfrak{k}_{p+1} \in \mathfrak{K}^\alpha{}_{p+1}$ and $\mathfrak{k}^*{}_p \in \mathfrak{K}^*{}_p{}^\alpha$. If F is given by a matrix η in terms of fixed bases in $\mathfrak{K}^\alpha{}_{p+1}$ and $\mathfrak{K}^\alpha{}_p$, and F^* by a matrix ζ in terms of the dual bases, by (10.1)

$$(F(\mathfrak{k}^i{}_{p+1}), \mathfrak{k}^*{}_p{}^j) = \eta_k{}^i(\mathfrak{k}_p{}^k, \mathfrak{k}^*{}_p{}^j) = \eta_j{}^i = (\mathfrak{k}^i{}_{p+1}, F^*(\mathfrak{k}^*{}_p{}^j)) = \zeta_k{}^j(\mathfrak{k}^i{}_{p+1}, \mathfrak{k}^{*k}_{p+1}) = \zeta_i{}^j.$$

Therefore ζ is the transpose of η.

If $\mathfrak{K}^\beta \to \mathfrak{K}^\alpha$, this homomorphism H defines in a similar way a dual H^* transforming $\mathfrak{K}^{*\alpha}$ into $\mathfrak{K}^{*\beta}$ satisfying

$$(10.2) \qquad (H(\mathfrak{k}^\beta), \mathfrak{k}^{*\alpha}) = (\mathfrak{k}^\beta, H^*(\mathfrak{k}^{*\alpha})).$$

In this way $\{\mathfrak{K}^{*\alpha}\}$ becomes a direct system. From the relation $FH = HF$ and the relations (10.1) and (10.2) we can deduce the dual relation

$$(10.3) \qquad F^*H^* = H^*F^*.$$

[12] The use of this notion is due to Pontrjagin [16]. Proofs of the statements of this paragraph can be found in his paper.

Since the matrix of H^* is the transpose of that of H, by a remark above, bases can be chosen in $\mathfrak{K}^{*\alpha}$ and $\mathfrak{K}^{*\beta}$ so that this matrix has the form $\| 1, 0 \|$. Therefore H^* maps $\mathfrak{K}^{*\alpha}$ isomorphically into a direct summand of $\mathfrak{K}^{*\beta}$. We shall use this to prove that the limit group $\mathfrak{K}^*$ of $\{\mathfrak{K}^{*\alpha}\}$ (No. 5) is a free group. We construct a free basis. Choose a free basis in $\mathfrak{K}^{*1}$ and form its image in $\mathfrak{K}^*$. These elements are independent in $\mathfrak{K}^*$; for the contrary would imply a refinement $\mathfrak{K}^{*\alpha}$ of $\mathfrak{K}^{*1}$ into which $\mathfrak{K}^{*1}$ is not isomorphically mapped. Suppose we have found an independent set S_β in $\mathfrak{K}^*$ generating a subgroup $\mathfrak{K}^*_\beta$ which contains the image of each $\mathfrak{K}^{*\alpha}$ for $\alpha < \beta$. Let $\mathfrak{K}^*{}_0{}^\beta$ be the subgroup of $\mathfrak{K}^{*\beta}$ imaging into $\mathfrak{K}^*_\beta$. It follows that there exist groups $\mathfrak{K}^{*\alpha_i}$ ($\alpha_i < \beta$, $i = 1, \cdots, k$) and a refinement $\mathfrak{K}^{*\gamma}$ of $\mathfrak{K}^{*\beta}$ and the $\mathfrak{K}^{*\alpha_i}$ in which the images of $\mathfrak{K}^{*\beta}$ and $\mathfrak{K}^{*\alpha_i}$ intersect in the image of $\mathfrak{K}^*{}_0{}^\beta$. Since these transformations are isomorphisms into direct summands of $\mathfrak{K}^{*\gamma}$, it follows that $\mathfrak{K}^*{}_0{}^\beta$ is a direct summand of $\mathfrak{K}^{*\beta}$. We can therefore find a set of independent elements of $\mathfrak{K}^{*\beta}$ not in $\mathfrak{K}^*{}_0{}^\beta$ which with the latter group generate $\mathfrak{K}^{*\beta}$. We form the image in $\mathfrak{K}^*$ of this set and adjoin it to S_β to obtain $S_{\beta+1}$. The remaining argument is obvious. It is also clear that $\mathfrak{K}^{*\alpha}$ images into a direct summand of $\mathfrak{K}^*$.

From the relation (10.3) it is seen that the system of homomorphisms $\mathfrak{K}^*{}_p{}^\alpha \to \mathfrak{K}^{*\alpha}_{p+1}$ defines a homomorphism F^* of $\mathfrak{K}^*_p$ into $\mathfrak{K}^*_{p+1}$ (if $\{\mathfrak{k}^*{}_p{}^\alpha\} \in \mathfrak{K}^*_p$, $F^*(\{\mathfrak{k}^*{}_p{}^\alpha\}) = \{F^*(\mathfrak{k}^*{}_p{}^\alpha)\} \in \mathfrak{K}^*_{p+1}$). As $\mathfrak{K}^*_{p+1}$ is a free group, $\mathfrak{K}^*_p$ decomposes into a direct sum $\mathfrak{C}^*_p + \mathfrak{D}^*_p$ such that $F^*(\mathfrak{C}^*_p) = 0$ and F^* maps $\mathfrak{D}^*_p$ isomorphically into a subgroup of $\mathfrak{K}^*_{p+1}$. We digress for the moment to state a theorem on the dual homology group.

If $\mathfrak{G}$ is an arbitrary abelian group (no topology assumed) the group $\mathfrak{K}^*_p(\mathfrak{G})$ of *dual chains over* $\mathfrak{G}$ of A mod B is the set of all finite linear forms in elements of $\mathfrak{K}^*_p$ with coefficients in $\mathfrak{G}$. The *boundary* of a dual chain over $\mathfrak{G}$ is defined $F^*(\mathfrak{g}_i\mathfrak{k}^{*i}) = \mathfrak{g}_i F(\mathfrak{k}^{*i})$. It is obvious that $\mathfrak{C}^*_p(\mathfrak{G})$ consists entirely of dual cycles; these are called the *pure cycles*. Certain elements of $\mathfrak{D}^*_p(\mathfrak{G})$ may be cycles; this subgroup is called the *group of impure cycles*. Further, since $F^*F^* = 0$, $\mathfrak{C}^*_p(\mathfrak{G})$ contains the subgroup of bounding cycles. As the *dual homology group* is obtained by reducing the group of cycles modulo the bounding cycles, we obtain

THEOREM 10.2. *The dual homology group over* $\mathfrak{G}$ *of* A *mod* B *is expressible as a direct sum of a group of impure cycles and the group of those elements representable as linear forms with coefficients from* $\mathfrak{G}$ *in dual cycles with integer coefficients.*[13]

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[13] This theorem together with an argument of Čech [9] provides a direct proof that

We return to the former discussion. Let us choose an independent basis for $\mathfrak{D}^*_p$. The group $\mathfrak{K}^*_p{}^a$ decomposes into a direct sum $\mathfrak{C}^*_p{}^a + \mathfrak{D}^*_p{}^a$ where $F^*(\mathfrak{C}^*_p{}^a) = 0$ and this does not hold for any non-zero element of $\mathfrak{D}^*_p{}^a$. The latter group may be chosen in many ways. We make the following selection. Choose those elements of $\mathfrak{K}^*_p{}^a$ which image into basis elements of $\mathfrak{D}^*_p$. These we call *permanent* basis elements of $\mathfrak{D}^*_p{}^a$. They generate a direct summand of $\mathfrak{K}^*_p{}^a$ which does not meet $\mathfrak{C}^*_p{}^a$. We may therefore adjoin further *temporary* basis elements so as to obtain a complete independent basis for $\mathfrak{K}^*_p{}^a$.

We now pass to a complete subsystem of $\{\mathfrak{K}^*_p{}^a\}$. The subsystem will contain all the groups of the system but will contain only a subclass of the homomorphisms. If $\mathfrak{K}^*_p{}^a \to \mathfrak{K}^*_p{}^\beta$, this homomorphism is included in the subsystem only if the image in $\mathfrak{K}^*_p{}^\beta$ of an arbitrary element of $\mathfrak{K}^*_p{}^a$ may be reduced to a cycle by the addition of a linear form in permanent basis element of $\mathfrak{D}^*_p{}^\beta$.

We prove the existence of such a refinement $\mathfrak{K}^*_p{}^\beta$ of $\mathfrak{K}^*_p{}^a$ as follows. Let $\bar{\mathfrak{D}}^*_p{}^a$ be the subgroup of $\mathfrak{D}^*_p{}^a$ imaging into $\mathfrak{C}^*_p$. As $\bar{\mathfrak{D}}^*_p{}^a$ has a finite basis, in some refinement $\mathfrak{K}^*_p{}^\gamma$ of $\mathfrak{K}^*_p{}^a$ the image of $\bar{\mathfrak{D}}^*_p{}^a$ lies in $\mathfrak{C}^*_p{}^\gamma$. The image of each element of $\mathfrak{K}^*_p{}^a$ in $\mathfrak{K}^*_p$ may be reduced to an element of $\mathfrak{C}^*_p$ by adding a finite form in basis elements of $\mathfrak{D}^*_p$. Since $\mathfrak{K}^*_p{}^a$ has a finite basis, a finite number of the basis elements of $\mathfrak{D}^*_p$ will suffice for all elements of $\mathfrak{K}^*_p{}^a$. There will exist therefore a $\mathfrak{K}^*_p{}^\delta$ whose image in $\mathfrak{K}^*_p$ contains these basis elements of $\mathfrak{D}^*_p$. Then a common refinement $\mathfrak{K}^*_p{}^\beta$ of $\mathfrak{K}^*_p{}^\gamma$ and $\mathfrak{K}^*_p{}^\delta$ obviously satisfies the above condition. To see that any two groups have a common refinement relative to the subsystem, we need only choose for each a proper refinement in the subsystem then a refinement of the latter two in the original system.

Let us perform this operation for $p = 0$. The subsystem obtained has a corresponding complete subsystem in $\{\mathfrak{K}^*_1{}^a\}$. In this subsystem we may perform the same operation. The new subsystem has corresponding subsystems of dimensions 0 and 2. In the latter, the same operation can be performed. In general, if $n \geqq 0$ is an integer, systems $\{\mathfrak{K}_p{}^a\}$ can be determined for $p \leqq n$ satisfying the above condition. Future considerations will be confined to these subsystems.

For each α, let us choose an independent basis for $\mathfrak{C}^{*\alpha}$. Suppose $\mathfrak{K}^{*\alpha} \to \mathfrak{K}^{*\beta}$; due to the conditions satisfied by $\mathfrak{K}^{*\beta}$, this homomorphism expressed in terms of the bases which have been defined in these groups has the matrix form

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the integers form a universal coefficient group for the dual homology theory. However this result can also be obtained from our Theorem 12 by passing to character groups.

$$
\begin{array}{cc|cc|l}
 & \mathfrak{C}^{*\beta} & \mathfrak{D}^{*\beta} & & \\
\mathfrak{C}^{*\alpha} & W' & 0 & 0 & \\
\hline
\mathfrak{D}^{*\alpha} & 0 & X' & 0 & \text{perm.} \\
 & Y' & Z' & 0 & \text{temp.} \\
 & & \text{perm.} & \text{temp.} &
\end{array}
\qquad X' = \| 1, 0 \|.
$$

In $\mathfrak{K}^\alpha$, for each α, let us choose the basis dual to the basis of $\mathfrak{K}^{*\alpha}$. Then the homomorphism $\mathfrak{K}^\beta \to \mathfrak{K}^\alpha$ assumes the form

$$
\begin{array}{cc|cc|l}
 & \mathfrak{C}^{\alpha} & \mathfrak{B}^{\alpha} & & \\
\mathfrak{C}^{\beta} & W & 0 & Y & \\
\hline
\mathfrak{B}^{\beta} & 0 & X & Z & \text{perm.} \\
 & 0 & 0 & 0 & \text{temp.} \\
 & & \text{perm.} & \text{temp.} &
\end{array}
\tag{10.4}
$$

where the basis of $\mathfrak{C}^\alpha$ is dual to that of $\mathfrak{C}^{*\alpha}$, and the basis of $\mathfrak{B}^\alpha$ is dual to that of $\mathfrak{D}^{*\alpha}$ and has been divided into two parts according to whether or not an element is dual to a permanent or temporary element.

The group $\mathfrak{B}_p{}^\alpha$ is the group of cycles which bound or have bounding multiples. To see this, let ρ be the rank of $\mathfrak{D}^*{}_p{}^\alpha$ ($=$ rank of $\mathfrak{B}_p{}^\alpha$). As F^* maps $\mathfrak{D}^*{}_p{}^\alpha$ isomorphically into $\mathfrak{K}^{*\alpha}_{p+1}$ the smallest direct summand $\mathfrak{B}^{*\alpha}_{p+1}$ of the latter group containing its image has the rank ρ. We choose an independent basis of $\mathfrak{K}^{*\alpha}_{p+1}$ containing a basis of $\mathfrak{B}^{*\alpha}_{p+1}$, and then we form the dual basis in $\mathfrak{K}^\alpha{}_{p+1}$. The group $\mathfrak{D}^\alpha{}_{p+1}$ generated by the chains dual to basis elements of $\mathfrak{B}^{*\alpha}_{p+1}$ has the rank ρ. Since F and F^* are dual, F maps $\mathfrak{D}^\alpha{}_{p+1}$ isomorphically into a subgroup of $\mathfrak{B}_p{}^\alpha$. As $\mathfrak{B}_p{}^\alpha$ has the rank ρ, the statement is proved.

Suppose $\mathfrak{K}^\beta$ is a proper refinement of $\mathfrak{K}^\alpha$. We shall introduce between these two groups a third auxiliary group $\mathfrak{K}^{\beta\alpha}$. It shall be the direct sum $\mathfrak{C}^{\beta\alpha} + \mathfrak{B}^{\beta\alpha}$ of two free groups on a finite basis. We assume a $1:1$ correspondence between the generators of $\mathfrak{C}^\beta$ and $\mathfrak{C}^{\beta\alpha}$ and between the generators of $\mathfrak{B}^{\beta\alpha}$ and the permanent generators of $\mathfrak{B}^\beta$. The homomorphism $\mathfrak{K}^\beta \to \mathfrak{K}^{\beta\alpha}$ is given by this correspondence

$$
\begin{array}{cc|c|l}
 & \mathfrak{C}^{\beta\alpha} & \mathfrak{B}^{\beta\alpha} & \\
\mathfrak{C}^{\beta} & 1 & 0 & \\
\hline
\mathfrak{B}^{\beta} & 0 & 1 & \text{perm.} \\
 & 0 & 0 & \text{temp.}
\end{array}
\tag{10.5}
$$

The homomorphism $\mathfrak{K}^{\beta\alpha} \to \mathfrak{K}^\alpha$ is given by

$$
\begin{array}{cc|cc}
 & \mathfrak{C}^{\alpha} & \mathfrak{B}^{\alpha} & \\
\mathfrak{C}^{\beta\alpha} & W & 0 & Y \\
\hline
\mathfrak{B}^{\beta\alpha} & 0 & X & Z \\
 & & \text{perm.} & \text{temp.}
\end{array}
\tag{10.6}
$$

where the submatrices W, X, Y and Z are those of (10.4). It is clear that the product of the homomorphisms $\mathfrak{K}^{\beta} \to \mathfrak{K}^{\beta\alpha} \to \mathfrak{K}^{\alpha}$ is the homomorphism $\mathfrak{K}^{\beta} \to \mathfrak{K}^{\alpha}$. We make of the enlarged collection of groups an inverse system as follows: If $\mathfrak{K}^{\beta\alpha} \to \mathfrak{K}^{\alpha} \to \mathfrak{K}^{\gamma}$, we define $\mathfrak{K}^{\beta\alpha} \to \mathfrak{K}^{\gamma}$ as the product. If $\mathfrak{K}^{\delta} \to \mathfrak{K}^{\beta} \to \mathfrak{K}^{\beta\alpha}$, we define $\mathfrak{K}^{\delta} \to \mathfrak{K}^{\beta\alpha}$ as the product. Then, if $\mathfrak{K}^{\epsilon\delta} \to \mathfrak{K}^{\beta} \to \mathfrak{K}^{\beta\alpha}$, we define $\mathfrak{K}^{\epsilon\delta} \to \mathfrak{K}^{\beta\alpha}$ as the product.

We shall define a boundary operation F mapping $\mathfrak{K}^{\beta\alpha}_{p+1}$ into $\mathfrak{K}_p{}^{\beta\alpha}$ and the latter into $\mathfrak{K}^{\beta\alpha}_{p-1}$. The boundary of any element of $\mathfrak{B}_q{}^{\beta\alpha}$ $(q = p, p+1)$ is defined to be zero. The boundary of a linear form in the basis of $\mathfrak{C}_q{}^{\beta\alpha}$ $(q = p, p+1)$ is the image in $\mathfrak{B}^{\beta\alpha}_{q-1}$ of the boundary of the same linear form in the corresponding basis elements of $\mathfrak{C}_q{}^{\beta}$. It is clear that the boundary operation commutes with the homomorphisms, and $\mathfrak{B}^{\beta\alpha}_{q-1}$ is the group of cycles which bound or have bounding multiples.

Let us consider the complete subsystem composed of those groups having two upper indices with all the homomorphisms between two such. In this system if $\mathfrak{K}^{\gamma\beta} \to \mathfrak{K}^{\alpha\delta}$ then $\mathfrak{C}^{\gamma\beta}$ images into $\mathfrak{C}^{\alpha\delta}$ and $\mathfrak{B}^{\gamma\beta}$ into $\mathfrak{B}^{\alpha\delta}$. This is proved by multiplying three matrices of the form (10.4), (10.5) and (10.6) respectively.

We introduce now the general topological group of coefficients $\mathfrak{G}$. $\mathfrak{K}^{\beta\alpha}(\mathfrak{G})$ is the group of linear forms in the basis elements of $\mathfrak{K}^{\beta\alpha}$ with coefficients from $\mathfrak{G}$. The boundary homomorphisms $\mathfrak{K}_p{}^{\beta\alpha}(\mathfrak{G}) \to \mathfrak{K}^{\beta\alpha}_{p-1}(\mathfrak{G})$ are defined in the usual way. Thus in $\mathfrak{K}_p{}^{\beta\alpha}(\mathfrak{G})$ we can distinguish a subgroup of cycles and a subgroup of bounding cycles, the latter being contained in the former. $\mathfrak{H}_p{}^{\beta\alpha}(\mathfrak{G})$ is obtained by reducing the group of cycles modulo the closure of the group of bounding cycles. A homomorphism $\mathfrak{K}_p{}^{\gamma\beta} \to \mathfrak{K}_p{}^{\alpha\delta}$ defines a homomorphism $\mathfrak{H}_p{}^{\gamma\beta}(\mathfrak{G}) \to \mathfrak{H}_p{}^{\alpha\delta}(\mathfrak{G})$ in the usual way (F commutes with $\mathfrak{K}^{\gamma\beta} \to \mathfrak{K}^{\alpha\delta}$). Thus $\{\mathfrak{H}_p{}^{\alpha\beta}(\mathfrak{G})\}$ is an inverse system. Furthermore it is equivalent with the system $\{\mathfrak{H}_p{}^{\alpha}(\mathfrak{G})\}$; for $\{\mathfrak{K}_q{}^{\alpha}\}$ and $\{\mathfrak{K}_q{}^{\alpha\beta}\}$ $(q = p-1, p, p+1)$ are equivalent and F commutes with the homomorphisms of the enlarged system containing both.

Now $\mathfrak{H}_p{}^{\alpha\beta}(\mathfrak{G})$ is the direct sum of a torsion group $\mathfrak{T}_p{}^{\alpha\beta}(\mathfrak{G})$ (elements representable by linear forms in basis elements of $\mathfrak{B}_p{}^{\alpha\beta}$) and a group $\mathfrak{S}_p{}^{\alpha\beta}(\mathfrak{G})$ of elements representable as linear forms in basis elements of $\mathfrak{C}_p{}^{\alpha\beta}$. Since $\mathfrak{K}^{\gamma\beta} \to \mathfrak{K}^{\alpha\delta}$ implies $\mathfrak{C}^{\gamma\beta} \to \mathfrak{C}^{\alpha\delta}$ and $\mathfrak{B}^{\gamma\beta} \to \mathfrak{B}^{\alpha\delta}$, $\{\mathfrak{T}_p{}^{\alpha\beta}(\mathfrak{G})\}$ and $\{\mathfrak{S}_p{}^{\alpha\beta}(\mathfrak{G})\}$ are inverse systems. Then their limit groups $\mathfrak{T}_p(\mathfrak{G})$ and $\mathfrak{S}_p(\mathfrak{G})$ respectively are subgroups of the limit group $\mathfrak{H}_p(\mathfrak{G})$ of $\{\mathfrak{H}_p{}^{\alpha\beta}(\mathfrak{G})\}$ and it is trivial that $\mathfrak{H}_p(\mathfrak{G})$ is their direct sum. This completes the proof.

Remark. Let us note that the canonical system of chains $\{\mathfrak{K}_p{}^{\alpha\beta}\}$ can be defined for any finite number of successive values of p, say $0 \leqq p \leqq n$, so that

the groups of the two systems $\{\mathfrak{K}_p^{\alpha\beta}\}$ and $\{\mathfrak{K}_{p+1}^{\alpha\beta}\}$ are in 1:1 correspondence according to their upper indices and there is defined a homomorphism F mapping $\mathfrak{K}_{p+1}^{\alpha\beta}$ into $\mathfrak{K}_p^{\alpha\beta}$ which commutes with the homomorphisms and satisfies $FF = 0$. If we pass to the corresponding systems over $\mathfrak{G}$, the group $\mathfrak{H}_p(\mathfrak{G})$ $(0 \leqq p \leqq n-1)$ defined above is the p-th homology group over $\mathfrak{G}$ of A mod B.

III. A universal coefficient group.

11. The construction. We shall begin with an arbitrary bicompact group $\mathfrak{H}$, and we shall construct in terms of $\mathfrak{H}$ and the group $\mathfrak{G}$ two topological groups $\mathfrak{T}$ and $\mathfrak{S}$. Assuming that $\mathfrak{H} = \mathfrak{H}_p(A, B, \mathfrak{X})$ for a topological space A and closed subset B, we prove that $\mathfrak{T} = \mathfrak{T}_{p-1}(A, B, \mathfrak{G})$ and $\mathfrak{S} = \mathfrak{S}_p(A, B, \mathfrak{G})$ (see Theorem 10.1). The construction of $\mathfrak{T}$ and $\mathfrak{S}$ is not invariant since it involves a number of choices. To prove that $\mathfrak{T}$ and $\mathfrak{S}$ are invariants of the pair of groups $\mathfrak{H}$, $\mathfrak{G}$ it is necessary to carry along with the construction a certain amount of algebraic argument.

By Theorem 7.3, there is an inverse system of elementary groups having $\mathfrak{H}$ as limit group. Let $S = \{\mathfrak{H}^\alpha\}$ be any such system. As $\mathfrak{H}^\alpha$ is an elementary group, we may express it as a direct sum $\mathfrak{C}^\alpha + \mathfrak{D}^\alpha$ of a finite group $\mathfrak{D}^\alpha$ and a toral group $\mathfrak{C}^\alpha$ of finite dimension $d(\alpha)$.

Let us express $\mathfrak{D}^\alpha$ as the direct sum of finite cyclic groups $\mathfrak{X}'^{\alpha i}$ $(i = 1, \cdots, \tau(\alpha))$ where $\theta^{\alpha i}$ is the order of $\mathfrak{X}'^{\alpha i}$. We shall suppose there is a fixed isomorphism between $\mathfrak{X}'^{\alpha i}$ and the subgroup of $\mathfrak{X}$ of order $\theta^{\alpha i}$. An element $\mathfrak{x} \in \mathfrak{X}$ $(\theta^{\alpha i}\mathfrak{x} = 0)$ has a correspondent in $\mathfrak{X}'^{\alpha i}$ which is denoted $\mathfrak{x}\mathfrak{X}'^{\alpha i}$.

Let us express $\mathfrak{C}^\alpha$ as the direct sum of circular groups $\mathfrak{X}^{\alpha i}$ $(i = 1, \cdots, d(\alpha))$. We choose a fixed continuous isomorphism between $\mathfrak{X}$ and $\mathfrak{X}^{\alpha i}$ and denote the correspondent of $\mathfrak{x} \in \mathfrak{X}$ by $\mathfrak{x}\mathfrak{X}^{\alpha i}$.

Let $\tilde{\mathfrak{X}}$ be a continuous homomorphic image of $\mathfrak{X}$ in $\mathfrak{C}^\alpha$. The image of $\mathfrak{x} \in \mathfrak{X}$ is denoted $\mathfrak{x}\tilde{\mathfrak{X}}$. For some elements $\mathfrak{x}_i(\mathfrak{x})$, we must have [9] $\mathfrak{x}\tilde{\mathfrak{X}} = \mathfrak{x}_i(\mathfrak{x})\mathfrak{X}^{\alpha i}$. We prove that there are unique integers u_i such that

$$\mathfrak{x}\tilde{\mathfrak{X}} = \mathfrak{x}u_i\mathfrak{X}^{\alpha i}.$$

The function $\mathfrak{x}_i(\mathfrak{x})$ is a continuous homomorphism of $\mathfrak{X}$ into itself (this follows from the definition of direct sum). It is well known that any such character of $\mathfrak{X}$ is given by an integer u_i satisfying $\mathfrak{x}_i(\mathfrak{x}) = u_i\mathfrak{x}$ for each $\mathfrak{x} \in \mathfrak{X}$.

The subgroup $\mathfrak{C}^\alpha$ of $\mathfrak{H}^\alpha$ is uniquely determined since it is the component of zero. The group $\mathfrak{D}^\alpha$ may in general be chosen in several ways. Suppose $\tilde{\mathfrak{D}}^\alpha$ is a finite subgroup so that $\mathfrak{H}^\alpha = \mathfrak{C}^\alpha + \tilde{\mathfrak{D}}^\alpha$. Let us decompose $\tilde{\mathfrak{D}}^\alpha$ into cyclic subgroups $\tilde{\mathfrak{X}}'^{\alpha i}$; and let us decompose $\mathfrak{C}^\alpha$ in some new way into circular subgroups $\tilde{\mathfrak{X}}^{\alpha i}$. Then the element $\mathfrak{x}\tilde{\mathfrak{X}}'^{\alpha i}$ $(\mathfrak{x} \in \mathfrak{X}, \tilde{\theta}^{\alpha i}\mathfrak{x} = 0)$ may be written

$$\mathfrak{x}\bar{\mathfrak{X}}'^{\alpha i} = \mathfrak{x}_j{}^i(\mathfrak{x})\mathfrak{X}^{\alpha j} + \mathfrak{x}'_j{}^i(\mathfrak{x})\mathfrak{X}'^{\alpha j}.$$

We shall prove that there are integers $v_j{}^i$ and $w_j{}^i$ which are unique mod $\bar{\theta}^{\alpha i}$ such that

$$\mathfrak{x}\bar{\mathfrak{X}}'^{\alpha i} = \mathfrak{x}v_j{}^i\mathfrak{X}^{\alpha j} + \mathfrak{x}w_j{}^i\mathfrak{X}'^{\alpha j}.$$

The function $\mathfrak{x}_j{}^i(\mathfrak{x})$ maps homomorphically the subgroup of $\mathfrak{X}$ with generator $1/\bar{\theta}^{\alpha i}$ into $\mathfrak{X}$ and therefore into a subgroup of itself. Thus there is an integer $v_j{}^i$ unique mod $\bar{\theta}^{\alpha i}$ such that $1/\bar{\theta}^{\alpha i}$ is imaged into $v_j{}^i$ times itself. In the same way we obtain $w_j{}^i$.

Thus between the two decompositions of $\mathfrak{H}^\alpha$ into the direct sums of circular groups and finite cyclic groups, we have the transformations

$$(11.1)\qquad \begin{cases} \mathfrak{x}\bar{\mathfrak{X}}^{\alpha i} = \mathfrak{x}u_i{}^{\alpha i}\mathfrak{X}^{\alpha j} \\ \mathfrak{x}\bar{\mathfrak{X}}'^{\alpha i} = \mathfrak{x}v_j{}^{\alpha i}\mathfrak{X}^{\alpha j} + \mathfrak{x}w_j{}^{\alpha i}\mathfrak{X}'^{\alpha j}, \end{cases}$$

$$(11.2)\qquad \begin{cases} \mathfrak{x}\mathfrak{X}^{\alpha k} = \mathfrak{x}\bar{u}_i{}^{\alpha k}\bar{\mathfrak{X}}^{\alpha i} \\ \mathfrak{x}\mathfrak{X}'^{\alpha k} = \mathfrak{x}\bar{v}_i{}^{\alpha k}\bar{\mathfrak{X}}^{\alpha i} + \mathfrak{x}\bar{w}_i{}^{\alpha k}\bar{\mathfrak{X}}'^{\alpha i}. \end{cases}$$

As each transformation is the inverse of the other we obtain

$$(11.3)\qquad \begin{cases} \bar{u}_j{}^{\alpha i}u_k{}^{\alpha j} = \delta_k{}^i \quad (\delta_i{}^i = 1,\ \delta_k{}^i = 0\ (i \neq k)), \\ \bar{v}_j{}^{\alpha i}u_k{}^{\alpha j} + \bar{w}_j{}^{\alpha i}v_k{}^{\alpha j} \equiv 0 \quad (\operatorname{mod} \theta^{\alpha i}), \\ \bar{w}_j{}^{\alpha i}w_k{}^{\alpha j} \equiv \delta_k{}^i \quad (\operatorname{mod} \theta^{\alpha i}). \end{cases}$$

If $\mathfrak{H}^\beta$ is a refinement of $\mathfrak{H}^\alpha$, $H_\beta{}^\alpha$ maps the circular subgroups $\mathfrak{X}^{\beta i}$ of $\mathfrak{H}^\beta$ into such in $\mathfrak{H}^\alpha$, and maps the finite groups $\mathfrak{X}'^{\beta i}$ into such in $\mathfrak{H}^\alpha$. Therefore we may adopt the notation just described and write $H_\beta{}^\alpha$ in the form

$$(11.4)\qquad \begin{cases} H_\beta{}^\alpha(\mathfrak{x}\mathfrak{X}^{\beta i}) = \mathfrak{x}x^{\alpha i}_{\beta j}\mathfrak{X}^{\alpha j} \\ H_\beta{}^\alpha(\mathfrak{x}\mathfrak{X}'^{\beta i}) = \mathfrak{x}y^{\alpha i}_{\beta j}\mathfrak{X}^{\alpha j} + \mathfrak{x}z^{\alpha i}_{\beta j}\mathfrak{X}'^{\alpha j} \end{cases}$$

where the integers $x^{\alpha i}_{\beta j}$ are uniquely determined, and the integers $y^{\alpha i}_{\beta j}$ and $z^{\alpha i}_{\beta j}$ are unique mod $\theta^{\beta i}$.

Similarly for the other bases

$$(11.5)\qquad \begin{cases} H_\beta{}^\alpha(\mathfrak{x}\bar{\mathfrak{X}}^{\beta i}) = \mathfrak{x}\bar{x}^{\alpha i}_{\beta j}\bar{\mathfrak{X}}^{\alpha j} \\ H_\beta{}^\alpha(\mathfrak{x}\bar{\mathfrak{X}}'^{\beta i}) = \mathfrak{x}\bar{y}^{\alpha i}_{\beta j}\bar{\mathfrak{X}}^{\alpha j} + \mathfrak{x}\bar{z}^{\alpha i}_{\beta j}\bar{\mathfrak{X}}'^{\alpha j}. \end{cases}$$

If we apply successively the transformations (11. 2) for $\mathfrak{H}^\beta$ then (11. 5) and finally (11. 1), we must obtain the transformation (11. 4). We have therefore the relations

$$(11.6)\qquad \begin{cases} \bar{u}_j{}^{\beta i}\bar{x}^{\alpha j}_{\beta k}u_l{}^{\alpha k} = x^{\alpha i}_{\beta l} \\ \bar{v}_j{}^{\beta i}\bar{x}^{\alpha j}_{\beta k}u_l{}^{\alpha k} + \bar{w}_j{}^{\beta i}y^{\alpha j}_{\beta k}u_l{}^{\alpha k} + \bar{w}_j{}^{\beta i}\bar{z}^{\alpha j}_{\beta k}v_l{}^{\alpha k} \equiv y^{\alpha i}_{\beta l} \quad (\operatorname{mod} \theta^{\beta i}) \\ \bar{w}_j{}^{\beta i}\bar{z}^{\alpha j}_{\beta k}w_l{}^{\alpha k} \equiv z^{\alpha i}_{\beta l} \quad (\operatorname{mod} \theta^{\beta i}) \end{cases}$$

where of course there is no summation on β.

In terms of $\mathfrak{G}$, $\{\mathfrak{H}^{\alpha}\}$ and the decomposition of $\mathfrak{H}^{\alpha}$ into subgroups $\mathfrak{X}^{\alpha i}$ and $\mathfrak{X}'^{\alpha i}$, we construct an inverse system $\{\mathfrak{T}^{\alpha}\}$ as follows. Let $\mathfrak{G}^{*\alpha i}$ $(i = 1, \cdots, \tau(\alpha))$ be a group bicontinuously isomorphic with the group obtained by reducing $\mathfrak{G}$ modulo the closure of the subgroup of elements divisible by $\theta^{\alpha i}$. Under this fixed homomorphism of $\mathfrak{G}$ into $\mathfrak{G}^{*\alpha i}$, denote the image of $\mathfrak{g} \in \mathfrak{G}$ by $\mathfrak{g}\mathfrak{G}^{*\alpha i}$. Let $\mathfrak{T}^{\alpha}$ be the direct sum $\sum_{i=1}^{\tau(\alpha)} \mathfrak{G}^{*\alpha i}$. The number $z_{\beta j}^{\alpha i}$ of (11.4) is such that $\mathfrak{x} z_{\beta j}^{\alpha i} \theta^{\alpha j}$ (not summed on j) is the zero of $\mathfrak{X}$ if $\theta^{\beta i}\mathfrak{x} = 0$. If we let $\mathfrak{x}$ be the element $1/\theta^{\beta i}$, we see that there is an integer $s_{\beta j}^{\alpha i}$ such that

$$s_{\beta j}^{\alpha i} \theta^{\beta i} = z_{\beta j}^{\alpha i} \theta^{\alpha j} \quad \text{(not summed on } j \text{ and } \beta). \tag{11.7}$$

We define a homomorphism $H_{\beta}{}^{\alpha}$ of $\mathfrak{T}^{\beta}$ into $\mathfrak{T}^{\alpha}$ by the relations

$$H_{\beta}{}^{\alpha}(\mathfrak{g}\mathfrak{G}^{*\beta i}) = \mathfrak{g} s_{\beta j}^{\alpha i} \mathfrak{G}^{*\alpha j} \qquad (i = 1, \cdots, \tau(\beta)). \tag{11.8}$$

Since $z_{\beta j}^{\alpha i}$ is unique mod $\theta^{\beta i}$, $s_{\beta j}^{\alpha i}$ is unique mod $\theta^{\alpha j}$, thus the right side is a unique element of $\mathfrak{T}^{\alpha}$.

Due to the ambiguity in representing an element of $\mathfrak{G}^{*\beta i}$, we must prove that $H_{\beta}{}^{\alpha}$ is uniquely defined. Suppose $\mathfrak{g}\mathfrak{G}^{*\beta i} = \mathfrak{g}'\mathfrak{G}^{*\beta i}$, then $\mathfrak{g} - \mathfrak{g}'$ is a limit of elements of $\mathfrak{G}$ divisible by $\theta^{\beta i}$. Therefore $(\mathfrak{g} - \mathfrak{g}')s_{\beta j}^{\alpha i}$ is a limit of elements divisible by $s_{\beta j}^{\alpha i}\theta^{\beta i}$. By (11.7), $(\mathfrak{g} - \mathfrak{g}')s_{\beta j}^{\alpha i}$ is a limit of elements divisible by $\theta^{\alpha j}$. Therefore $\mathfrak{g}s_{\beta j}^{\alpha i}$ and $\mathfrak{g}'s_{\beta j}^{\alpha i}$ image into the same element of $\mathfrak{G}^{*\alpha j}$, and $H_{\beta}{}^{\alpha}$ is unique.

To see that $H_{\beta}{}^{\alpha}$ is continuous, let U be a neighborhood of zero in $\mathfrak{T}^{\alpha}$. Then U is given as the product space of neighborhoods U^{j} $(j = 1, \cdots, \tau(\alpha))$ of neighborhoods of zero in the $\mathfrak{G}^{*\alpha j}$. Let V be a neighborhood of zero in $\mathfrak{G}$ such that V is imaged into U^{j} $(j = 1, \cdots, \tau(\alpha))$ under the transformation sending $\mathfrak{g} \in \mathfrak{G}$ into $\mathfrak{g}s_{\beta j}^{\alpha i}\mathfrak{G}^{*\alpha j}$ (not summed on j). The image of V in $\mathfrak{G}^{*\beta i}$ under the homomorphism $\mathfrak{g} \to \mathfrak{g}\mathfrak{G}^{*\beta i}$ is a neighborhood U'^{i} of zero in $\mathfrak{G}^{*\beta i}$. The product space U' of the U'^{i} is a neighborhood of zero in $\mathfrak{T}^{\beta}$ mapped by $H_{\beta}{}^{\alpha}$ into U.

If the homomorphism $H_{\beta}{}^{\alpha}$ of $\mathfrak{T}^{\beta}$ into $\mathfrak{T}^{\alpha}$ be defined for every pair α, β of ordinals such that $\mathfrak{H}^{\beta}$ is a refinement of $\mathfrak{H}^{\alpha}$, we obtain an inverse system $\{\mathfrak{T}^{\alpha}\}$. (The verification of axiom c) of No. 2 is immediate. Since $\{\mathfrak{H}^{\alpha}\}$ satisfies axioms a) and b), we deduce corresponding properties for the integers x, y, and z of (11.4) from which it follows that a) and b) hold in $\{\mathfrak{T}^{\alpha}\}$). Let us prove that $\{\mathfrak{T}^{\alpha}\}$ is independent of the decomposition of $\mathfrak{H}^{\alpha}$ into subgroups. If $\mathfrak{H}^{\alpha}$ be decomposed into the subgroups $\bar{\mathfrak{X}}^{\alpha i}$ and $\bar{\mathfrak{X}}'^{\alpha i}$, we construct groups $\bar{\mathfrak{G}}^{*\alpha i}$ in a similar way and obtain a new group $\bar{\mathfrak{T}}^{\alpha}$. From (11.1) and (11.2) we deduce integers $t_{j}{}^{\alpha i}$ and $\bar{t}_{j}{}^{\alpha i}$ such that

$$\left.\begin{aligned} t_j{}^{\alpha i}\bar{\theta}^{\alpha i} &= w_j{}^{\alpha i}\theta^{\alpha j} \\ \bar{t}_j{}^{\alpha i}\theta^{\alpha i} &= \bar{w}_j{}^{\alpha i}\theta^{\alpha j} \end{aligned}\right\} \text{ (not summed on } j).$$

From these relations and (11. 3), we deduce

$$\bar{t}_j{}^{\alpha i}t_k{}^{\alpha j} \equiv \delta_k{}^i \pmod{\theta^{\alpha k}}.$$

Then the transformation

$$f^\alpha(\mathfrak{g}\bar{\mathfrak{G}}^{*\alpha i}) = \mathfrak{g}t_j{}^{\alpha i}\mathfrak{G}^{*\alpha j}$$

has the inverse

$$\bar{f}^\alpha(\mathfrak{g}\mathfrak{G}^{*\alpha j}) = \mathfrak{g}\bar{t}_k{}^{\alpha j}\bar{\mathfrak{G}}^{*\alpha k},$$

and is therefore a bicontinuous isomorphism between $\mathfrak{T}^\alpha$ and $\bar{\mathfrak{T}}^\alpha$. By (11. 5), there are integers $\bar{z}^{\alpha i}_{\beta j}$ satisfying

$$z^{\alpha i}_{\beta j}\bar{\theta}^{\beta i} = \bar{z}^{\alpha i}_{\beta j}\bar{\theta}^{\alpha j} \quad \text{(not summed on } j).$$

By (11. 6) and (11. 7), we obtain

$$\bar{t}_i{}^{\beta h}z^{\alpha i}_{\beta j}t_k{}^{\alpha j} \equiv s^{\alpha h}_{\beta k} \pmod{\theta^{\alpha k}}.$$

Therefore the homomorphism

$$\bar{H}_\beta{}^\alpha(\mathfrak{g}\bar{\mathfrak{G}}^{*\beta i}) = \mathfrak{g}s^{\alpha i}_{\beta j}\bar{\mathfrak{G}}^{*\alpha i}$$

carries over into the homomorphism $H_\beta{}^\alpha$ under the isomorphisms f^α: $H_\beta{}^\alpha = f^\alpha\bar{H}_\beta{}^\alpha\bar{f}^\beta$. Thus the inverse system $\{\bar{\mathfrak{T}}^\alpha\}$ is isomorphic with $\{\mathfrak{T}^\alpha\}$. It follows that $\{\mathfrak{T}^\alpha\}$ is uniquely defined by $\mathfrak{G}$ and $\{\mathfrak{H}^\alpha\}$.

A complete subsystem of $S = \{\mathfrak{H}^\alpha\}$ defines in the same way a complete subsystem of $\{\mathfrak{T}^\alpha\}$. Thus equivalent systems S and $\bar{S}$ determine equivalent systems $\{\mathfrak{T}^\alpha\}$ and $\{\bar{\mathfrak{T}}^\alpha\}$. By Theorem 7. 4, it follows that $\{\mathfrak{T}^\alpha\}$ is determined up to equivalent systems by $\mathfrak{G}$ and $\mathfrak{H}$. Hence by Theorem 4. 1, the limit group $\mathfrak{T}$ of $\{\mathfrak{T}^\alpha\}$ is determined up to bicontinuous isomorphisms by $\mathfrak{G}$ and $\mathfrak{H}$.

In terms of $\mathfrak{G}$, $\{\mathfrak{H}^\alpha\}$, and the decomposition of $\mathfrak{H}^\alpha$ into subgroups $\mathfrak{X}^{\alpha i}$ and $\mathfrak{X}'^{\alpha i}$, we construct an inverse system $\{\mathfrak{S}^\alpha\}$ as follows. Let $\mathfrak{G}^{\alpha i}$ $(i = 1, \cdots, d(\alpha))$ be a group bicontinuously isomorphic with $\mathfrak{G}$. Let $\mathfrak{G}'^{\alpha i}$ $(i = 1, \cdots, \tau(\alpha))$ be a group bicontinuously isomorphic with the subgroup of elements of $\mathfrak{G}$ of orders dividing $\theta^{\alpha i}$. As usual the element of $\mathfrak{G}^{\alpha i}(\mathfrak{G}'^{\alpha i})$ corresponding to $\mathfrak{g} \in \mathfrak{G}$ is denoted $\mathfrak{g}\mathfrak{G}^{\alpha i}$ $(\mathfrak{g}\mathfrak{G}'^{\alpha i})$. Let $\mathfrak{S}^\alpha$ be the direct sum

$$\mathfrak{S}^\alpha = \sum_{i=1}^{d(\alpha)} \mathfrak{G}^{\alpha i} + \sum_{i=1}^{\tau(\alpha)} \mathfrak{G}'^{\alpha i}.$$

Let the homomorphism $H_\beta{}^\alpha$ of $\mathfrak{S}^\beta$ into $\mathfrak{S}^\alpha$ be defined by

$$(11.9) \qquad \begin{cases} H_\beta{}^\alpha(\mathfrak{g}\mathfrak{G}^{\beta i}) = \mathfrak{g}x^{\alpha i}_{\beta j}\mathfrak{G}^{\alpha j} \\ H_\beta{}^\alpha(\mathfrak{g}\mathfrak{G}'^{\beta i}) = \mathfrak{g}y^{\alpha i}_{\beta j}\mathfrak{G}^{\alpha j} + \mathfrak{g}z^{\alpha i}_{\beta j}\mathfrak{G}'^{\alpha j} \end{cases}$$

where the integers x, y and z are those of (11.4). Since $y^{\alpha i}_{\beta j}$ and $z^{\alpha i}_{\beta j}$ are unique mod $\theta^{\beta i}$ and $\mathfrak{g}$ in the second equation is of order $\theta^{\beta i}$, the transformation is uniquely defined. It is verified without difficulty that $\{\mathfrak{S}^\alpha\}$ is an inverse system.

Let us prove that $\{\mathfrak{S}^\alpha\}$ is independent of the decomposition of $\mathfrak{H}^\alpha$ into subgroups. If $\mathfrak{H}^\alpha$ be decomposed into the subgroups $\tilde{\mathfrak{X}}^{\alpha i}$ and $\tilde{\mathfrak{X}}'^{\alpha i}$, we construct groups $\tilde{\mathfrak{G}}^{\alpha i}$ and $\tilde{\mathfrak{G}}'^{\alpha i}$ in a similar way, and obtain a new group $\tilde{\mathfrak{S}}^\alpha$. Using (11.1), the transformation f^α defined by

$$f^\alpha(\mathfrak{g}\tilde{\mathfrak{G}}^{\alpha i}) = \mathfrak{g}u_j{}^{\alpha i}\mathfrak{G}^{\alpha j}$$
$$f^\alpha(\mathfrak{g}\tilde{\mathfrak{G}}'^{\alpha i}) = \mathfrak{g}v_j{}^{\alpha i}\mathfrak{G}^{\alpha j} + \mathfrak{g}w_j{}^{\alpha i}\mathfrak{G}'^{\alpha\alpha j}$$

has the inverse

$$\bar{f}^\alpha(\mathfrak{g}\mathfrak{G}^{\alpha j}) = \mathfrak{g}\bar{u}_k{}^{\alpha j}\tilde{\mathfrak{G}}^{\alpha k}$$
$$\bar{f}^\alpha(\mathfrak{g}\mathfrak{G}'^{\alpha j}) = \mathfrak{g}\bar{v}_k{}^{\alpha j}\tilde{\mathfrak{G}}^{\alpha k} + \mathfrak{g}\bar{w}_k{}^{\alpha j}\tilde{\mathfrak{G}}'^{\alpha k}.$$

This follows from the relations (11.3). And the homomorphism $\tilde{H}_\beta{}^\alpha$ defined by

$$\tilde{H}_\beta{}^\alpha(\mathfrak{g}\tilde{\mathfrak{G}}^{\beta i}) = \mathfrak{g}\tilde{x}^{\alpha i}_{\beta j}\tilde{\mathfrak{G}}^{\alpha j}$$
$$\tilde{H}_\beta{}^\alpha(\mathfrak{g}\tilde{\mathfrak{G}}'^{\beta i}) = \mathfrak{g}\tilde{y}^{\alpha i}_{\beta j}\tilde{\mathfrak{G}}^{\alpha j} + \mathfrak{g}\tilde{z}^{\alpha i}_{\beta j}\tilde{\mathfrak{G}}'^{\alpha j}$$

satisfies, in view of (11.6), the relation $H_\beta{}^\alpha = f^\alpha\tilde{H}_\beta{}^\alpha\bar{f}^\beta$. And we see that $\{\mathfrak{S}^\alpha\}$ is determined up to isomorphic systems by $\{\mathfrak{H}^\alpha\}$ and $\mathfrak{G}$.

Just as above we can prove that the limit group $\mathfrak{S}$ of $\{\mathfrak{S}^\alpha\}$ is determined up to bicontinuous isomorphisms by $\mathfrak{G}$ and $\mathfrak{H}$.

12. The isomorphism. We now assume that $\mathfrak{H} = \mathfrak{H}_p(A, B, \mathfrak{X})$ for a topological space A and closed subset B. We shall prove that $\mathfrak{T} = \mathfrak{T}_{p-1}(A, B, \mathfrak{G})$ and $\mathfrak{S} = \mathfrak{S}_p(A, B, \mathfrak{G})$.

We suppose that $\{\mathfrak{K}_q{}^\alpha\}$ $(q = 0, 1, \cdots, p+1)$ is the canonical inverse system of chains constructed in No. 10 for the space A. We suppose, for each α, bases have been chosen in the groups $\mathfrak{K}_p{}^\alpha$ and $\mathfrak{K}^\alpha{}_{p-1}$ so that the boundary relations for the chains of $\mathfrak{K}_p{}^\alpha$ are in quasi-canonical form. We suppose in particular that we have split these basis elements into five classes so that the relations (8.4) hold.

If we pass to the corresponding homology groups over $\mathfrak{G}$, we find by Theorem 8 that

$$\mathfrak{T}^\alpha{}_{p-1}(\mathfrak{G}) = \sum \mathfrak{G}^{*\alpha i} \qquad (i = 1, \cdots, \tau_{p-1}(\alpha))$$

where $\mathfrak{G}^{*\alpha i}$ is obtained by reducing $\mathfrak{G}$ modulo the closure of the subgroup of elements divisible by $\theta^{\alpha i}_{p-1}$, and

$$\mathfrak{S}_p{}^{\alpha}(\mathfrak{G}) = \sum \mathfrak{G}^{\alpha j} + \sum \mathfrak{G}'^{\alpha k} \quad (j = 1, \cdots, R_p(\alpha)\,;\ k = 1, \cdots, \tau_{p-1}(\alpha))$$

where $\mathfrak{G}^{\alpha j}$ is isomorphic with $\mathfrak{G}$ and $\mathfrak{G}'^{\alpha k}$ with the subgroup of $\mathfrak{G}$ of order $\theta^{\alpha k}_{p-1}$. In particular

$$\mathfrak{H}_p{}^{\alpha}(\mathfrak{X}) = \sum \mathfrak{X}^{\alpha j} + \sum \mathfrak{X}'^{\alpha k}. \tag{12.1}$$

Since $\mathfrak{H}_p(A, B, \mathfrak{X})$ is the limit group of $\{\mathfrak{H}_p{}^{\alpha}(\mathfrak{X})\}$, we may suppose that the latter is the system used in No. 11 for constructing $\mathfrak{T}$ and $\mathfrak{S}$, and we may suppose moreover that the decomposition there used of each $\mathfrak{H}_p{}^{\alpha}(\mathfrak{X})$ into a direct sum is the decomposition (12.1). It is then immediately clear that $\mathfrak{T}^{\alpha} = \mathfrak{T}^{\alpha}{}_{p-1}(\mathfrak{G})$ and $\mathfrak{S}^{\alpha} = \mathfrak{S}_p{}^{\alpha}(\mathfrak{G})$. These isomorphisms are set up in an obvious way. We have only to prove that under these isomorphisms the homomorphisms of the systems $\{\mathfrak{T}^{\alpha}\}$ and $\{\mathfrak{S}^{\alpha}\}$ carry over into those of $\{\mathfrak{T}^{\alpha}{}_{p-1}(\mathfrak{G})\}$ and $\{\mathfrak{S}_p{}^{\alpha}(\mathfrak{G})\}$ respectively. The latter case is trivial. For the former, suppose the homomorphism of $\mathfrak{K}_p{}^{\beta}$ into $\mathfrak{K}_p{}^{\alpha}$ has on the basis element $d_p{}^{\beta k}$ of class four (see (8.4)) the value

$$y^{\alpha k}_{\beta j} c_p{}^{\alpha j} + z^{\alpha k}_{\beta j} d_p{}^{\alpha j} + w^{\alpha k}_{\beta j} e_p{}^{\alpha j},$$

(we recall that, in a canonical system of chains, $\mathfrak{C}_p{}^{\beta}$ is mapped into $\mathfrak{C}_p{}^{\alpha}$). Then, for the homomorphism of $\mathfrak{H}_p{}^{\beta}(\mathfrak{X})$ into $\mathfrak{H}_p{}^{\alpha}(\mathfrak{X})$,

$$H_{\beta}{}^{\alpha}(\mathfrak{x}\mathfrak{X}'^{\beta k}) = \mathfrak{x} y^{\alpha k}_{\beta j} \mathfrak{X}^{\alpha j} + \mathfrak{x} z^{\alpha k}_{\beta j} \mathfrak{X}'^{\alpha j}$$

As $FH = HF$, we have

$$\theta^{\beta k}_{p-1} H_{\beta}{}^{\alpha}(b^{\beta k}_{p-1}) = H_{\beta}{}^{\alpha} F(d_p{}^{\beta k}) = F H_{\beta}{}^{\alpha}(d_p{}^{\beta k}) = z^{\alpha k}_{\beta j} \theta^{\alpha j}_{p-1} b^{\alpha j}_{p-1} + w^{\alpha k}_{\beta j} a^{\alpha j}_{p-1}.$$

It follows that $z^{\alpha k}_{\beta j} \theta^{\alpha j}_{p-1}$ (not summed on j) is divisible by $\theta^{\beta k}_{p-1}$. If $s^{\alpha k}_{\beta j}$ denotes the quotient, then

$$H_{\beta}{}^{\alpha}(b^{\beta k}_{p-1}) \sim s^{\alpha k}_{\beta j} b^{\alpha j}_{p-1}.$$

Hence

$$H_{\beta}{}^{\alpha}(\mathfrak{g}\mathfrak{G}^{*\beta k}) = \mathfrak{g} s^{\alpha k}_{\beta j} \mathfrak{G}^{*\alpha j}.$$

As this is the homomorphism (11.8), we have proved

THEOREM 12. *The group $\mathfrak{H}_p(A, B, \mathfrak{G})$ is the direct sum $\mathfrak{T}_p + \mathfrak{S}_p$ where $\mathfrak{T}_p$ is an invariant of the pair $\mathfrak{G}$, $\mathfrak{H}_{p+1}(A, B, \mathfrak{X})$ and $\mathfrak{S}_p$ is an invariant of the pair $\mathfrak{G}$, $\mathfrak{H}_p(A, B, \mathfrak{X})$. Thus the group $\mathfrak{X}$ is a universal coefficient group for the Čech homology theory of a topological space.*

It is well known that in a compact metric space the Čech theory and the Vietoris [18] theory are equivalent; hence

COROLLARY. *The group $\mathfrak{X}$ is universal for the Vietoris homology theory of a compact metric space.*

REMARK. Čech has pointed out to me the following simple invariant definition of the group $\mathfrak{S}$ in terms of the groups $\mathfrak{G}$ and $\mathfrak{H}$. Let $\mathfrak{H}^*$ be the discrete group of continuous characters of $\mathfrak{H}$. Then $\mathfrak{S}$ is defined to be the group of all homomorphic mappings of $\mathfrak{H}^*$ into $\mathfrak{G}$. If U is a neighborhood of zero in $\mathfrak{G}$ and $h_1, h_2, \cdots, h_k$ are a finite number of elements of $\mathfrak{H}^*$, the set V of elements of $\mathfrak{S}$ mapping each h_i into U is called a neighborhood of zero in $\mathfrak{S}$. That $\mathfrak{S}$ is the group constructed in No. 11 is obvious if $\mathfrak{H}$ is an elementary group. Otherwise let $\mathfrak{H}$ be a limit group of the inverse system $\{\mathfrak{H}^a\}$ of elementary groups. Let $\mathfrak{S}^a$ be the group of homomorphic mappings of $\mathfrak{H}^{*a}$ into $\mathfrak{G}$. As $\{\mathfrak{H}^{*a}\}$ is a direct system (No. 5), it is easy to see that $\{\mathfrak{S}^a\}$ is an inverse system and that its limit group is the group of homomorphic mappings of $\mathfrak{H}^*$ into $\mathfrak{G}$. In this way the case of a general bicompact group $\mathfrak{H}$ is reduced to that of an elementary group. If one could find an equally simple invariant definition of $\mathfrak{T}$ in terms of $\mathfrak{G}$ and $\mathfrak{H}$, the argument of the preceding two sections could be greatly simplified.

IV. The infinite complex.

13. Infinite cycles. Let K be an infinite complex with a countable number of cells. We require that K be locally finite in the sense that the star of any vertex is finite. Let L be a closed subcomplex of K and let the p-cells of $K - L$ be ordered in a sequence: $E_p{}^i$ $(i = 1, 2, \cdots)$. A p-chain over $\mathfrak{G}$ of K mod L is an infinite linear form: $\mathfrak{g}_i E_p{}^i$. These constitute a group $\mathfrak{K}_p(K, L, \mathfrak{G})$ in which a topology is introduced as follows. If U is a neighborhood of zero in $\mathfrak{G}$ and n is an integer, the set V of those chains whose first n coefficients lie in U is a neighborhood of zero in $\mathfrak{K}$. As in the case of a finite complex, the boundary operator F is a continuous homomorphism of $\mathfrak{K}_p$ into $\mathfrak{K}_{p-1}$. Then cycles and bounding cycles can be distinguished. The group $\mathfrak{H}_p(K, L, \mathfrak{G})$ is obtained by reducing the group of cycles modulo the closure of the subgroup of bounding cycles.

14. Universal group theorem. If we can show that the homology theory of an infinite complex can be analyzed in terms of inverse systems of homology groups of finite complexes, the proof that $\mathfrak{X}$ is a universal coefficient group for infinite cycles is clearly contained in the preceding sections. The definition just given of $\mathfrak{H}_p(K, L, \mathfrak{G})$ is standard since it follows closely the spirit of the definition of Lefschetz [15; p. 299]. Consider the following alternative definition.

Let $\{L^\alpha\}$ be a sequence of closed subcomplexes of K such that $\prod L^\alpha = L$, $L^\alpha \supset L^{\alpha+1}$, and each $K - L^\alpha$ is finite. Let $\mathfrak{K}_p{}^\alpha(\mathfrak{G}) = \mathfrak{K}_p(K, L^\alpha, \mathfrak{G})$. Suppose $L^\alpha \supset L^\beta$. To each chain of $\mathfrak{K}_p{}^\beta(\mathfrak{G})$ we associate the chain of $\mathfrak{K}_p{}^\alpha(\mathfrak{G})$ obtained by omitting its terms involving cells on L^α. This is a homomorphism of $\mathfrak{K}_p{}^\beta(\mathfrak{G})$ into $\mathfrak{K}_p{}^\alpha(\mathfrak{G})$. It preserves cycles and bounding cycles, and therefore induces a continuous homomorphism of $\mathfrak{H}_p{}^\beta(\mathfrak{G})$ into $\mathfrak{H}_p{}^\alpha(\mathfrak{G})$. In this way $\{\mathfrak{H}_p{}^\alpha(\mathfrak{G})\}$ constitutes an inverse system with a limit group $\mathfrak{H}'_p(K, L, \mathfrak{G})$. It is clear that $\mathfrak{H}'_p(\mathfrak{G})$ does not depend on the particular sequence $\{L^\alpha\}$; for any other sequence $\{L'^\alpha\}$ satisfying the same conditions determines an equivalent system $\{\mathfrak{H}'_p{}^\alpha(\mathfrak{G})\}$. We will see under what conditions the group $\mathfrak{H}'_p(K, L, \mathfrak{G})$ is $\mathfrak{H}_p(K, L, \mathfrak{G})$.

We define a continuous homomorphism of $\mathfrak{H}_p(\mathfrak{G})$ into $\mathfrak{H}'_p(\mathfrak{G})$ as follows. If $\mathfrak{c}_p$ is an infinite cycle mod L, let $\mathfrak{c}_p{}^\alpha$ be the cycle mod L^α obtained by omitting terms of $\mathfrak{c}_p$ involving cells on L^α. Suppose $\mathfrak{h} \epsilon \mathfrak{H}_p(\mathfrak{G})$ has $\mathfrak{c}_p$ as a representative cycle. Then $\mathfrak{c}_p{}^\alpha$ is representative of a class $\mathfrak{h}^\alpha \epsilon \mathfrak{H}_p{}^\alpha(\mathfrak{G})$. Clearly $\{\mathfrak{h}^\alpha\}$ is an element $\mathfrak{h}' \epsilon \mathfrak{H}'_p(\mathfrak{G})$. If $\mathfrak{c}'_p \sim \mathfrak{c}_p$, then $\mathfrak{c}'_p{}^\alpha \sim \mathfrak{c}_p{}^\alpha$, and $\mathfrak{h}'$ is independent of the representative $\mathfrak{c}_p$ of $\mathfrak{h}$. Thus $f(\mathfrak{h}) = \mathfrak{h}'$ is a continuous homomorphism of $\mathfrak{H}_p(\mathfrak{G})$ into $\mathfrak{H}'_p(\mathfrak{G})$. In fact f is an isomorphism of $\mathfrak{H}_p(\mathfrak{G})$ into a subgroup of $\mathfrak{H}'_p(\mathfrak{G})$. For suppose $f(\mathfrak{h}) = 0$. Then each $\mathfrak{c}_p{}^\alpha$ is a limit of bounding cycles mod L^α. That is: to a neighborhood V of $\mathfrak{c}_p{}^\alpha$ corresponds a chain $\mathfrak{k}_{p+1}$ such that $F(\mathfrak{k}_{p+1})$ mod L^α lies in V. If W is a neighborhood of $\mathfrak{c}_p$ determined by U in $\mathfrak{G}$ and the integer n, choose L^α so that it does not contain the first n p-cells of $K - L$. Let V be the neighborhood of $\mathfrak{c}_p{}^\alpha$ determined by U in $\mathfrak{G}$. Then $F(\mathfrak{k}_{p+1})$ mod L lies in W. Thus $\mathfrak{c}_p$ is a limit of bounding cycles, and $\mathfrak{h} = 0$.

If the group $\mathfrak{G}$ has the division-closure property (No. 8), we shall prove that the inverse of f is defined over the whole of $\mathfrak{H}'_p(\mathfrak{G})$. Suppose $\mathfrak{h}' = \{\mathfrak{h}^\alpha\}$ is an arbitrary element of $\mathfrak{H}'_p(\mathfrak{G})$. Let $\mathfrak{c}_p{}^\alpha$ be a representative of $\mathfrak{h}^\alpha$. Then $\mathfrak{c}_p{}^{\alpha+1} - \mathfrak{c}_p{}^\alpha$ reduced mod L^α is a limit of bounding cycles. As $\mathfrak{G}$ has the division-closure property, the group of bounding cycles of K mod L^α is closed. Hence there is a chain $\mathfrak{k}^\alpha{}_{p+1}$ such that

$$F(\mathfrak{k}^\alpha{}_{p+1}) = \mathfrak{c}_p{}^{\alpha+1} - \mathfrak{c}_p{}^\alpha \text{ mod } L^\alpha.$$

Consider the sequence of chains

$$\mathfrak{c}_p{}^1, \mathfrak{c}_p{}^2 - F(\mathfrak{k}^1{}_{p+1}), \cdots, \mathfrak{c}_p{}^\alpha - \sum_{i=1}^{\alpha-1} F(\mathfrak{k}^i{}_{p+1}), \cdots$$

where $F(\mathfrak{k}^i{}_{p+1})$ is the boundary mod L of $\mathfrak{k}^i{}_{p+1}$. The $(\alpha + 1)$-st term is the α-th when reduced mod L^α. The sequence therefore converges to a cycle $\mathfrak{c}_p$ of

K mod L. Furthermore $\mathfrak{c}_p \sim \mathfrak{c}_p{}^a$ mod L^a; hence $\mathfrak{c}_p$ represents a class $\mathfrak{h} \epsilon \mathfrak{H}_p(\mathfrak{G})$ such that $f(\mathfrak{h}) = \mathfrak{h}'$. It is not difficult to show that the inverse of f when defined is continuous.

Thus the two definitions of $\mathfrak{H}_p(K, L, \mathfrak{G})$ are equivalent if $\mathfrak{G}$ has the division-closure property. If we prefer the second definition then we can assert without qualification that $\mathfrak{X}$ is universal for the infinite cycles of an infinite complex.

It would be interesting to determine whether or not the two definitions are always equivalent.

Appendix I.

15. The homology groups of a bicompact connected group. We propose to establish the following

THEOREM 15. *If $\mathfrak{A}$ is a bicompact connected group, then $\mathfrak{A}$ and $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X})$ are bicontinuously isomorphic.*

The theorem is fairly trivial for a toral group. The general case is obtained by a limiting process to which the next few sections are devoted. We base our definition of homology groups on finite coverings by open sets. This we may do, for a topological group satisfies the separation axiom 5 of Hausdorff; so a bicompact group is normal (Remark I, No. 9). In Theorem 17, open coverings are used in an essential way, we have no proof of the theorem if the homology groups are based on closed coverings.

16. The induced homomorphism. Let A^1 and A^2 be topological spaces and f a continuous mapping of A^1 into a subset of A^2. Let $\{\phi^{i\alpha}\}$ $(i = 1, 2)$ be a complete system of finite coverings by open sets of A^i. Let $K^{i\alpha}$ be the nerve of $\phi^{i\alpha}$, and $\mathfrak{H}_p{}^{i\alpha} = \mathfrak{H}_p(K^{i\alpha}, \mathfrak{G})$ and $\mathfrak{H}_p{}^i = \mathfrak{H}_p(A^i, \mathfrak{G})$.

By means of f we shall define an inverse system S^{12} which includes $S^2 = \{\mathfrak{H}_p{}^{2\alpha}\}$ as a subsystem and $S^1 = \{\mathfrak{H}_p{}^{1\alpha}\}$ as a complete subsystem. S^{12} shall consist of the groups of the systems S^1 and S^2; it shall include all the homomorphisms of S^1 and S^2 and certain additional ones defined as follows. An open set in A^2 has an open set as its inverse image in A^1. Thus $\phi^{2\alpha}$ has as its inverse image a finite covering by open sets $\psi^{1\alpha}$ of A^1. Let $\phi^{1\beta}$ be a refinement of $\psi^{1\alpha}$. Then the projections $\phi^{1\beta} \to \psi^{1\alpha} \to \phi^{2\beta}$ determines a simplicial mapping of $K^{1\beta}$ into $K^{2\alpha}$. Let $H^{\alpha\beta}$ be the continuous homomorphism of $\mathfrak{H}_p{}^{1\beta}$ into $\mathfrak{H}_p{}^{2\alpha}$ induced by this simplicial mapping (No. 9). Let us include among the homomorphisms of S^{12} the homomorphisms $H^{\alpha\beta}$ for all α and related β satisfying the condition that $\phi^{1\beta}$ is a refinement of $\psi^{1\alpha}$. It is not difficult to prove that S^{12} is an inverse system.

As S^1 is a complete subsystem of S^{12}, $\mathfrak{H}_p{}^1$ is the limit group of S^{12}. The coördinates of an element $\mathfrak{h}^1$ of $\mathfrak{H}_p{}^1$ in the groups of S^2 make up the coördinates of an element $\mathfrak{h}^2$ of $\mathfrak{H}_p{}^2$. The correspondence $\bar{f}(\mathfrak{h}^1) = \mathfrak{h}^2$ is a homomorphism of $\mathfrak{H}_p{}^1$ into $\mathfrak{H}_p{}^2$. It is continuous; for, if V^2 is a neighborhood of $\mathfrak{h}^2$ determined by a neighborhood V^{2a} of $\mathfrak{h}^{2a}$, then the neighborhood V^1 of $\mathfrak{h}^1$ determined by V^{2a} is such that $\bar{f}(V^1) \subset V^2$.

The homomorphism $\bar{f}$ of $\mathfrak{H}_p{}^1$ into $\mathfrak{H}_p{}^2$ is said to be induced by f.

We must show that $\bar{f}$ is independent of the systems $\{\phi^{1a}\}$ and $\{\phi^{2a}\}$ used in its definition. But this is trivial; for new systems may be included with the old as complete subsystems of larger systems.

LEMMA 16. *If f and g are continuous mappings of A^1 into A^2 and A^2 into A^3, respectively, then the induced homomorphisms satisfy:* $\overline{(gf)} = \bar{g}\bar{f}$.

In S^{12} there are no homomorphisms of a group of S^2 into one of S^1. Similarly in S^{23}. Hence we may form the logical sum of S^{12} and S^{23} and obtain a new inverse system S^{123} which contains S^{13} as a complete subsystem. In S^{123} we may compare $\overline{(gf)}$ and $\bar{g}\bar{f}$; and the assertion of the lemma follows.

17. On the homology groups of a limit space. Let $\{A^a\}$ be an inverse system of bicompact spaces and A its limit space. We shall assume that, for all α, each point of A^a is the coördinate of an element of A. The set of groups $\{\mathfrak{H}_p{}^a\}$ $(\mathfrak{H}_p{}^a = \mathfrak{H}_p(A^a, \mathfrak{G}))$ together with the homomorphisms induced by the mappings of the system constitutes, by Lemma 16, an inverse system. Let $\mathfrak{H}_p = \mathfrak{H}_p(A, \mathfrak{G})$. Then

THEOREM 17. *$\mathfrak{H}_p$ is the limit group of $\{\mathfrak{H}_p{}^a\}$.*

Let $\{\phi^{(a)\beta}\}$ be a complete system of finite coverings by open sets of A^a ((α) indicates that α is fixed). Each open set of $\phi^{a\beta}$ has an open inverse image in A. Hence $\phi^{a\beta}$ determines a finite covering $\psi^{a\beta}$ of A by open sets. The double system $\{\{\psi^{a\beta}\}\}$ is complete. For let ψ be an arbitrary finite covering of A by open sets. As A is bicompact, ψ has a refinement ψ' consisting of neighborhoods. Let the open set V^i $(i = 1, \cdots, k)$ of ψ' be defined by the neighborhood V^{a_i} in A^{a_i}. Let A^a be a common refinement of $A^{a_1}, \cdots, A^{a_k}$. Then the images V^{ai} of the V^i in A^a constitute a finite covering of A^a by open sets. Let $\phi^{a\beta}$ be a refinement of this covering; then clearly $\psi^{a\beta}$ is a refinement of ψ.

The nerve $K^{a\beta}$ of $\phi^{a\beta}$ is likewise the nerve of $\psi^{a\beta}$, for the image of A in A^a covers A^a. Let $\mathfrak{H}_p{}^{a\beta} = \mathfrak{H}_p(K^{a\beta}, \mathfrak{G})$. Then $\mathfrak{H}_p$ is the limit group of the double system $\{\{\mathfrak{H}_p{}^{a\beta}\}\}$. As $\psi^{a\beta_1} \to \psi^{a\beta_2}$ if and only if $\phi^{a\beta_1} \to \phi^{a\beta_2}$, $\mathfrak{H}_p{}^a$ is the limit group of the subsystem $\{\mathfrak{H}_p{}^{(a)\beta}\}$.

The coördinates in the subsystem $\{\mathfrak{H}_p^{(\alpha)\beta}\}$ of an element $\mathfrak{h} \in \mathfrak{H}_p$ make up an element $\mathfrak{h}^\alpha \in \mathfrak{H}_p^\alpha$. By the definition of the induced homomorphisms, the elements $\mathfrak{h}^\alpha$ are the coördinates of an element $\mathfrak{h}'$ of the limit group $\mathfrak{H}'_p$ of $\{\mathfrak{H}_p^\alpha\}$. The correspondence $f(\mathfrak{h}) = \mathfrak{h}'$ so defined is a bicontinuous isomorphism. It is trivial that f is a homomorphism. If $\mathfrak{h}'$ is an arbitrary element of $\mathfrak{H}'_p$, its coördinate $\mathfrak{h}^\alpha$ in $\mathfrak{H}_p^\alpha$ has a coördinate $\mathfrak{h}^{\alpha\beta}$ in $\mathfrak{H}_p^{\alpha\beta}$. Suppose $\mathfrak{H}_p^{\gamma\delta} \rightarrow \mathfrak{H}_p^{\alpha\beta}$. Let A^ϵ be a refinement of A^γ and A^α; and let $\psi^{\epsilon\lambda}$ be a refinement of $\psi^{\alpha\beta}$ and $\psi^{\gamma\delta}$. Then, by the definition of the induced homomorphism of $\mathfrak{H}_p^\epsilon$ into $\mathfrak{H}_p^\alpha$, we have $\mathfrak{h}^{\epsilon\lambda} \rightarrow \mathfrak{h}^{\alpha\beta}$. Similarly $\mathfrak{h}^{\epsilon\lambda} \rightarrow \mathfrak{h}^{\gamma\delta}$. From the uniqueness of projections $\mathfrak{h}^{\gamma\delta} \rightarrow \mathfrak{h}^{\alpha\beta}$. It follows that $\{\{\mathfrak{h}^{\alpha\beta}\}\}$ is an element $\mathfrak{h} \in \mathfrak{H}_p$ such that $f(\mathfrak{h}) = \mathfrak{h}'$. Therefore $f(\mathfrak{H}_p)$ covers $\mathfrak{H}'_p$.

If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are distinct in $\mathfrak{H}_p$, then, for some pair α, β, $\mathfrak{h}_1^{\alpha\beta}$ and $\mathfrak{h}_2^{\alpha\beta}$ are distinct. Then $\mathfrak{h}_1^\alpha$ and $\mathfrak{h}_2^\alpha$ are distinct; and, finally $\mathfrak{h}'_1$ and $\mathfrak{h}'_2$ are distinct. Thus f is an isomorphism.

Let V' be a neighborhood of $\mathfrak{h}'$ in $\mathfrak{H}'_p$ determined by a neighborhood V^α of $\mathfrak{h}^\alpha$ in $\mathfrak{H}_p^\alpha$ which, in turn, is determined by a neighborhood $V^{\alpha\beta}$ of $\mathfrak{h}^{\alpha\beta}$ in $\mathfrak{H}_p^{\alpha\beta}$. Then $V^{\alpha\beta}$ determines a neighborhood V of $\mathfrak{h}$ in $\mathfrak{H}_p$ such that $f(V) \subset V'$. So f is continuous. On the other hand, if V is a neighborhood of $\mathfrak{h}$ in $\mathfrak{H}_p$ determined by a neighborhood $V^{\alpha\beta}$ of $\mathfrak{h}^{\alpha\beta}$ in $\mathfrak{H}_p^{\alpha\beta}$, $V^{\alpha\beta}$ determines a neighborhood V^α of $\mathfrak{h}^\alpha$ in $\mathfrak{H}_p^\alpha$, and this in turn determines a neighborhood V' of $\mathfrak{h}'$ in $\mathfrak{H}'_p$ satisfying $f(V) \supset V'$. So f is inversely continuous, and the theorem is proved.

18. Proof of Theorem 15. By Theorem 7.3, $\mathfrak{A}$ may be represented as the limit group of an inverse system $\{\mathfrak{A}^\alpha\}$ of elementary groups so that the image of $\mathfrak{A}$ in $\mathfrak{A}^\alpha$ covers the latter. As $\mathfrak{A}$ is connected, so is $\mathfrak{A}^\alpha$; hence $\mathfrak{A}^\alpha$ is a finite dimensional toral group. Let us express $\mathfrak{A}^\alpha$ as the direct sum $\sum_{i=1}^{k} \mathfrak{X}^{\alpha i}$ of groups isomorphic with $\mathfrak{X}$. The point set $\mathfrak{X}^{\alpha i}$ is a simple closed curve $\Gamma^{\alpha i}$ on $\mathfrak{A}^\alpha$. Giving to $\mathfrak{X}$ a definite orientation gives to each simple closed curve an orientation so that the set of 1-cycles (over $\mathfrak{J}$) so obtained form a 1-dimensional homology basis in $\mathfrak{A}^\alpha$.

If $\mathfrak{X}^\alpha$ is a homomorphic image of $\mathfrak{X}$ in $\mathfrak{A}^\alpha$, we may express $\mathfrak{X}^\alpha$ as a linear form $a_i\mathfrak{X}^{\alpha i}$ in the basis subgroups with integer coefficients (see No. 11). If Γ^α is the singular image on $\mathfrak{A}^\alpha$ of the basic 1-cycle of $\mathfrak{X}$, we assert that

$$\Gamma^\alpha \sim a_i\Gamma^{\alpha i}. \tag{18.1}$$

Let $\bar{\Gamma}^\alpha$ be the 1-cycle which is the singular image of the basic 1-cycle of $\mathfrak{X}$ under the mapping which sends $\mathfrak{x} \in \mathfrak{X}$ into $\mathfrak{x}\bar{\mathfrak{X}}^\alpha = \mathfrak{x}a_2\mathfrak{X}^{\alpha 2} + \cdots + \mathfrak{x}a_k\mathfrak{X}^{\alpha k}$. Let T be the 2-simplex in the (x, y)-plane having the vertices $(0,0)$, $(0,1)$, and $(1,1)$.

Let C_1, C_2, C_3 be the edges $[(0,0),(0,1)]$, $[(0,1),(1,1)]$ $[(0,0),(1,1)]$ respectively. If T is suitably oriented, then $F(T) = C_1 + C_2 - C_3$. The transformation

$$f(x,y) = ya_1\mathfrak{X}^{a1} + x\bar{\mathfrak{X}}^{a} \qquad (0 \leqq x \leqq 1;\ x \leqq y \leqq 1)$$

of T into $\mathfrak{A}^a$ (on the right side x and y are real numbers mod 1) is continuous over T and maps C_1 into $a_1\Gamma^{a1}$, C_2 into $\bar{\Gamma}^a$, and C_3 into Γ^a. Thus $\Gamma^a \sim a_1\Gamma^{a1} + \bar{\Gamma}^a$. We may treat $\bar{\Gamma}^a$ in the same way. Performing this operation k times, we obtain the relation (18.1).

We suppose that we have chosen basis subgroups $\mathfrak{X}^{ai}$ $(i = 1, \cdots, k(\alpha))$ and corresponding 1-cycles Γ^{ai} on each group $\mathfrak{A}^a$ of the system. If $\mathfrak{A}^\beta \to \mathfrak{A}^a$, this homomorphism f may be written (No. 11)

(18.2) $$f(\mathfrak{x}\mathfrak{X}^{\beta i}) = \mathfrak{x}a_j{}^i\mathfrak{X}^{aj}.$$

Then, by (18.1),

(18.3) $$f(\Gamma^{\beta i}) \sim a_j{}^i\Gamma^{aj}.$$

Let $\mathfrak{H}^a = \mathfrak{H}_1(\mathfrak{A}^a, \mathfrak{X})$. $\mathfrak{H}^a$ may be represented as the group of linear forms $\mathfrak{x}_i\Gamma^{ai}$ where $\mathfrak{x}_i \epsilon \mathfrak{X}$. Let I^a be the isomorphism which pairs $\mathfrak{x}_i\Gamma^{ai}$ of $\mathfrak{H}^a$ with $\mathfrak{x}_i\mathfrak{X}^{ai}$ of $\mathfrak{A}^a$.

If $\mathfrak{A}^\beta \to \mathfrak{A}^a$, the induced homomorphism $\mathfrak{H}_p{}^\beta \to \mathfrak{H}_p{}^a$, by (18.3), is

$$\bar{f}(\mathfrak{x}_i\Gamma^{\beta i}) = \mathfrak{x}_i a_j{}^i\Gamma^{aj}.$$

It follows that the system of isomorphisms $\{I^a\}$ establishes an isomorphism between the inverse systems $\{\mathfrak{A}^a\}$ and $\{\mathfrak{H}^a\}$. The limit group $\mathfrak{H}$ of $\{\mathfrak{H}^a\}$ is therefore bicontinuously isomorphic with $\mathfrak{A}$. By Theorem 17, $\mathfrak{H}$ is $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X})$. This completes the proof.

Appendix II.

Example 1. We shall prove that the integers do not form a universal coefficient group for the homology theory of a compact space.

Let A^m $(m = 1, 2, \cdots)$ be a simple closed curve, and $M^m{}_{m+1}$ a continuous mapping of A^{m+1} into A^m of degree 2 (i. e. A^{m+1} is wrapped twice around A^m). Let A be the limit space of this inverse sequence.[14] Then A is 1-dimensional, bicompact, and it has the 2nd countability axiom. It may therefore be imbedded homeomorphically in euclidean 3-space. $\mathfrak{H}_1(A^m, \mathfrak{J})$ ($\mathfrak{J}$ = group of integers) is a free group on one generator. The induced homomorphism of $\mathfrak{H}_1(A^{m+1}, \mathfrak{J})$ into $\mathfrak{H}_1(A^m, \mathfrak{J})$ maps the generator of the first group into twice

[14] This example was considered by Vietoris 18.

the generator of the second. One proves readily that the limit group of $\{\mathfrak{H}_1(A^m, \mathfrak{J})\}$ reduces to the zero. By Theorem 17, this limit group is $\mathfrak{H}_1(A, \mathfrak{J})$. On the other hand, $\mathfrak{H}_1(A^m, \mathfrak{X})$ is isomorphic with $\mathfrak{X}$ and the homomorphism of $\mathfrak{H}_1(A^{m+1}, \mathfrak{X})$ into $\mathfrak{H}_1(A^m, \mathfrak{X})$ has the degree 2. It follows, by Lemma 2. 1, that $\mathfrak{H}_1(A, \mathfrak{X})$ is not the zero group. In fact if $M^m{}_{m+1}$ is sufficiently smooth, it is easy to see that $\mathfrak{H}_1(A, \mathfrak{X})$ and A are homeomorphic. Since one cannot deduce the structure of $\mathfrak{H}_1(A, \mathfrak{X})$ from that of $\mathfrak{H}_1(A, \mathfrak{J})$ the statement is proved.

Example 2. We shall prove that $\mathfrak{X}$ is not a universal coefficient group for the homology theory of the finite cycles of an infinite complex.

Let A^m $(m = 1, 2, \cdots)$ be a simple closed curve and $M^m{}_{m+1}$ a continuous mapping of A^{m+1} into A^m of degree $m + 1$. Let A be the limit space of this sequence, and let A be imbedded homeomorphically in euclidean 3-space E_3. It is not difficult to prove that the 1-dimensional homology group over $\mathfrak{J}$ of the finite cycles of $E_3 - A$ is isomorphic with the group of rational numbers. Let A' be homeomorphic with A, and let it be imbedded in $E_3 - A$. Then the 1-dimensional homology group over $\mathfrak{J}$ of the finite cycles of $E_3 - (A + A')$ is the direct sum of two rational groups. Thus the two homology groups have ranks 1 and 2 respectively. However if we apply the theorem of Čech [9] to compute the 1-dimensional groups over $\mathfrak{X}$ of the finite cycles of $E_3 - A$ and $E_3 - (A + A')$ it is found that the first group is obtained by reducing $\mathfrak{X}$ modulo its subgroup of elements of finite order, and the second group is obtained by reducing a 2-dimensional toral group modulo its subgroup of elements of finite order. Both of these groups are isomorphic with the direct sum of groups of rational numbers equal in number to the power of the continuum. This proves the statement.

Example 3. We shall prove that, even if the coefficient group has the division-closure property, the group of bounding cycles in an infinite complex may not be closed. This is in contrast with the case of a finite complex (No. 8).

Let L' be the product complex of a circle by a line segment. We choose one of the two circles bounding L' and identify triplets of equally spaced points. The resulting complex consists of two 1-cycles Γ^1 and Γ^2 and a 2-chain L such that $F(L) = \Gamma^2 - 3\Gamma^1$. Let L_i $(i = 1, 2, \cdots)$ be a sequence of such complexes; and let us identify $\Gamma_i{}^1$ with $\Gamma^2{}_{i-1}$ $(i = 2, 3, \cdots)$ and give this cycle the new notation Γ_i. Let us add a 2-cell L_0 whose boundary is Γ_1. Then $F(L_0) = \Gamma_1$ and $F(L_i) = \Gamma_{i+1} - 3\Gamma_i$ $(i = 1, 2, \cdots)$ are the bounding relations in the resulting infinite complex K. As

$$\Gamma_i \sim 3\Gamma_{i-1} \sim 3^2\Gamma_{i-2} \sim \cdots \sim 3^{i-1}\Gamma_1 \sim 0,$$

the infinite 1-cycle $\sum_{i=1}^{\infty} \Gamma_i$ is a limit of the bounding cycles $\sum_{i=1}^{k} \Gamma_i$ $(k = 1, 2, \cdots)$. Suppose as is impossible, that there are integers x_i $(i = 0, 1, 2, \cdots)$ such that $F(\sum_{i=0}^{\infty} x_i L_i) = \sum_{i=1}^{\infty} \Gamma_i$. By a simple computation

$$F(\sum_{i=0}^{\infty} x_i L_i) = \sum_{i=1}^{\infty} (x_{i-1} - 3x_i)\Gamma_i.$$

It follows that $3x_i = x_{i-1} - 1$. Thus $|x_i| < |x_{i-1}| < \cdots < |x_0|$. As the x's are integers, this cannot hold for every integer i. The contradiction proves the statement.

Example 4. We have proved in No. 10, that the p-th homology group of a topological space decomposes into a direct sum of a torsion group and a reduced homology group. In the special case of a finite complex, we have seen in No. 8 that the reduced homology group admits a further decomposition into the direct sum of two groups. We shall show by an example that this further decomposition does not occur in general in the homology groups of a compact metric space.

For a finite complex K, $\mathfrak{T}_p(K, \mathfrak{X})$ reduces to zero and $\mathfrak{H}_p(K, \mathfrak{X})$ is the direct sum of its component of zero and a finite group. We shall construct a 2-dimensional compact metric space A such that the component of zero of $\mathfrak{H}_2(A, \mathfrak{X})$ is not a direct summand of the entire group.

Let $\mathfrak{H}^*$ be the discrete group [15] generated by $e_1, e_2, \cdots$, and $a_0, a_1, a_2, \cdots$ subject to the relations $2^{2n} e_n = 0$ and $2a_n = a_{n-1} + e_n$ $(n = 1, 2, \cdots)$.

Let us prove that the subgroup $\mathfrak{E}$ of $\mathfrak{H}^*$ of elements of finite order is generated by $e_1, e_2, \cdots$. Suppose $a = \sum_{1}^{n} \alpha_i e_i + \sum_{0}^{n} \beta_i a_i$ is of finite order. Then $a' = \sum_{0}^{n} \beta_i a_i$ is of some finite order $k > 0$. It follows that there exist integers λ_i, μ_i $(i = 1, \cdots, m)$ giving the identity

$$k \sum_{0}^{n} \beta_i a_i \equiv \sum_{1}^{m} 2^{2i} \lambda_i e_i + \sum_{1}^{m} \mu_i (2a_i - a_{i-1} - e_i).$$

Comparing coefficients, we have

$$\mu_i = 0 \ (i > n), \quad k\beta_n = 2\mu_n, \quad k\beta_i = 2\mu_i - \mu_{i+1} \ (i = 1, \cdots, n-1),$$
$$k\beta_0 = -\mu_1, \quad 2^{2i}\lambda_i - \mu_i = 0 \qquad (i = 1, \cdots, m).$$

We find that μ_1 is divisible by k; then, inductively, we find that μ_i

[15] For the construction of this group and the proof of its properties the author is indebted to Dr. Reinhold Baer.

$(i=1,\cdots,n)$ is divisible by k. Let $\mu_i/k=\tau_i$. Then $\beta_0=-\tau_1$ $\beta_i=2\tau_i-\tau_{i+1}$, $\beta_n=2\tau_n$, and

$$\sum_0^n \beta_i a_i = -\tau_1 a_0 + \sum_1^{n-1}(2\tau_i-\tau_{i+1})a_i + 2\tau_n a_n = \sum_1^n \tau_i(2a_i-a_{i-1}) = \sum_1^n \tau_i e_i.$$

Hence $a=\sum_1^n(\alpha_i+\tau_i)e_i$; as was to be proved.

We prove now that $\mathfrak{E}$ is not a direct summand of $\mathfrak{H}^*$. The difference group $\mathfrak{H}^*-\mathfrak{E}$ has $\bar{a}_0,\bar{a}_1,\cdots$ as generators with the relations $2\bar{a}_n=\bar{a}_{n-1}$ $(n=1,2,\cdots)$. If, to the contrary, $\mathfrak{E}$ is a direct summand, then there are elements f_i $(i=0,1,2,\cdots)$ in $\mathfrak{E}$ such that $2(a_i+f_i)=a_{i-1}+f_{i-1}$ $(i=1,2,\cdots)$. Using the relations in $\mathfrak{H}^*$, we find that

$$f_{i-1}=2f_i+e_i \qquad (i=1,2,\cdots).$$

Then, for each integer n,

$$f_0=2^n f_n+\sum_1^n 2^{i-1}e_i.$$

As f_n is an element of $\mathfrak{E}$ we can set $f_n=\sum_1^{m(n)}\alpha_i{}^n e_i$. We may assume without loss of generality that $m(n)\geqq n$. Then, if k is the order of f_0,

$$k2^n\sum_1^{m(n)}\alpha_i{}^n e_i + k\sum_1^n 2^{i-1}e_i = 0. \tag{1}$$

Let n be even and $i=n/2$. Then $k\equiv 0 \pmod{2^{n/2+1}}$. As this holds for every even integer n, we find that $k=0$ which contradicts the fact that k is the order of f_0 (if $f_0=0$, (1) holds for any integer k). This proves that $\mathfrak{E}$ is not a direct summand of $\mathfrak{H}^*$.

Let $\mathfrak{H}$ be the group of characters of $\mathfrak{H}^*$. The annihilator of $\mathfrak{E}$ in $\mathfrak{H}$ is the component of zero of $\mathfrak{H}$ (see [17], p. 386, Corollary 1c). As $\mathfrak{E}$ is not a direct summand of $\mathfrak{H}^*$, the component of zero of $\mathfrak{H}$ is not a direct summand of $\mathfrak{H}$ ([17], p. 382, Theorem 1b). We shall construct a compact metric space A such that $\mathfrak{H}=\mathfrak{H}_2(A,\mathfrak{X})$.

We first construct a direct sequence $\{\mathfrak{H}^{*m}\}$ whose limit group is $\mathfrak{H}^*$. $\mathfrak{H}^{*m}$ has $e_i{}^m$ $(i=1,\cdots,m)$ and c^m as its generators with the relations $2^{2i}e_i{}^m=0$ $(i=1,\cdots,m)$. The homomorphism H^* of $\mathfrak{H}^{*m}$ into $\mathfrak{H}^{*m+1}$ is defined by

$$H^*(c^m)=2c^{m+1}-e^{m+1}_{m+1}, \qquad H^*(e_i{}^m)=e_i{}^{m+1} \qquad (i=1,\cdots,m).$$

If we form the inverse sequence $\{\mathfrak{H}^m\}$ dual to $\{\mathfrak{H}^{*m}\}$ (No. 6), we find that

$$\mathfrak{H}^m=\mathfrak{X}^m+\sum_{i=1}^m \mathfrak{X}'^{mi}$$

where $\mathfrak{X}^m$ is isomorphic with $\mathfrak{X}$ and $\mathfrak{X}'^{mi}$ is isomorphic with the subgroup of $\mathfrak{X}$ of order 2^{2i}. The homomorphism H of $\mathfrak{H}_p{}^{m+1}$ into $\mathfrak{H}_p{}^m$ is given by

$$H(\mathfrak{x}\mathfrak{X}^{m+1}) = 2\mathfrak{x}\mathfrak{X}^m, \qquad H(\mathfrak{x}\mathfrak{X}'^{m+1,m+1}) = -\mathfrak{x}\mathfrak{X}^m,$$
$$H(\mathfrak{x}\mathfrak{X}'^{m+1,i}) = \mathfrak{x}\mathfrak{X}'^{mi} \qquad (i = 1, \cdots, m).$$

Let Q denote the point set in the plane consisting of the unit circle C and its interior. On C let us identify the point having the angular coördinate $\theta + p(2\pi/k)$ $(p = 1, \cdots, k;\ 0 \leqq \theta < 2\pi/k)$ with the point θ. In this way Q is converted into a complex Q^k such that $\mathfrak{H}_2(Q^k, \mathfrak{X})$ is isomorphic with the subgroup $\mathfrak{X}'^k$ of $\mathfrak{X}$ of order k. Let K^m $(m = 1, 2, \cdots)$ be a complex composed of a 2-sphere P^m and the complexes Q^{2i} $(i = 1, \cdots, m)$; we now denote these latter by $Q^{m,2i}$. Let π be a continuous mapping of K^{m+1} into K^m which maps P^{m+1} onto P^m with degree 2, $Q^{m+1,2i}$ onto $Q^{m,2i}$ $(i = 1, \cdots, m)$ with degree $+1$, and $Q^{m+1,2(m+1)}$ onto P^m with degree -1. It follows that $\mathfrak{H}_2(K^m, \mathfrak{X}) = \mathfrak{H}^m$ and the homomorphism of $\mathfrak{H}_2(K^{m+1}, \mathfrak{X})$ into $\mathfrak{H}_2(K^m, \mathfrak{X})$ induced by π is the homomorphism H of $\mathfrak{H}^{m+1}$ into $\mathfrak{H}^m$. If A is the limit space of the inverse sequence $\{K^m\}$, A is bicompact and has the 2nd countability axiom. As A possesses arbitrarily small mappings into 2-dimensional complexes, A is at most 2-dimensional. By Theorem 17, $\mathfrak{H}_2(A, \mathfrak{X})$ is the limit group of

$$\{\mathfrak{H}_2(K^m, \mathfrak{X})\} = \{\mathfrak{H}^m\}.$$

Thus $\mathfrak{H} = \mathfrak{H}_2(A, \mathfrak{X})$, and the component of zero of $\mathfrak{H}_2(A, \mathfrak{X})$ is not a direct summand. As $\mathfrak{H}_2(A, \mathfrak{X})$ is not zero, A is 2-dimensional.

Let A be imbedded in euclidean 5-space E_5. By the Pontrjagin theorem of duality [17], the 2-dimensional group over the integers of the finite cycles of $E_5 - A$ is the group $\mathfrak{H}^*$. In this the dual case the torsion group is not a direct summand of the homology group.

Example 5. We shall prove that $\mathfrak{X}$ considered as a discrete group is not a universal coefficient group. Thus the emphasis we have placed on the notion of a topologized homology group is essential.

Let $\mathfrak{A}$ be the character group of the discrete group of rational numbers. It is compact, connected, metric and 1-dimensional, and it contains no elements of finite order. If we construct in this group a Hamel basis, we find that $\mathfrak{A}$ is isomorphic with the direct sum of a set of groups each isomorphic with the group of rational numbers, the number of summands being the power of the continuum. The same is true of the direct sum $\mathfrak{A} + \mathfrak{A}$. Let $\mathfrak{X}_0$ denote the group $\mathfrak{X}$ with the discrete topology. As both $\mathfrak{X}_0$ and $\mathfrak{X}$ have the division-closure property, $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X})$ and $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X}_0)$ are isomorphic (though not continuously so). By Theorem 15, $\mathfrak{A}$ and $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X})$ are isomorphic, likewise $\mathfrak{A} + \mathfrak{A}$ and

$\mathfrak{H}_1(\mathfrak{A} + \mathfrak{A}, \mathfrak{X})$. Then $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{X}_0)$ and $\mathfrak{H}_1(\mathfrak{A} + \mathfrak{A}, \mathfrak{X}_0)$ are isomorphic. However, if $\mathfrak{R}$ is the group of rational numbers, it can be shown that $\mathfrak{H}_1(\mathfrak{A}, \mathfrak{R})$ and $\mathfrak{H}_1(\mathfrak{A} + \mathfrak{A}, \mathfrak{R})$ have ranks 1 and 2 respectively. So $\mathfrak{X}_0$ is not universal.

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ON CLOSED SPACES OF CONSTANT MEAN CURVATURE.

By T. Y. THOMAS.

Theorems on spaces of constant mean curvature (Einstein spaces) defined analytically by the equations

$$B_{\alpha\beta} = \lambda g_{\alpha\beta}, \qquad (B_{\alpha\beta} = \Sigma B^{\nu}{}_{\alpha\beta\nu}),$$

where λ is the mean curvature and the B's are the components of the curvature tensor, have been given by Kasner, Schouten and Struik.[1] These theorems are all of *local* character. In particular it has been shown that if the space is of dimensionality $n = 2, 3$ it must be of constant curvature. The following paper deals with *closed*[2] hypersurfaces S (without boundary) of constant mean curvature $\lambda > 0$ and dimensionality $n \geqq 2$ in a euclidean space E of $n + 1$ dimensions. In view of the above mentioned result, such spaces must be of constant curvature if $n = 2, 3$. We shall here prove that they must likewise be of constant curvature for $n \geqq 4$. As our proof depends on the fact that S is closed this result appears essentially as a theorem in the large and in this sense is distinguished from the local theorems of the above writers.

Our work is based on equations established in the paper by T. Y. Thomas, "On the variation of curvature in Riemann spaces of constant mean curvature," *Annali di Mathematica*, vol. 13 (1934-35), p. 227 and the paper by C. B. Allendoerfer, "Einstein spaces of class one," to appear in the *Bulletin of the American Mathematical Society*. From the latter of these we take the equations

$$(1) \qquad b_{\beta\gamma}b_{\alpha\delta} = k_1 g_{\beta\gamma}g_{\alpha\delta} + k_2 \sum_{a,b=1}^{n} g^{ab} \sum_{\mu,\nu=1}^{n} g^{\mu\nu}[2B_{a\alpha\nu\delta}B_{b\beta\gamma\mu} + B_{a\mu\delta\beta}B_{b\nu\gamma\alpha}],$$

where

$$ds^2 = \sum_{a,b=1}^{n} g_{\alpha\beta}dx^{\alpha}dx^{\beta}, \qquad \psi = \sum_{a,b=1}^{n} b_{\alpha\beta}dx^{\alpha}dx^{\beta}$$

[1] E. Kasner, "The impossibility of Einstein fields immersed in flat space of five dimensions," *American Journal of Mathematics*, vol. 43 (1921), p. 126; "Finite representations of the solar gravitational field in flat space of six dimensions," *ibid.*, p. 130; "Geometrical theorems on Einstein's cosmological equations," *ibid.*, p. 217.

J. A. Schouten and D. J. Struik, "On some properties of general manifolds relating to Einstein's theory of gravitation," *American Journal of Mathematics*, vol. 43 (1921), p. 213.

[2] By saying briefly that the hypersurface S is closed, we mean in the terminology of the point set theory that S is compact and closed with respect to E. Hence S is contained in a finite portion of the euclidean space E.

are the first and second fundamental forms of the hypersurface S and

$$k_1 = \frac{\lambda}{n-2}, \qquad k_2 = \frac{1}{2\lambda(n-2)}.$$

From the former of these papers we select the equations

$$(2) \qquad \sum_{\mu,\nu=1}^{n} g^{\mu\nu} K_{,\mu\nu} - (4/3)\lambda K + \sum_{\mu,\nu=1}^{n} g^{\mu\nu} \sum_{\alpha,\beta,\gamma,\delta=1}^{n} B_{\alpha\beta\gamma\delta,\mu\nu} \lambda_1{}^{\alpha} \lambda_2{}^{\beta} \lambda_1{}^{\gamma} \lambda_2{}^{\delta} = 0.$$

Here $K_{,\mu\nu}$ are the components of the second extension of the sectional curvature K determined by the orthogonal unit vectors λ_1 and λ_2 at any point P of the hypersurface S and the $B_{\alpha\beta\gamma\delta,\mu\nu}$ are the components of the second extension of the curvature tensor B. We have the relations [3]

$$(3) \qquad \sum_{\mu,\nu=1}^{n} g^{\mu\nu} B_{\alpha\beta\gamma\delta,\mu\nu} = (2/3)\lambda B_{\alpha\beta\gamma\delta} + \sum_{a,b=1}^{n} g^{ab} \sum_{\mu,\nu=1}^{n} g^{\mu\nu} \times [B_{a\mu\alpha\beta} B_{b\nu\gamma\delta} + 2B_{a\beta\delta\mu} B_{b\alpha\gamma\nu} + 2B_{a\alpha\mu\delta} B_{b\beta\gamma\nu}].$$

Now interchange the indices γ, δ in (1) and subtract. When use is made of the Gauss equations relating the coefficients $b_{\alpha\beta}$ of the second fundamental form of the hypersurface S to the components of the curvature tensor we obtain

$$\begin{aligned}(4) \qquad B_{\alpha\beta\gamma\delta} &= k_1(g_{\beta\gamma} g_{\alpha\delta} - g_{\beta\delta} g_{\alpha\gamma}) \\ &+ k_2 \sum_{a,b=1}^{n} g^{ab} \sum_{\mu,\nu=1}^{n} g^{\mu\nu} [B_{a\mu\alpha\beta} B_{b\nu\gamma\delta} + 2B_{a\beta\delta\mu} B_{b\alpha\gamma\nu} + 2B_{a\alpha\mu\delta} B_{b\beta\gamma\nu}] \\ &+ k_2 \sum_{a,b=1}^{n} g^{ab} \sum_{\mu,\nu=1}^{n} g^{\mu\nu} [B_{a\mu\delta\beta} B_{b\nu\gamma\alpha} + B_{a\mu\gamma\beta} B_{b\nu\alpha\delta} + B_{a\mu\alpha\beta} B_{b\nu\delta\gamma}].\end{aligned}$$

We observe that when the last set of terms in (4) is multiplied by $\lambda_1{}^{\alpha} \lambda_2{}^{\beta} \lambda_1{}^{\gamma} \lambda_2{}^{\delta}$ and summed on repeated indices the expression vanishes identically. Hence in consequence of (2), (3), (4) and the equations defining the sectional curvature K [4] we have

$$\sum_{\mu,\nu=1}^{n} g^{\mu\nu} K_{,\mu\nu} = (2\lambda + (1/k_2))K - (k_1/k_2); \text{ or}$$

$$(5) \qquad \sum_{\mu,\nu=1}^{n} g^{\mu\nu} K_{,\mu\nu} = 2\lambda[(n-1)K - \lambda]$$

when we substitute the above values of the constants k_1 and k_2.

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[3] These relations appear as equations (3.7) in Thomas (*loc. cit.*). Attention is here called to several typographical errors in the derivation of these relations. The term $B_{\alpha\beta\delta\epsilon,\gamma,\zeta}$ in the right member of (3.2) should be replaced by $B_{\alpha\beta\delta\epsilon,\gamma\zeta}$. Also in the equations at the bottom of p. 230 the term $B_{\alpha\beta\gamma\delta,\zeta,\epsilon}$ in the left member should be replaced by $B_{\alpha\beta\gamma\delta,\epsilon,\zeta}$ and the term $B_{\alpha\beta\gamma\delta,\epsilon,\zeta}$ in the right member should be replaced by $B_{\alpha\beta\gamma\delta,\epsilon\zeta}$.

[4] The equations (1.2) of Thomas (*loc. cit.*).

Since S is closed in the euclidean space E there exists a point P of S at which K assumes its maximum value. Since S is without boundary the point P will be an inner point of S and hence at P the left member of (5) will be $\leqq 0$. Suppose that S is not a space of constant curvature. Then at P the bracket expression will be > 0. Since $\lambda > 0$ by hypothesis it follows that the right member of (5) will be positive at the point P thus giving a contradiction. Hence the hypersurface S must have constant curvature.

We observe that the above requirements regarding continuity and differentiability are met if the hypersurface S is defined by functions $\phi^i(x)$ where $i = 1, \cdots, n+1$ which are continuous and possess continuous partial derivatives to the fourth order.[5] Such a surface is said by some writers to be of class C^4. Using this terminology we have proved the following theorem.

THEOREM. *Any hypersurface S of class C^4 of constant mean curvature $\lambda > 0$ and dimensionality $n \geqq 4$ in a euclidean space E of $n+1$ dimensions, the hypersurface S being closed but without boundary, is a space of constant curvature.*

In view of the above mentioned local theorems our theorem is proved if $n = 2, 3$ for hypersurfaces S of class C^3. We do not consider the question of whether the above theorem is valid for hypersurfaces of less restrictive class.

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[5] In fact under this hypothesis the coefficients of the second fundamental form of S are continuous and possess continuous first and second derivatives. Cp. § 1 and § 2 of T. Y. Thomas, "Riemann spaces of class one and their characterization," to appear shortly in the *Acta Mathematica*. Hence it follows from the Gauss equations that the components of the curvature tensor are continuous with continuous first and second derivatives and this permits the definition of the second extension of the curvature tensor whose components occur in the equations (2) and (3).

ON CONTINUA OF CONDENSATION.

By G. T. Whyburn.

A continuum M is said to have property N[1] provided that for every $\epsilon > 0$ there exists a finite collection G of disjoint non-degenerate subcontinua of M such that every subcontinuum of M of diameter $> \epsilon$ contains some continuum of G. Recently a study has been made by R. L. Moore[2] of this property in connection with various sorts of continua of condensation. Among other results, Moore proves that "The set of all regular curves with property N includes the set of all dendrons and all regular curves without continua of condensation and is included in the set of all regular curves with no essential continuum of condensation" (*loc. cit.*, p. 72).

In this paper I shall prove the following theorem[3] which yields exactly the relation between property N and the existence of continua of condensation in a given continuum:

Theorem. *In order that a compact metric continuum M have property N it is necessary and sufficient that M be locally connected and that no cyclic element of M have a continuum of condensation.*

Proof. The condition is necessary. In the first place, since clearly if M has property N it cannot contain an infinite sequence of disjoint continua all of diameter greater than some $d > 0$, it follows that not only M but every subcontinuum of M must be locally connected. Thus the first condition is necessary.

Now let E be any true cyclic element of M, and suppose, contrary to our theorem, that E has a continuum of condensation K. Now obviously any subcontinuum of M must also have property N, so E has property N. Since, by the above, K must be locally connected, we can suppose without loss of generality that K is a simple arc ab. Let $\epsilon = \frac{1}{2}\rho(a, b)$. Since K is a continuum

[1] See R. L. Moore, "Fundamental point set theorems," *Rice Institute Pamphlet*, vol. 23, no. 1 (1936), see p. 67.

[2] *Loc. cit.* See also an abstract in *Bulletin of the American Mathematical Society*, vol. 42 (1936), p. 35.

[3] This theorem was recently communicated to R. L. Moore, who states that he had not thought of the theorem but that he can prove it with the aid of some results which he has found but not yet published.

of condensation of E, it follows by a theorem of the author's [4] that the non-local separating points of E are dense on K. Now let $W_1, W_2, \cdots, W_n$ be a set of disjoint continua such that every subcontinuum of E of diameter $> \epsilon$ contains a continuum W_i. Let $W_{n_1}, W_{n_2}, \cdots, W_{n_j}$ be the continua W_i which lie wholly in K and let Y be the sum of the remaining ones. Then for each $i \leq j$, W_{n_i} is a subarc $a_i b_i$ of ab; hence W_{n_i} contains an inner point which is not a local separating point of E, and accordingly there exists an arc $\overline{x_i y_i}$ in E such that x_i, y_i are in the order a, a_i, x_i, y_i, b_i, b, $\overline{x_i y_i} \cdot K = x_i + y_i$, and $\overline{x_i y_i} \cdot Y = 0$. Now for each $i \leq j$, replace the arc $x_i y_i$ of ab by the arc $\overline{x_i y_i}$ and call the continuum thus formed H. Then since $H \supset a + b$, we have $\delta(H) > \epsilon$. But since for no $i \leq j$ does H contain the arc $x_i y_i$ of K, H contains no set W_{n_i}; and since $\overline{x_i y_i} \cdot Y = 0$ for each $i \leq j$, H can contain no set of Y. Hence H contains no set W_i whatever. Thus the supposition that the second condition is not necessary leads to a contradiction.

The proof for the sufficiency of the conditions will be given in five steps.

(1) *Any continuum M having no continuum of condensation has property N.*[5]

Proof. Let $\epsilon > 0$ be given. We can write [6]

$$M = F + \sum_{i=1,2,\ldots} \widehat{a_i x_i b_i} \tag{i}$$

where F is closed and totally disconnected and each set $\widehat{a_i x_i b_i}$ is an open free arc. Since at most a finite number of the arcs $\widehat{a_i x_i b_i}$ are of diameter $\geq \epsilon/5$, by adding a finite number of points on these to F clearly we can obtain the decomposition (i) so that all arcs $\widehat{a_i x_i b_i}$ are of diameter $< \epsilon/5$. Let us suppose this has been done.

Now since F is totally disconnected and closed, it follows at once that there exists an integer k such that every component of $M - \sum_{i=1}^{k} \widehat{a_i x_i b_i}$ is of diameter $< \epsilon/5$. For otherwise we could find a monotone decreasing sequence of continua $K_1, K_2, K_3, \cdots$ such that for each k, $\delta(K_k) \geq \epsilon/5$ and K_k is a component of $M - \sum_{i=1}^{k} \widehat{a_i x_i b_i}$; this is impossible since then ΠK_i would be a continuum in F of diameter $\geq \epsilon/5$.

For each $i \leq k$, let W_i be any closed arc contained wholly in $\widehat{a_i x_i b_i}$. Then

[4] See *Mathematische Annalen*, vol. 102 (1929), p. 320.

[5] See Moore, *loc. cit.* The proof is given here merely for the sake of completeness.

[6] See Urysohn, *Verhandelingen der Akademie te Amsterdam*, vol. 13, no. 3 (1927), p. 57.

every subcontinuum of M of diameter $> \epsilon$ must contain one of these arcs W_i. For let Q be a subcontinuum of M of diameter $> \epsilon$. Then Q intersects at least two components of $M - \sum_{i=1}^{k} \widehat{a_i x_i b_i}$ since if L is any one such component, then $L +$ all arcs $\widehat{a_i x_i b_i}$ having an endpoint on L is a set of diameter $< 3\epsilon/5$. Let $a, b \in Q$, where a and b lie in different components C_a and C_b of $M - \sum_{i=1}^{k} \widehat{a_i x_i b_i}$. Let ab be an arc in Q. Let a' be the last point of C_a on ab in the order a, b and let b' be the first point following a' which belongs to $M - \sum_{i=1}^{k} \widehat{a_i x_i b_i}$. Then clearly $\widehat{a'b'}$ is one of the arcs $\widehat{a_i x_i b_i}$ for some $i \leq k$. Whence $Q \supset a'b' \supset W_i$.

(2) *If no true cyclic element of a locally connected continuum S has a continuum of condensation, neither does any cyclic chain of S.*

For let Q be a subcontinuum of a cyclic chain $C(a, b)$. Then Q contains an arc pq, and pq has a segment xy which either belongs to the set K of all points separating a and b in $C(a, b)$ or to a true cyclic element C_i in $C(a, b)$. If $xy \subset K$, clearly $xy - (x + y)$ is open in $C(a, b)$. If $xy \subset C_i$, then $[xy - (K \cdot xy)] \cdot \overline{C(a, b) - C_i} = 0$; and since xy is not a continuum of condensation of C_i, some subarc $x'y'$ of xy is such that $x'y' \cdot \overline{C(a, b) - Q} = 0$. Thus in either case Q is not a continuum of condensation of $C(a, b)$.

(3) *If no true cyclic element of a locally connected continuum S has a continuum of condensation and if we express*

$$S = \sum_{i=1}^{k} C(p_i, q_i) + \sum_{i=k+1}^{\infty} C(p_i, q_i) + H$$

as in the cyclic chain approximation theorem,[7] *then for no k does $H_k = \sum_{i=1}^{k} C(p_i, q_i)$ have a continuum of condensation. Thus every H_k has property N.*

(4) *If every H_k in a locally connected continuum S has property N, so also does S.*

Proof. Let $\epsilon > 0$. Then there exists a k such that every component of $S - H_k$ is of diameter $< \epsilon/3$. Since H_k has property N, there exists a finite

[7] See Kuratowski and Whyburn, *Fundamenta Mathematicae*, t. 16 (1930), pp. 305-331. Here $C(p_i, q_i)$ is a cyclic chain.

number of disjoint continua $W_1, W_2, \cdots, W_n$ in H_k such that any subcontinuum of H_k of diameter $> \epsilon/3$ contains one of these continua W_i.

Now let Q be an arbitrary subcontinuum of S of diameter $> \epsilon$. Let $p, q \varepsilon Q$ be chosen so that $\rho(p, q) > \epsilon$. Let $p' = p$ if $p \varepsilon H_k$ and if not let p' be the boundary of the component Q_p of $S - H_k$ containing p. Similarly let $q' = q$ if $q \varepsilon H_k$ and otherwise let q' be the boundary of the component Q_q of $S - H_k$ containing q. Then since

$$\rho(p, p') + \rho(q, q') \leq \delta(Q_p) + \delta(Q_q) < \epsilon/3 + \epsilon/3 = 2\epsilon/3,$$

we have

$$\rho(p', q') > \epsilon/3.$$

Thus since $p' + q' \subset Q \cdot H_k$, we have

$$\delta(Q \cdot H_k) > \epsilon/3.$$

But $Q \cdot H_k$ is a continuum. Whence $Q \cdot H_k \supset W_j$ for some j. Accordingly, S has property N.

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CONTINUOUS TRANSFORMATIONS PRESERVING ALL TOPOLOGICAL PROPERTIES.

By JAMES F. WARDWELL.

1. *Introduction.* This paper concerns itself with a solution to the following problem:[1] If A and B are any compact metric spaces and $T(A) = B$ is a continuous transformation, under what conditions will B be homeomorphic with A, that is, what continuous transformations will preserve all topological properties of A? In view of the fact that continuous transformations and upper semi-continuous decompositions[2] are known to be equivalent[3] for compact metric spaces, the above problem can be stated in this way: If we have an upper semi-continuous decomposition of a compact metric space A, under what conditions will the hyperspace B of this decomposition be the same kind of space as A, that is, be homeomorphic with A?

The solution to this problem, for the case when A is a plane or sphere, is due to R. L. Moore.[4] It may be stated as follows: *If A is a topological sphere and $T(A) = B$ is a monotone transformation*[5] *such that, for any $b \varepsilon B$, $T^{-1}(b)$ does not separate A, then B is a topological sphere.*

However, no conditions have heretofore been found which yield the desired result for general compact metric spaces or even for any compact Euclidean spaces of higher dimension than two.

2. *Conditions.* A little investigation makes it clear that the conditions which are to be imposed must be conditions on the complements of the sets of the decomposition (that is, the sets $T^{-1}(b)$, for $b \varepsilon B$) in the space as well as

[1] This problem was suggested by G. T. Whyburn to whom the author is greatly indebted for his helpful suggestions and criticism in the preparation of this paper. See the abstract of his paper "Analytic topology" in *American Mathematical Monthly*, vol. 42 (1935), p. 190.

[2] See R. L. Moore, *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 416-428.

[3] See P. Alexandroff, *Mathematische Annalen*, vol. 96 (1927), pp. 551-571, and C. Kuratowski, *Fundamenta Mathematicae*, vol. 11 (1928), pp. 169-185.

[4] See *loc. cit.*

[5] A continuous transformation $T(A) = B$ is said to be a *monotone transformation* when each set $T^{-1}(b)$, for $b \varepsilon B$, is connected. See C. B. Morrey, Jr., *American Journal of Mathematics*, vol. 57 (1935), pp. 17-50, and G. T. Whyburn, *American Journal of Mathematics*, vol. 56 (1934), no. 2, pp. 294-302.

on these sets themselves. It seemed that the following condition might yield the desired result:

I. *For any* $b \varepsilon B$ *and any* $x \varepsilon A$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$.

However we have not been able to show that, for arbitrary compact metric spaces A and B, B is homeomorphic with A when this condition is satisfied, even when almost all of the sets $T^{-1}(b)$ are degenerate. The same was true for this condition:

II. *For any* $b \varepsilon B$ *and any* $x \varepsilon T^{-1}(b)$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$.

This last condition is less restrictive than the first one in that it does not make the space A homogeneous while condition I does. If II is satisfied and if A is homogeneous, then I is obviously satisfied.

We finally found that if condition II was further restricted to give:

III. *For any* $\epsilon > 0$, *any* $b \varepsilon B$, *and any* $x \varepsilon T^{-1}(b)$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$ *which is stationary*[6] *outside of the* ϵ*-neighborhood of* $T^{-1}(b)$.

the desired results are obtained for the case stated in section 7 below.

3. *Some effects of these conditions.*

If the compact metric space A *is connected, conditions I and III each imply that* $T(A) = B$ *is a monotone transformation.*

In order to demonstrate this result, let us assume the contrary in both cases. Then there exists some set $T^{-1}(b) = C$, where $C = C_1 + C_2$, mutually separated. Now $\bar{C}_1 \cdot \bar{C}_2 = 0$, since C is closed. Take neighborhoods U_1 and U_2 of C_1 and C_2 respectively so that $\bar{U}_1 \cdot \bar{U}_2 = 0$. Let $U = U_1 + U_2$. There exists a closed cutting S in $A - U$ which separates $\bar{U}_1$ and $\bar{U}_2$ in A, that is, $A - S = A_1 + A_2$, mutually separated, where

$$A_1 \supset \bar{U}_1 \supset C_1 \quad \text{and} \quad A_2 \supset \bar{U}_2 \supset C_2.$$

For condition I, take any $x \varepsilon A - S$. Now there exists a homeomorphism $W(A - x) = A - C$. If we apply W^{-1} to the set

$$A - C - S = (A_1 - C_1) + (A_2 - C_2)$$

[6] A transformation W is said to be *stationary* over all points y for which $W(y) = y$.

we obtain a separation

$$A - x - W^{-1}(S) = W^{-1}(A_1 - C_1) + W^{-1}(A_2 - C_2).$$

Now $x + W^{-1}(S)$ is a closed cutting of A, and x is an isolated point of this cutting. Furthermore x is a limit point of both $W^{-1}(A_1 - C_1)$ and $W^{-1}(A_2 - C_2)$. Therefore, by an established theorem,[7] x is a local separating point [7] of A. Now for any point y in $A - x$, there exists a homeomorphism $R(A - x) = A - y$. Hence every point of A is a local separating point of A. Now A is homogenous, and hence every point of A is of the same Menger order. Accordingly no point of A is of order greater than 2 because, by the Local Separating Point–Order Theorem,[7] there exists at most a countable number of local separating points of order greater than 2. Since A consists of more than one point and is connected, it follows that every point of A is of order exactly 2. Therefore A is a simple closed curve. From this it follows that $C_1 + C_2$ separates A; and thus x separates A, which is a contradiction, since no single point separates a simple closed curve. Hence, for condition I, every set $T^{-1}(b)$ is connected, that is, T is a monotone transformation.

For condition III, take any point $x \varepsilon C_1$. By hypothesis, there exists a homeomorphism $W(A - x) = A - C$ which is stationary outside of U. Let $R(A) = A$ be the transformation such that, for any $y \varepsilon A - C$, $R(y) = W^{-1}(y)$ while $R(C) = x$. Now R is continuous. Let the cutting $S = A - U$. Then we have $A - S = A - (A - U) = U_1 + U_2$, mutually separated. Applying R to $A - S$ we have: $A - R(S) = R(U_1) + R(U_2)$. However, $R(S) = W^{-1}(S) = S$, since W is stationary in $A - U$. Therefore, $A - S = R(U_1) + R(U_2)$. Let P be the set of all points p of U_2 so that $R(p) \subset U_1$. Consequently $R(U_2 - P) \subset U_2$. If we let $Q = S + U_2 - P$, then $A = Q + U_1 + P$. Now it can easily be seen that $Q + U_1$ and P are mutually separated sets. Hence we have a separation of the connected set A; and this is a contradiction. Therefore T is a monotone transformation also in this case.

Since conditions I and III each make the transformation T a monotone one when A is connected, it follows that each of these conditions reduces to those of Moore for the case of the plane.

Condition II does not make T a monotone transformation even when A is connected.

This result is illustrated by the following example. Take a line L in a plane and a segment pq on this line. In the interior of pq take a point p_1,

[7] See G. T. Whyburn, *Monatshefte für Mathematik und Physik*, vol. 36 (1929), pp. 305-314.

and in the interior of p_1q take a point q_1. In pp_1 take a sequence $\{p_i\}$ of points converging to p, ordering these points so that $p_{i+1} \subset pp_i$, for each i. Take a similar sequence $\{q_i\}$ of points in q_1q converging to q. On one side of the line L construct some simple arcs as follows: Take a simple arc C from p_1 to q_1 which meets pq only in the points p_1 and q_1. For each i, take a simple arc C_i from p_i to p_{i+1} which meets the set $pq + C + \sum_{j=1}^{i-1} C_j$ only in p_i and p_{i+1}, and such that $\delta(C_i) \to 0$. For the points of $\{q_i\}$, for each i, take a simple arc D_i from q_i to q_{i+1} which meets $pq + C + \sum_{k=1}^{\infty} C_k + \sum_{j=1}^{i-1} D_j$ only in the points q_i and q_{i+1}, and such that $\delta(D_i) \to 0$. On the other side of L construct sequences of simple closed curves as follows: At each point p_i take a sequence $\{E_k{}^i\}$ of simple closed curves converging to p_i, every two of which intersect only in p_i, and so that $E_k{}^i \cdot E_m{}^j = 0$, for $i \neq j$, and for all k and m. At each point q_i take a sequence $\{F_k{}^i\}$ of simple closed curves converging to q_i, every two of which intersect only in q_i, and so that $F_k{}^i \cdot F_m{}^j = 0$, for $i \neq j$, and for all k and all m, and no one of which intersects any of the $E_k{}^i$, for all i and all k. Let A represent the set

$$pq + C + \sum_{i=1}^{\infty} (C_i + D_i) + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (E_k{}^i + F_k{}^i).$$

Now A is a connected compact metric space. Take a decomposition of A into the set $p_1 + q_1$ and the points of $A - (p_1 + q_1)$. This is obviously an upper semi-continuous decomposition. We must now show that $A - (p_1 + q_1)$ is homeomorphic with $A - p_1$ and with $A - q_1$. Now $A - (p_1 + q_1)$ consists of the sets:

$$(pp_1 + \sum_{i=1}^{\infty} C_i + \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} E_k{}^i) - p_1,$$

$$(q_1q + \sum_{i=1}^{\infty} D_i + \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} F_k{}^i) - q_1,$$

and the countable collection of free open arcs:

$$C - (p_1 + q_1),\ p_1q_1 - (p_1 + q_1),\ E_k{}^1 - p_1, \text{ and } F_k{}^1 - q_1, \text{ for all } k.$$

Furthermore, $A - p_1$ and $A - q_1$ each consist of the same number and same types of sets as $A - (p_1 + q_1)$, and it is easily seen that both $A - p_1$ and $A - q_1$ are homeomorphic with $A - (p_1 + q_1)$.

If A is a 2-dimensional manifold, conditions I and III are equivalent on A.

For, if condition III is given, condition I is satisfied because A is homo-

geneous. If condition I is given, we have, for any set $T^{-1}(b)$, for $b \varepsilon B = T(A)$, and any $x \varepsilon A$, a homeomorphism $W(A - x) = A - T^{-1}(b)$. Now take the point x in $T^{-1}(b)$. The set $T^{-1}(b)$ is connected since A is connected. Take a monotone sequence $\{U_i\}$ of neighborhoods closing down on x such that $\bar{U}_i$ is a closed 2-cell, for each i. Let $W(U_i - x) = V_i^\circ$ and $V_i^\circ + T^{-1}(b) = V_i$, for each i. The neighborhoods V_i close down on $T^{-1}(b)$ since the neighborhoods U_i close down on x. Let $C_i = U_i - \bar{U}_{i+1}$ and $D_i = V_i - \bar{V}_{i+1}$, for each i. Now $W[F(U_i)] = F(V_i)$,[8] for each i, and W is a homeomorphism. Hence for each i, $C_i + F(U_i) + F(U_{i+1})$ and $D_i + F(V_i) + F(V_{i+1})$ are each homeomorphic with a circular ring. For any $\epsilon > 0$ take an ϵ-neighborhood U_ϵ of $T^{-1}(b)$ so that $U_\epsilon - T^{-1}(b)$ is homeomorphic with a 2-cell V_ϵ of x minus x. Pick a k large enough so that $\bar{V}_k \subset U_\epsilon$ and $\bar{U}_k \subset U_\epsilon$. Now there exists a homeomorphism

$$R_\epsilon[(U_\epsilon - \bar{U}_k) + F(U_k) + F(U_\epsilon)] = (U_\epsilon - \bar{V}_k) + F(V_k) + F(U_\epsilon),$$

where $R_\epsilon(p) = p$ for $p \varepsilon F(U_\epsilon)$, since each of these sets is a circular ring and clearly any homeomorphism between the two outer curves of two such rings can be extended to the whole rings.[9] Let $R_\epsilon = S_{k-1}$. Similarly there exists a homeomorphism

$$S_k[C_k + F(U_k) + F(U_{k+1})] = D_k + F(V_k) + F(V_{k+1})$$

such that $S_k(x) = S_{k-1}(x)$ for $x \varepsilon F(U_k)$. Likewise there exists a homeomorphism

$$S_{k+1}[C_{k+1} + F(U_{k+1}) + F(U_{k+2})] = D_{k+1} + F(V_{k+1}) + F(V_{k+2})$$

such that $S_{k+1}(x) = S_k(x)$ for $x \varepsilon F(U_{k+1})$, and so on. In general for $j \geq -1$ we have a homeomorphism

$$S_{k+j}[C_{k+j} + F(U_{k+j}) + F(U_{k+j+1})] = D_{k+j} + F(V_{k+j}) + F(V_{k+j+1})$$

such that $S_{k+j}(x) = S_{k+j-1}(x)$ for $x \varepsilon F(U_{k+j})$.

Now define a transformation S as follows: $S(p) = p$ if $p \varepsilon M - U_\epsilon$, $S(p) = R_\epsilon(p)$ if $p \varepsilon (U_\epsilon - \bar{U}_k) + F(U_k) + F(U_\epsilon)$, $S(p) = S_j(p)$ for $j \geq k$ where j is the least integer such that p is a point of $C_j + F(U_j) + F(U_{j+1})$. Now $S(A - x) = A - T^{-1}(b)$ is a homeomorphism which is stationary outside of U_ϵ. Hence condition III is satisfied.

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[8] The boundary $\bar{P} - P$ of any open set P is represented by $F(P)$.

[9] See Schoenflies, *Mathematische Annalen*, vol. 62 (1906), p. 324.

4. *Notation.* If A and B are any compact metric spaces and $T(A) = B$ is a continuous transformation, let $G = G_0$ denote the collection of all non-degenerate sets $T^{-1}(b)$, for $b \varepsilon B$ in A. Let G_1 denote the collection of all sets of G which intersect $L = L_0 = \lim. \sup. G$. Let G_2 be the collection of all sets of G_1 which intersect $L_1 = \lim. \sup. G_1$, etc.

If the collection G is countable, let $\{g_i^k\}$ represent the sets of $G_{k-1} - G_k$, for $k = 1, 2, \cdots$. Cover each g_i^k by a neighborhood U_i^k so that: (a) $U_i^k \cdot g_j^k = 0$, for $j \neq i$; (b) $U_i^k \cdot \sum_{j=1}^{i-1} \bar{U}_j^k = 0$; (c) no set of G_k intersects U_i^k, for any i; and (d) $\bar{U}_i^k$ is contained in the $\rho(g_i^k, L_{k-1})/2$-neighborhood of g_i^k, for each i.

5. LEMMA. *If A and B are compact metric spaces and $T(A) = B$ is continuous and such that for any $\epsilon > 0$, any $b \varepsilon B$, and any $x \varepsilon T^{-1}(b)$, there exists a homeomorphism $W(A - x) = A - T^{-1}(b)$ which is stationary outside of the ϵ-neighborhood of $T^{-1}(b)$, and if there exists an integer k so that $G_k = 0$, then A is homeomorphic with B.*

Proof. 1°. If $G_i = 0$, for any integer i, then G_{i-1} is a null sequence.[10] For, if not, there exists a number $d > 0$ such that there is an infinite collection $\{h_i\}$ of sets of G_{i-1} the diameter $\delta(h_i)$ of each of which is greater than d. Hence, if we take a convergent subsequence $\{h'_i\}$ of $\{h_i\}$, $\delta(\lim. \sup. \{h'_i\}) \geq d$. Now $\lim. \sup. \{h'_i\} \subset T^{-1}(p)$ for some $p \varepsilon B$. Therefore $\delta(T^{-1}(p)) \geq d$, and thus $T^{-1}(p) \varepsilon G_i$, since L_{i-1} contains $\lim. \sup. \{h_i\}$. This contradicts the fact that $G_i = 0$.

We shall make the proof by induction.

2°. We shall first demonstrate the result for the case $k = 1$. From 1° it follows that G is a null sequence, since $G_1 = 0$. Now $\{g_i^1\}$ denotes the collection of sets of G. Take a point $y_i \varepsilon g_i^1$, for each i. Let $T(g_i^1) = b_i$, for each i. By hypothesis, there exists a homeomorphism[11] $W_1(A - y_1) = A - g_1^1$ which is stationary in $A - U_1^1$. Now $T W_1(A - y_1) = B - b_1$. Let $T_1(A) = B$ be the transformation such that for any $x \varepsilon A$, $T_1(x) = T W_1(x)$ when $x \varepsilon A - y_1$, and $T_1(y_1) = b_1$. Now T_1 is univalued and continuous, since T and W_1 are, and since A is compact. Now T_1 is such that $T_1^{-1}(b_i) = g_i^1$, for $i = 2, \cdots$, since $T_1 = T$ over $A - U_1^1$. Similarly, for

[10] A sequence $\{M_i\}$ of sets is said to be a *null sequence* provided that, for any $\epsilon > 0$, there are at most a finite number of the sets which have a diameter greater than ϵ.

[11] The method used here was suggested by a proof given by Mrs. Lucille Whyburn for a certain finite case of our problem.

each n, there exists a homeomorphism $W_n(A - y_n) = A - g_n{}^1$ which is stationary in $A - U_n{}^1$. Moreover $g_n{}^1 \equiv T^{-1}_{n-1}(b_n)$, and thus

$$W_n(A - y_n) = A - T^{-1}_{n-1}(b_n).$$

Hence $T_{n-1}W_n(A - y_n) = B - b_n$. Let $T_n(A) = B$ be the transformation such that, for $x \varepsilon A - y_n$, $T_n(x) = T_{n-1}W_n(x)$, and $T_n(y_n) = b_n$. Now T_n is also univalued and continuous. Furthermore $T_n{}^{-1}(b_i) = g_i{}^1$, for $i > n$, and $T_n{}^{-1}(b_j)$ is a single point for each $j \leq n$. From our definition of $T_n(A) = B$ it follows that, for any n, and any $x \varepsilon A$:

$$\begin{array}{ll}
T_n(x) = T_{n-1}W_n(x) & \text{except for } x = y_n, \\
T_n(x) = T_{n-2}W_{n-1}W_n(x) & \text{except for } x = y_n \text{ or } y_{n-1}, \\
\cdot \qquad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \qquad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \qquad \cdot & \cdot \quad \cdot \quad \cdot \\
\cdot \qquad \cdot & \cdot \quad \cdot \quad \cdot \\
T_n(x) = T_1W_2 \cdots W_n(x) & \text{except for } x \varepsilon \sum_{i=2}^{n} y_i, \\
T_n(x) = T\,W_1W_2 \cdots W_n(x) & \text{except for } x \varepsilon \sum_{i=1}^{n} y_i.
\end{array}$$

Let $S = \lim_{n=\infty}. \{T_n\}$. We will now prove that $\{T_n\}$ is uniformly convergent. Since all T_n are continuous over A, it will follow that S is also continuous. Since T is continuous and is defined over a compact space, it is uniformly continuous. Hence, for any $\epsilon > 0$, there exists a δ_ϵ such that, if $\rho(x, y) < \delta_\epsilon$, for $x, y \varepsilon A$, $\rho(T(x), T(y)) < \epsilon$. Take any $\epsilon > 0$. Let $\delta = \delta_{\epsilon/2}$. Cover each point of L by a $\delta/2$-neighborhood $V_{\delta/2}$ and by a δ-neighborhood V_δ. Since L is closed and compact we can find a finite number of the neighborhoods $V_{\delta/2}$, say $V^1{}_{\delta/2}, V^2{}_{\delta/2}, \cdots V^h{}_{\delta/2}$, whose sum covers L. Let $V_\delta{}^1, V_\delta{}^2, \cdots V_\delta{}^h$, be the corresponding δ-neighborhoods. Now $V = \sum_{i-1}^{h} V_\delta{}^i \supset L$. Since $\{g_i{}^1\}$ is a null sequence and because of our definition of the neighborhoods $U_i{}^1$, it follows that $\{U_i{}^1\}$ s a null sequence and lim. sup. $\{U_i{}^1\} = L$. Hence there exists an integer N such that $\delta(U_n{}^1) < \delta/2$, for all $n > N$. There also exists an integer M so that $U_m \cdot \sum_{i=1}^{h} V^i{}_{\delta/2} \neq 0$, for all $m > M$. For, if not, there would be an infinite collection of the sets of $\{U_i{}^1\}$ which did not intersect $\sum_{i=1}^{h} V^i{}_{\delta/2}$. Since A is compact, this collection would possess a limit point p which is contained in L but not contained in $\sum_{i=1}^{h} V^i{}_{\delta/2}$, which is a contradiction.

Let N_ϵ be the larger of the two numbers N and M. For any $n > N_\epsilon$, U_n^1 is contained in some V_δ^r, for $r = 1, 2, \cdots$, or h. For $U_n^1 \cdot V^r{}_{\delta/2} \neq 0$, for some r, since $n > M$. Also $\delta(U_n^1) < \delta/2$ since $n > N$. Hence $U_n^1 \subset V_\delta^r$. For any point $x \in (A - \sum_{i=N_\epsilon+1}^{\infty} U_i^1)$, $T_n(x) = T_{N_\epsilon} W_{N_\epsilon+1} \cdots W_n(x)$ and $T_{n+p}(x) = T_{N_\epsilon} W_{N_\epsilon+1} \cdots W_{n+p}(x)$, for $n > N_\epsilon$. Now all W_j, for $j > N_\epsilon$, are stationary outside of $\sum_{i=N_\epsilon+1}^{\infty} U_i^1$. Hence $T_n(x) = T_{N_\epsilon}(x) = T_{n+p}(x)$. Therefore, for any such x, $\rho(T_n(x), T_{n+p}(x)) = 0$, for $n > N_\epsilon$, and $p = 1, 2, \cdots$. For any $x \in \sum_{i=N_\epsilon+1}^{\infty} (U_i^1 - y_i)$, we have that $T_n(x) = T\, W_{N_\epsilon+1} \cdots W_n(x)$, since, by our definition of the neighborhoods U_i^1, all W_j, for $j \leq N_\epsilon$ are stationary for all such points x. Now x is contained in U_h^1, for some h; and U_h^1 is contained in V_δ^r, for some r. Each W_k, for $N_\epsilon < k \leq n$, transforms a point of U_h^1 only into some other point of U_h^1. Therefore $y = W_{N_\epsilon+1} \cdots W_n(x)$ is contained in U_h^1 which is contained in V_δ^r. Likewise

$$z = W_{N_\epsilon+1} \cdots W_{n+p}(x) \subset U_h^1 \subset V_\delta^r.$$

Hence

$$\rho(y, z) < \delta_{\epsilon/2}.$$

However

$$T_n(x) = T\, W_{N_\epsilon+1} \cdots W_n(x) = T(y)$$

and

$$T_{n+p}(x) = T\, W_{N_\epsilon+1} \cdots W_{n+p}(x) = T(z).$$

Accordingly

$$\rho(T_n(x), T_{n+p}(x)) = \rho(T(y), T(z)) < \epsilon, \quad \text{for } n > N_\epsilon, \text{ and } p = 1, 2, \cdots,$$

since T is uniformly continuous. For any y_i, $T_i(y_i) = b_i$, by definition. But $b_i = T(y_i)$. Hence $T_i(y_i) = T(y_i)$, for all i. If $x = y_i$, for any $i > N_\epsilon$, $T_n(x) = T_i W_{i+1} \cdots W_n(x)$, for $i > N_\epsilon$, $n > N_\epsilon$, and $n > i$. But all W_j, for $i < j \leq n$, are stationary on y_i. Therefore

$$T_n(x) = T_i(x) = T(x), \text{ for all } i < n.$$

If $n \leq i$, $T_n(x) = T(x)$, by definition. In the same way $T_{n+p}(x) = T(x)$, for $i > N_\epsilon$, $n > N_\epsilon$. Hence

$$\rho(T_n(x), T_{n+p}(x)) = 0, \text{ for } n > N_\epsilon,\ p = 1, 2, \cdots, \text{ for } x \subset \sum_{i=N_\epsilon+1}^{\infty} y_i.$$

Therefore, for any $\epsilon > 0$, there exists an N_ϵ such that for any $n > N_\epsilon$, and for any

$$x \in A,\ \rho(T_n(x),\ T_{n+p}(x)) < \epsilon, \text{ for } p = 1, 2, \cdots.$$

Hence $\{T_n\}$ converges uniformly to S. Consequently S is continuous.

Furthermore S is univalued both ways. For take any two distinct points p and q of A. There exists an integer N such that $p + q \subset A - \sum_{i=N}^{\infty} U_i^1$ Moreover T_N is univalued both ways over $A - \sum_{i=N}^{\infty} U_i^1$. Therefore

$$T_N(p) \neq T_N(q).$$

Now for any point

$$x \varepsilon A - \sum_{i=N}^{\infty} U_i^1,\ T_{N+j}(x) \equiv T_N(x), \text{ for all } j.$$

Hence

$$S(p) = T_{N+j}(p) = T_N(p) \neq T_N(q) = T_{N+j}(q) = S(q).$$

Accordingly $S(A) = B$ is a homeomorphism.

3°. We now assume that the lemma is true when $G_{k-1} = 0$ and proceed to prove that it is then true when $G_{k-1} \neq 0$ but $G_k = 0$.

Take a decomposition of A into the sets of G_{k-1} and the points of $A - G_{k-1}$. This is obviously an upper semi-continuous decomposition. Let its hyperspace be C and let $t(A) = C$ be the continuous transformation associated with the decomposition. The sets of G_{k-1} form a null sequence $\{g_i^k\}$ since $G_k = 0$. Let $t(g_i^k) = c_i \varepsilon C$, for each i. Now for any point $c \varepsilon C$, $t^{-1}(c)$ is a single point unless $c = c_i$, for some i; and $t^{-1}(c_i) = g_i^k = T^{-1}(b)$, a non-degenerate set, for some $b \varepsilon B$. Hence $\{t^{-1}(c_i)\}$ is the null sequence $\{g_i^k\}$. Furthermore, lim. sup. $\{t^{-1}(c_i)\}$ = lim. sup. $\{g_i^k\}$ which is L_{k-1}, and the collection of sets of G which intersect L_{k-1} is vacuous, by hypothesis. Moreover, for any $\epsilon > 0$, any $t^{-1}(c)$, for $c \varepsilon C$, and any $y \varepsilon t^{-1}(c)$, there exists a homeomorphism $W(A - y) = A - t^{-1}(c)$ which is stationary outside of the ϵ-neighborhood of $t^{-1}(c)$, because any non-degenerate $t^{-1}(c)$ is a set $T^{-1}(b)$, for some $b \varepsilon B$. Hence the conditions of this lemma are satisfied by $t(A) = C$ for the case demonstrated in part 2°. Therefore A is homeomorphic with C.

Let $Z(C) = T\,t^{-1}(C) = B$. Then $Z(C) = B$ is a univalued transformation. Moreover it is continuous, since T and t are each continuous, and A is compact. All $Z^{-1}(b)$, for $b \varepsilon B$, are degenerate except when $T^{-1}(b)$ is a set of $G - G_{k-1}$. Let H be the collection of all non-degenerate sets $Z^{-1}(b)$, for $b \varepsilon B$. Let H_1 be the collection of all sets of H which intersect M = lim. sup. H. Let H_2 be the collection of all sets of H_1 which intersect M_1 = lim. sup. H_1, etc. Wherefore, $H = t(G)$, and $H_i = t(G_i)$, for each $i < k$. However $t(G_{k-1})$ consists of the collection $\{c_i\}$ of single points of C. Hence $H_{k-1} = 0$.

Take any $\epsilon > 0$, any non-degenerate $Z^{-1}(b)$, for $b \varepsilon B$, and any point $y \varepsilon Z^{-1}(b)$. We must show that there exists a homeomorphism

$$W(C-y) = C - Z^{-1}(b)$$

which is stationary outside of the ϵ-neighborhood U_ϵ of $Z^{-1}(b)$ in C.

Now t is uniformly continuous, since A is compact. Hence there exists a δ_ϵ so that if $\rho(r, s) < \delta_\epsilon$, for $r, s \varepsilon A$, then $\rho(t(r), t(s)) < \epsilon$. Take a neighborhood V of $T^{-1}(b)$ in A such that V is contained in the δ_ϵ-neighborhood of $T^{-1}(b)$, and such that $\bar{V} \cdot g_i^k = 0$, for all i. We can do this because the sets $T^{-1}(p)$, for $p \varepsilon B$, are closed and disjoint, and $T^{-1}(b) \cdot \bar{G}_{k-1} = 0$, since $Z^{-1}(b)$ is non-degenerate in C. If we let $x = t^{-1}(y)$, then $x \varepsilon T^{-1}(b)$. Now, by hypothesis, there exists a homeomorphism $R(A - x) = A - T^{-1}(b)$ which is stationary in $A - V$. However, $A - x = t^{-1}(C-y)$. Hence

$$R\, t^{-1}(C-y) = A - T^{-1}(b).$$

We now designate by W the transformation

$$t\, R\, t^{-1}(C-y) = C - Z^{-1}(b).$$

Now $W(C-y) = C - Z^{-1}(b)$ is univalued. For take any point $q \varepsilon C - y$. If $q \neq c_i$, for any i, $W(q) = t\, R\, t^{-1}(q)$ is a single point of $C - Z^{-1}(b)$, because t is one-to-one for such points q and R is a homeomorphism. If $q = c_i$, for some i, then $t^{-1}(q) = g_i^k$. Furthermore $R\, t^{-1}(q) = t^{-1}(q)$, since $g_i^k \cdot V = 0$ and R is stationary outside of V. Therefore $W(q) = t\, t^{-1}(q) = q$. By a similar proof we see that W^{-1} is also univalued. Hence W is one-to-one.

Furthermore, W and W^{-1} are both continuous since t and R are continuous and A is compact. Therefore W is a homeomorphism.

Now W is stationary outside of U_ϵ. For if we take any point $q \varepsilon C - U_\epsilon$, then $t^{-1}(q) \cdot V = 0$. Hence, since R is stationary outside of V,

$$R\, t^{-1}(q) = t^{-1}(q).$$

Thus we have

$$W(q) = t\, R\, t^{-1}(q) = t\, t^{-1}(q) = q.$$

Accordingly the homeomorphism

$$W(C-y) = C - Z^{-1}(b)$$

is stationary outside of U_ϵ.

Hence the conditions of the lemma are satisfied by $Z(C) = B$ for the case $k-1$. We have assumed that for this case the lemma is true. Wherefore, C is homeomorphic with B. However, we have seen that A is homeomorphic with C. Therefore A is homeomorphic with B.

6. *Corollaries.* There are some properties of the homeomorphism, say $S(A)=B$, whose existence is established in the above lemma, which follow from the proof of the lemma. These will be stated and verified in the following two corollaries. We will assume that the conditions of the lemma are satisfied.

COROLLARY I. *The homeomorphism $S(A)=B$, whose existence is established in the lemma, is such that $S\equiv T$ over all points of A lying in* $A-\sum_{j=1}^{k}\sum_{i=1}^{\infty}U_i{}^j$.

Proof. We shall make the proof by induction. Take any point p in $A-\sum_{j=1}^{k}\sum_{i=1}^{\infty}U_i{}^j$. If $k=1$, then $\{g_i{}^1\}$ is a null sequence. Now $W_j(p)=p$, for all j, since W_j is stationary outside of $U_j{}^1$, for every j. Now, for all m, $T_m(p)=T\,W_1\cdots W_m(p)$. Hence $T_m(p)=T(p)$, for all m. Therefore $S(p)\equiv T(p)$ for this case.

Now let us assume that this is also true for $k-1$, and prove that it is then true for k. The sequence $\{g_i{}^k\}$ is a null sequence, since $G_k=0$. Take a decomposition of A into the sets of $\{g_i{}^k\}$ and the points of $A-\sum_{i=1}^{\infty}g_i{}^k$. This is an upper semi-continuous decomposition. Let its hyper-space be C and let $t(A)=C$ be the continuous transformation associated with the decomposition. Now $t(A)=C$ satisfies the conditions of the lemma for the case $k=1$. Hence there exists a homeomorphism $R_1(A)=C$ so that $R_1\equiv t$ over all points of the set $A-\sum_{i=1}^{\infty}U_i{}^k$, as we have just demonstrated above. Now we established in part 3° of the proof of the lemma that the conditions of the lemma are satisfied for the case $k-1$ by the spaces C and B and the transformation $Z(C)=T\,t^{-1}(C)=B$, where the non-degenerate sets in C are the sets $t(g_i{}^j)$ and the corresponding neighborhoods are the neighborhoods $t(U_i{}^j)$ for all i, and for $j<k$. From our assumption it then follows that there exists a homeomorphism $R_2(C)=B$ such that $R_2\equiv Z$ over $C-\sum_{j=1}^{k-1}\sum_{i=1}^{\infty}t(U_i{}^j)$. Now $S(A)=R_2R_1(A)=B$. Furthermore $S(p)=R_2R_1(p)=R_2t(p)$, since $R_1(p)\equiv t(p)$. Now $t(p)\subset C-\sum_{j=1}^{k-1}\sum_{i=1}^{\infty}t(U_i{}^j)$. Hence $R_2t(p)=Z\,t(p)$. However $Z\,t(p)=T\,t^{-1}t(p)=T(p)$. Thus $S(p)=T(p)$ for any point p in $A-\sum_{j=1}^{k}\sum_{i=1}^{\infty}U_i{}^j$.

COROLLARY II. *The homeomorphism $S(A)=B$, whose existence is established in the lemma, is such that, for any $p\,\varepsilon\,A$, if $p\subset U_d{}^a$, for any*

$\varkappa \leq k$, and any d, $S^{-1}T(p) \subset U_r^\rho$, *for some* $\rho \leq k$ *and some* r, *such that there is a chain of neighborhoods* $U_d^\alpha, U_e^\beta, \cdots, U_r^\rho$ *so that each two consecutive neighborhoods in this chain intersect each other.*

Proof. We shall make the proof by induction.

If $k = 1$, then $\varkappa = 1$. Now, for any j, $S(U_j^1) = T_j(U_j^1)$, since, for any point $x \,\varepsilon\, U_j^1$, $T_n(x) = T_j(x)$ for all $n > j$. Furthermore

$$T_j(U_j^1 - y_j) = T\, W_j(U_j^1 - y_j)$$

by definition, and because all W_i are stationary over U_j^1, for $i < j$, since $U_j^1 \cdot U_i^1 = 0$ for $i \neq j$. Now, since W_j is stationary in $A - U_j^1$, we have

$$W_j(U_j^1 - y_j) = U_j^1 - g_j^1.$$

Hence we have

$$S(U_j^1) - S(y_j) = S(U_j^1 - y_j) = T(U_j^1 - g_j^1) = T(U_j^1) - T(g_j^1).$$

But $S(y_j) = b_j = T(g_j^1)$. Therefore $S(U_j^1) = T(U_j^1)$. Thus, for any point $p \subset U_j^1$, for any j, $S^{-1}T(p) \subset U_j^1$.

We now assume that the statement is true for $k - 1$ and proceed to prove it for k. The sequence $G_{k-1} = \{g_i^k\}$ is a null sequence, since $G_k = 0$. Take a decomposition of A into the sets of G_{k-1} and the points of $A - G_{k-1}$, and thus obtain the hyperspace C, the transformations

$$t(A) = C,\; Z(C) = T\, t^{-1}(C) = B$$

and the homeomorphisms

$$R_1(A) = C,\; R_2(C) = B \text{ and } S(A) = R_2R_1(A) = B$$

as in the proof of Corollary I. From our assumption it follows that $R_2(C) = B$ is such that $R_2^{-1}Z(C) = C$ satisfies the results of this corollary for the case $k - 1$. Now $T(A) = Z\, t(A) = B$. Hence we have that

$$S^{-1}T(A) = R_1^{-1}R_2^{-1}Z\, t(A) = A.$$

Take any point p of A. If $p \subset U_i^j$, for any $j \leq k$, and any i, then

$$q = t(p) \subset t(U_i^j).$$

According to our assumption, if $q \subset t(U_d^\alpha)$, for any $\varkappa \leq k - 1$, and any d, $R_2^{-1}Z(q)$ is in a neighborhood $t(U_r^\rho)$ for some ρ and some r, such that there is a chain of neighborhoods $t(U_d^\alpha), t(U_e^\beta), \cdots, t(U_r^\rho)$ such that each two consecutive neighborhoods in this chain intersect each other. Now if

$$R_2^{-1}Z(q) \not\subset \sum_{i=1}^{\infty} t(U_i^k),$$

then

$$S^{-1}T(p) = R_1^{-1}[R_2^{-1}Z(q)] \subset U_r,$$

since

$$R_1(x) = t(x) \text{ for any } x \varepsilon A - \sum_{i=1}^{\infty} U_i^k.$$

Now the chain of neighborhoods $U_i^j, U_d^a, \cdots, U_r^\rho$ is such that each two consecutive neighborhoods of the chain intersect each other. For $U_i^j \cdot U_d^a \neq 0$ since $t(U_i^j) \cdot t(U_d^a) \supset q$; and any other two consecutive ones intersect because their images under t do. Hence the corollary is established for this case. Now if $R_2^{-1}Z(q) \subset t(U_m^k)$, for some m, then we have that

$$S^{-1}T(p) = R_1^{-1}[R_2^{-1}Z(q)] \subset U_m^k,$$

since, for all m, $R_1(U_m^k) = t(U_m^k)$, by the first part of this proof. The chain $U_i^j, U_d^a, \cdots, U_r^\rho, U_m^k$ is obviously such that each two consecutive neighborhoods of the chain intersect. For

$$U_r^\rho \cdot U_m^k \neq 0 \text{ since } t(U_r^\rho) \cdot t(U_m^k) \supset R_2^{-1}Z(q),$$

and by the above argument. Hence the corollary is also established for this case.

7. THEOREM. *If A and B are compact metric spaces and $T(A) = B$ is continuous and satisfies the conditions: (1) for any $\epsilon > 0$, any set $T^{-1}(b)$, for $b \varepsilon B$, and any $x \varepsilon T^{-1}(b)$, there exists a homeomorphism*

$$W(A - x) = A - T^{-1}(b)$$

which is stationary outside of the ϵ-neighborhood of $T^{-1}(b)$; (2) there exists some number α of the first or second number class such that $G_\alpha = 0$; and (3) $\prod_0^\infty L_i$ is a zero-dimensional set, then A is homeomorphic with B.

Proof. The theorem is true for any finite number α, by the lemma. We will now prove that the theorem is true when $\alpha = \omega$. When this is established, it will follow that it is true for any $\alpha > \omega$. For, if α is an isolated number, the result is obtained by induction as in the lemma. If α is a limit number, we again use induction, that is, we suppose the theorem true for all $\beta < \alpha$, and then prove that it is also true for α by the same method of proof we shall use here for $\alpha = \omega$.

Let H_1 be the collection of all sets g_j^1 of $G - G_1$ such that $U_j^1 \cdot \sum_{r=2}^{\infty} \sum_{i=1}^{\infty} U_i^r = 0$. Let $\{h_i^1\}$ represent the collection H_1. Let V_i^1 denote the neighborhood U_k^1 which covers h_i^1, for each i. Let H_2 be the collection of all sets of $(G - H_2) - G_2$ whose neighborhoods U do not intersect $\sum_{r=3}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Let $\{h_i^2\}$ represent the collection H_2; and let V_i^2 denote the neighborhood U_k^s, $s = 1$ or 2, which covers h_i^2, for each i. Let H_3 be the collection of all sets of $(G - \sum_{k=1}^{2} H_k) - G_3$ whose neighborhoods U do not intersect $\sum_{r=4}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Let $\{h_i^3\}$ denote the collection H_3; and let V_i^3 denote the neighborhood U_k^s, $s = 1, 2$, or 3, which covers h_i^3, for each i. Continue in this way indefinitely.

Now $\sum_{i=1}^{\infty} H_i = G$. For take any $g \varepsilon G$. There exists a finite integer N which is the largest number so that $g \varepsilon G_N$. Now, by definition, the neighborhood U of g is contained in the $\rho(g, L_N)/2$-neighborhood of g, thus $\bar{U} \cdot L_N = 0$. Take a neighborhood V of L_N so that $V \cdot \bar{U} = 0$. Now there exists an integer M so that $U_i^m \subset V$, for all $m > M$ and all i, since lim. sup. $\sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} U_i^j = L_N$. Therefore U does not intersect $\sum_{r=M+1}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Hence $g \varepsilon H_s$ for some $s \leq M$.

Take a decomposition of A into the sets of H_1 and the points of $A - H_1$. This is obviously an upper semi-continuous decomposition. Let its hyperspace be C_1 and let $T_1(A) = C_1$ be the continuous transformation associated with the decomposition. Now no sets of H_1 intersect lim. sup. H_1. Therefore $T_1(A) = C_1$ satisfies the conditions of the lemma for the case $k = 1$. Hence, by Corollary I section 6, there exists a homeomorphism $S_1(A) = C_1$ such that $S_1 \equiv T_1$ over $A - \sum_{i=1}^{\infty} V_i^1$. Let $Z_1(A) = S_1^{-1} T_1(A) = A$. Now Z_1 is univalued and continuous; $Z_1(h_i^1)$ is a point of A, for each i; and $Z_1(h_i^k) = h_i^k$, for all $k > 1$ and for all i.

Take a decomposition of A into the sets of H_2 and the points of $A - H_2$. This is an upper semi-continuous decomposition. Let C_2 be its hyperspace and $T_2(A) = C_2$ its associated continuous transformation. Now $T_2(A) = C_2$ satisfies the conditions of the lemma for the case $k = 2$. Hence there exists a homeomorphism $S_2(A) = C_2$ such that $S_2 \equiv T_2$ over $A - \sum_{i=1}^{\infty} V_i^2$. Let $Z_2(A) = S_2^{-1} T_2 Z_1(A) = A$. Now Z_2 is univalued and continuous and such that: $Z_2(h_i^j)$ is a point of A, for $j = 1$ or 2 and for each i; and Z_2 is the identity transformation over $A - \sum_{k=1}^{2} \sum_{i=1}^{\infty} V_i^k$. Hence $Z_2(h_i^k) = h_i^k$, for all

$k > 2$ and for all i. Continuing in this way indefinitely we get a sequence $\{Z_n\}$ of such transformations.

Let $Z = \lim_{n=\infty} \{Z_n\}$. We shall now prove that $\{Z_n\}$ converges uniformly to Z. It will then follow that Z is continuous. Take any $\epsilon > 0$. Now since $\prod_0^\infty L_i$ is closed, compact, and zero-dimensional, we can cover it by a finite number of disjoint $\epsilon/2$-neighborhoods, say $W_1, W_2, \cdots, W_h$. Let W denote $\sum_{i=1}^h W_i$. Now there exists an M such that $\sum_{j=M+1}^\infty \sum_{i=1}^\infty V_i{}^j \subset W$. For, if not, then no matter what M we take, there exists some $m > M$ and some i so that $V_i{}^m \not\subset W$. Take a sequence $M_1 < M_2 < M_3 < \cdots$ of integers and take the corresponding $V_i{}^{m_j} \not\subset W$. Take a point p_{m_j} of $V_i{}^{m_j}$ but not contained in W. Now $\{p_{m_j}\}$ contains a convergent subsequence $\{p'_{m_j}\}$ which converges to a point p. Now $p \subset L$ because $\lim. \sup. \sum_{j=1}^\infty \sum_{i=1}^\infty V_i{}^j = L$. Also, for any finite integer n, infinitely many of the points of $\{p'_{m_j}\}$ have subscripts greater than n. Therefore $p \subset L_n$ for every finite n. Hence $p \subset \prod_0^\infty L_i$, which is a contradiction. Furthermore there exists an integer $N \geq M$ so that for all $n > N$ and all i, $V_i{}^n \subset W_r$ for some r. For, if not, then no matter what N we take, there exists some $n > N$ and some i so the $V_i{}^n \not\subset W_r$, for any r. Take a sequence $\{N_j\}$ of integers such that $M < N_1 < N_2 < N_3 < \cdots$, and take the corresponding $V_{i_j}^{n_j} \not\subset W_r$, for any r. Now no $V_{i_j}^{n_j}$ will intersect both W_s and W_r, for $r, s = 1, 2, \cdots$, or h, unless the corresponding set $h_{i_j}^{n_j}$ intersects both W_s and W_r, since $W_i \cdot W_j = 0$, for $i \neq j$, and since each $V_{i_j}^{n_j} \subset W$. Hence there exists some u and some v so that each set of some subcollection $\{h_k\}$ of the collection $\{h_{i_j}^{n_j}\}$ have points in both W_u and W_v. Take a point p_k in $h_k \cdot W_u$ and a point q_k in $h_k \cdot W_v$, for each k. Now $\{p_k\}$ contains a convergent subsequence $\{p'_k\}$ which converges to a point $p \subset \prod_0^\infty L_i$, and $\{q_k\}$ contains a convergent subsequence $\{q'_k\}$ which converges to a point $q \subset \prod_0^\infty L_i$. Obviously p and q are distinct points. Furthermore, since p'_k and q'_k are on the same set $h_k = T^{-1}(b_k)$, for some $b_k \varepsilon B$, and since T is continuous, it follows that $p + q \subset T^{-1}(b)$, for some $b \varepsilon B$. Hence this set $T^{-1}(b)$ is a set of G_ω, which contradicts our assumption that $G_\omega = 0$.

The integer N depends upon ϵ. Take any $x \varepsilon A$. Now

$$Z_n(x) = S_n^{-1} T_n \cdots S_{N+1}^{-1} T_{N+1} Z_N(x)$$

and

$$Z_{n+p}(x) = S^{-1}_{n+p}T_{n+p} \cdots S_n^{-1}T_n \cdots S^{-1}_{N+1}T_{N+1}Z_N(x).$$

If $Z_N(x) \not\subset \sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} V_i^j$, then $S_k^{-1}T_kZ_N(x) = Z_N(x)$, for all $k > N$. Therefore $Z_n(x) = Z_N(x)$. Likewise $Z_{n+p}(x) = Z_N(x)$, for $p = 1, 2, \cdots$. Hence $\rho(Z_n(x), Z_{n+p}(x)) = 0$, for $n > N$ and $p = 1, 2, \cdots$, in this case. If $Z_N(x) \subset \sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} V_i^j$, then $Z_N(x) \subset W_r$ for some $r = 1, 2, \cdots$, or h. If $Z_N(x) \subset V^{N+1}_{d_1}$, for some d_1, then $S^{-1}_{N+1}T_{N+1}Z_N(x)$ is a point of $V^{N+1}_{m_1}$, for some m_1, where $V^{N+1}_{m_1} \subset W_r$. For, by Corollary II, there is a chain of neighborhoods $V^{N+1}_{d_1}, V^{N+1}_{e_1}, \cdots, V^{N+1}_{m_1}$, so that each two consecutive neighborhoods in the chain intersect. Hence $V^{N+1}_{m_1} \subset W_r$ since none of the V^{N+1}'s lying in any of the other W_i's will intersect those in W_r, as $W_i \cdot W_j = 0$, for $i \neq j$. Therefore $S^{-1}_{N+1}T_{N+1}Z_N(x) \subset W_r$ for this case. If $Z_N(x) \not\subset \sum_{i=1}^{\infty} V_i^{N+1}$, then we have that $S^{-1}_{N+1}T_{N+1}Z_N(x) = Z_N(x) \subset W_r$. If $S^{-1}_{N+1}T_{N+1}Z_N(x) \subset V^{N+2}_{d_2}$, for some d_2, then $S^{-1}_{N+2}T_{N+2}S^{-1}_{N+1}T_{N+1}Z_N(x)$ is a point of $V^{N+2}_{m_2}$, for some m_2, where $V^{N+2}_{m_2} \subset W_r$, by the same argument as above. If $S^{-1}_{N+1}T_{N+1}Z_N(x) \not\subset \sum_{i=1}^{\infty} V_i^{N+2}$, then we have that $S^{-1}_{N+2}T_{N+2}S^{-1}_{N+1}T_{N+1}Z_N(x) = S^{-1}_{N+1}T_{N+1}Z_N(x) \subset W_r$. Continue in this same way. Finally if the set $S^{-1}_{n-1}T_{n-1} \cdots S^{-1}_{N+1}T_{N+1}Z_N(x) \subset V^n_{d_n}$, for some d_n, then $S_n^{-1}T_nS^{-1}_{n-1}T_{n-1} \cdots Z_N(x) \subset V^n_{m_n}$, for some m_n, where $V^n_{m_n} \subset W_r$. If

$$S^{-1}_{n-1}T_{n-1} \cdots S^{-1}_{N+1}T_{N+1}Z_N(x) \not\subset \sum_{i=1}^{\infty} V_i^n$$

then

$$S_n^{-1}T_nS^{-1}_{n-1}T_{n-1} \cdots Z_N(x) = S^{-1}_{n-1}T_{n-1} \cdots Z_N(x) \subset W_r.$$

Therefore $Z_n(x)$ is a point p of W_r, and likewise

$$Z_{n+p}(x) = S^{-1}_{n+p}T_{n+p} \cdots S^{-1}_{n+1}T_{n+1}(p) = q \,\varepsilon\, W_r.$$

Accordingly

$$\rho(Z_n(x), Z_{n+p}(x)) = \rho(p, q) < \epsilon, \quad \text{for } n > N,\ p = 1, 2, \cdots;$$

and so $\{Z_n\}$ converges uniformly to Z. Hence Z is continuous.

Now Z transforms any set $g \,\varepsilon\, G$ into a single point of A. Furthermore, for any point $a \,\varepsilon\, A$, $Z^{-1}(a)$ is either a point p of A or a set $g \,\varepsilon\, G$ in A. Consider $T Z^{-1}(A) = B$. If $Z^{-1}(a) = p \,\varepsilon\, A$, $T Z^{-1}(a) = T(p) \,\varepsilon\, B$. If $Z^{-1}(a) = g$, where $g = T^{-1}(b)$, for some $b \,\varepsilon\, B$, then $T Z^{-1}(a) = T(g)$ which

is the point b. Hence $T Z^{-1}(A) = B$ is univalued. It is obviously continuous, since T and Z are continuous and A is compact.

Now $[T Z^{-1}]^{-1}(B) = Z T^{-1}(B) = A$ is also univalued. For take any point $b \varepsilon B$. If $T^{-1}(b) = p \varepsilon A$, then $Z T^{-1}(b) = Z(p)$ which is a point of A. since Z is univalued. If $T^{-1}(b) = g \varepsilon G$ in A, then $Z T^{-1}(b) = Z(g)$ which is also a point of A by the definition of Z.

Therefore $T Z^{-1}(A) = B$ is continuous and is univalued both ways. Hence, since A is compact, it is a homeomorphism.

8. *Applications.* The following two theorems classify some of the types of sets into which we can decompose the compact Euclidean 3-space,[12] so that, if the decomposition is upper semi-continuous and conditions (2) and (3) of the last theorem are satisfied, the hyperspace of the decomposition is also a compact Euclidean 3-space.

THEOREM a. *Let A be a compact Euclidean 3-space and $T(A) = B$ a continuous transformation such that each non-degenerate set $T^{-1}(b)$, for $b \varepsilon B$, is an isotopic sphere*[13] *plus its interior. Define the collections G and G_i and the sets L_i, for each i, as before. If there exists a number α of the first or second number class so that $G_\alpha = 0$; and if $\prod_0^\infty L_i$ is a zero-dimensional set; then B is also a compact Euclidean 3-space.*

Proof. Take any $\epsilon > 0$, any set $g \varepsilon G$, any $x \varepsilon g$, and any geometric sphere C in A. Let $S = C +$ its interior. There exists a homeomorphism $Z(A) = A$ so that $Z(S) = g$, since g is an isotopic sphere plus its interior. Let $Z^{-1}(x) = y$. Now $y \subset S$. It is well known that there exists a homeomorphism $R(A - y) = A - S$ which is stationary outside of the δ_ϵ-neighborhood U_δ of S. Let $W(A - x) = Z R Z^{-1}(A - x) = A - g$. Now W is a homeomorphism which is stationary outside of $V_\epsilon(g)$ by the proof used in the lemma for similar transformations. Therefore the conditions of the theorem in the last section are satisfied. Accordingly A is homeomorphic with B.

THEOREM b. *Let A be a compact Euclidean 3-space and $T(A) = B$ a continuous transformation such that each non-degenerate $T^{-1}(b)$, for $b \varepsilon B$, is a convex set in A. Define the collections G and G_i, and the sets L_i, for each i, as before. If there exists a number α of the first or second number*

[12] That is, the ordinary Euclidean 3-space compactified by adding the points at infinity.

[13] A set S in A is said to be an *isotopic sphere* if, for any geometric sphere C in A there exists a homeomorphism $R(A) = A$ such that $R(C) = S$.

class so that $G_a = 0$; and if $\prod_0^\infty L_i$ is a zero-dimensional set: then B is also a compact Euclidean 3-space.

Proof. We again take any $\epsilon > 0$, any set $g \varepsilon G$, and any $x \varepsilon g$ in A. Also take a convex ϵ-neighborhood V_ϵ of g in A. Take a system of spherical coördinates in A with the point x as origin. Choose any θ and any ϕ. The ray determined by this choice will intersect $F(V_\epsilon)$ in a single point, say (d, θ, ϕ), and it will intersect $F(g)$ in a single point, say (c, θ, ϕ). On this ray set up the following homeomorphism: $r' = \frac{(d-r) \cdot c}{d} + r$, if $r \leq d$, and $r' = r$, if $r > d$, where (r, θ, ϕ) represents any point on this ray. If we let θ and ϕ vary over all possible values, we get a homeomorphism $W(A - x) = A - g$ which is stationary outside of V_ϵ. Therefore, by the theorem of section 7, A is homeomorphic with B.

Since conditions I and III of section 2 were shown in section 3 to be equivalent when A is a 2-dimensional manifold, and in view of the theorem of section 7, we have the following:

THEOREM C. *The hyperspace B of any upper semi-continuous decomposition G of a 2-dimensional manifold A, where G is such that, for any $g \varepsilon G$ and any $x \varepsilon A$, there exists a homeomorphism $W(A - x) = A - g$, and such that conditions (2) and (3) of the theorem of section 7 are satisfied, is the same kind of 2-dimensional manifold, i. e. is homeomorphic with A.*

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ERGODIC CURVES.

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The problem of determining the least time required by the point of some motion to come within a distance ϵ of every point of the phase space was first considered by Birkhoff.[1] More recently Errera[2] has treated the problem in the plane of determining the length of the shortest curve, such that the distance of every point of a given domain D is at a distance $\leqq \epsilon$ from some point of the curve. He has solved the problem in case the domain D can be "swept out" by moving the center of a circle of radius ϵ along a Jordan curve of a certain special type. However, his methods fail to yield any information for an arbitrary domain D and it is not even certain, *a priori*, that a curve of shortest length, possessing the required property, exists.

In this paper this existence question is taken up in a slightly more general form, the domain D being replaced by a bounded point set M. The set of continuous, rectifiable curves, which are such that an arbitrary point of M lies at a distance $\leqq \epsilon$ from some point of the curve, is proved to contain at least one curve whose length equals the greatest lower bound, $\Lambda(\epsilon)$, of the lengths of the curves in the set. The set of such curves is shown to be closed. $\Lambda(\epsilon)$ is proved to be a monotone non-increasing function of ϵ which is continuous on the right for any positive ϵ. Whether $\Lambda(\epsilon)$ is continuous on the left, is an open question. The relation of the behavior of $\Lambda(\epsilon)$ in the neighborhood of $\epsilon = 0$ to the structure of M is investigated.

The author is indebted to Professor Tamarkin for pointing out the usefulness of a result of Tonelli (Lemma 4 in the present paper), the author's own proof having been invalid at this point.

1. *Notation and definitions.* Unless explicitly stated otherwise, ϵ denotes a positive number throughout the paper. x, y denote ordinary Cartesian coördinates in the Euclidean plane E, and M a bounded point set in E containing more than one point of E. The Euclidean distance between two points P, Q of E is denoted by $| PQ |$.

If P is a fixed point in E, the set of those points Q in E for which

[1] G. D. Birkhoff, "Probability and physical systems," *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 375-377.

[2] A. Errera, "Un problème de géométrie infinitésimale," *Memoires Académie royale de Belgique*, vol. 12, no. 1441 (1932).

$|PQ| \leqq \rho$ will constitute a *circular region.* ρ is the *radius* of the circular region and P its *center.* A circular region Γ *encloses* M if every point of M is a point of Γ.

A *continuous curve* [3] is an ordered set of points in E defined by the equations

$$x = f(t), \qquad y = g(t),$$

for t varying from a to b, where $f(t)$, $g(t)$ are real, continuous, single-valued functions of t defined in the closed interval $[a, b]$. If $f(t)$, $g(t)$ have bounded variation in $[a, b]$, the curve is *rectifiable.* A continuous curve is *ϵ-ergodic to M* if an arbitrary point of M is at a distance $\leqq \epsilon$ from some point of the curve. The set of points on a continuous curve is bounded, perfect and connected. In particular, it contains all of its limit points. Hence, if a continuous curve C be ϵ_m-ergodic to M and $\lim_{m\to\infty} \epsilon_m = 0$, it follows that C contains all the points of M. More generally, if C be ϵ_m-ergodic to M and $\lim_{m\to\infty} \epsilon_m = \epsilon$, it is seen that C is ϵ-ergodic to M. Two continuous curves

$$C_i: \qquad x = f_i(t), \qquad y = g_i(t), \qquad a \leqq t \leqq b, \qquad (i = 1, 2),$$

will be said to *lie in the η-neighborhood of each other,*[4] if the points on them corresponding to the same value of t always lie at a distance $\leqq \eta$ from each other.

Consider a succession

$$A_1, A_2, \cdots, A_n, \cdots,$$

of sets of points in E. A point P of E will be a *point of accumulation of the succession* [5] if for any positive number η, there are infinitely many sets A_n of the succession having at least one point P_n satisfying the condition $|PP_n| \leqq \eta$. Given a set of continuous curves

$$S: \qquad \{x = f(t), y = g(t)\}, \qquad 0 \leqq t \leqq 1,$$

a continuous curve

$$C: \qquad x = F(t), \qquad y = G(t), \qquad 0 \leqq t \leqq 1,$$

is a *curve of accumulation of S,*[6] if for any positive number η, there are infinitely many curves C' of S such that C, C' lie in the η-neighborhood of each other. S is said to be *closed* if it contains all of its curves of accumulation.

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[3] Cf., for example, L. Tonelli, *Fondamenti di calcolo delle variazioni,* vol. 1, p. 34.

[4] Cf. Tonelli, *op. cit.*, p. 72.

[5] Cf. M. Cipolla, "Sul postulato di Zermelo e la teoria dei limiti delle funzioni," *Atti dell'Accademia Gioenia in Catania,* Serie 5ª, vol. 6, p. 3, or Tonelli, *op. cit.*, p. 73.

[6] Tonelli, *op. cit.*, pp. 73-74.

Given a succession

$$S_1, S_2, \cdots, S_n, \cdots,$$

of sets of continuous curves, all defined in the interval $0 \leqq t \leqq 1$, the curve C is a *curve of accumulation of the succession*,[6] if for any positive number η, there are infinitely many sets S_n of the succession having at least one curve C_n such that C, C_n lie in the η-neighborhood of each other. The curve C will be a *limit curve of the succession*,[7] if for any positive number η, there exists an N_η such that C and an arbitrary member of S_n lie in the η-neighborhood of each other for $n > N_\eta$. The succession is then said to *converge uniformly towards* C.[7]

The adoption of the above definitions enables us later on to avoid the use of Zermelo's general selection principle.

2. *Preliminary lemmas.* In this section we collect four lemmas for later reference.

LEMMA 1. *Among the circular regions enclosing a given M, there is one, Γ, whose radius ρ ($\rho > 0$) is less than that of any other.*

Let $F(P)$ denote the least upper bound of $|PQ|$, where P is a fixed point in E and Q an arbitrary point of M. $F(P)$ is defined and continuous at any P. The value of the greatest lower bound of $F(P)$ is taken at least once by $F(P)$, say at R.

A circular region of radius $F(R)$ with R as center encloses M, and is the circular region Γ of least radius. For, $F(P) \geqq F(R)$, and the equality sign holds only if P coincides with R. To prove this, suppose $F(P) = F(R)$ with P and R different. Take P and R as centers of circular regions of radii $F(R)$. The two circular regions so obtained must intersect, since each encloses M. The part common to the two circular regions contains M and can be enclosed by a circular region of radius less than $F(R)$. But this is impossible, since $F(R)$ is the greatest lower bound of $F(P)$.

LEMMA 2. *If a succession $S'_1, S'_2, \cdots, S'_m, \cdots$ of sets of continuous curves ϵ-ergodic to M converges uniformly towards a limit curve C, then C is also ϵ-ergodic to M.*[8]

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[7] Tonelli, *op. cit.*, p. 76.

[8] When the convergence is not uniform, C is not necessarily ϵ-ergodic to M. For example, let M comprise the points of the two intervals on $y = 0$, $y = 1$ for which $0 \leqq x \leqq \epsilon$ ($\epsilon < 1/2$), and let S'_m contain the single curve

$$x = t, \qquad y = em^2\epsilon^{-2}t^2e^{-m^2\epsilon^{-2}t^2}, \qquad 0 \leqq t \leqq 1.$$

Here C is

$$x = t, \qquad y = 0, \qquad 0 \leqq t \leqq 1,$$

and is not ϵ-ergodic to M.

On the other hand C may be ϵ-ergodic to M even if the convergence be not uniform, as is the case in the example just given, when the points on $y = 1$ are removed from M.

Let P be an arbitrary point of M and γ be the circular region with center P and radius ϵ. Denote by Σ_m the set of points in which γ is intersected by the members of S'_m. Since no Σ_m is vacuous, the succession $\Sigma_1, \Sigma_2, \cdots, \Sigma_m, \cdots$ possesses[9] at least one point of accumulation on γ, say Q. If η denotes an arbitrarily small positive number, there are infinitely many sets Σ_m each having at least one point P_m, such that $| QP_m | < \eta/2$. If m be taken sufficiently large, due to the hypothesis of uniform convergence, there is a point R_m on C such that $| P_mR_m | < \eta/2$. Hence $| QR_m | < \eta$ and Q is a limit point for the points comprising C. Q therefore belongs to C, which establishes the lemma.

Lemmas 3 and 4 are due to Tonelli.[10]

Lemma 3. *In a set S of infinitely many continuous, rectifiable curves [succession $S_1, S_2, \cdots, S_n, \cdots$ of sets of continuous, rectifiable curves], all contained in a finite domain and all having a length less than a fixed number, the following hold:*

(a) *there exists a parametric representation of the type*

$$x = f(t), \qquad y = g(t), \qquad 0 \leqq t \leqq 1,$$

for each curve of the set [succession];

(b) *the set [succession] possesses at least one continuous and rectifiable curve of accumulation*

$$C: \quad x = F(t), \qquad y = G(t), \qquad 0 \leqq t \leqq 1;$$

(c) *there exists a process which makes a well-defined curve of accumulation C correspond to the given set [succession] and which determines a succession $S'_1, S'_2, \cdots, S'_m, \cdots$ which converges uniformly towards C, where S'_m is a set of curves extracted from $S[S_n(n \geqq m)]$.*

Lemma 4. *For a set of continuous, rectifiable curves all of whose lengths are less than L, the curves of accumulation of the set are all rectifiable and have lengths not greater than L.*

3. *Ergodic curves and the ergodic function* $\Lambda(\epsilon)$. We begin with

Definition 1. The greatest lower bound of the lengths of the con-

[9] M. Cipolla, *loc. cit.*, p. 3, proves this for linear point sets. The proof for point sets in the plane is quite simple and is omitted.

[10] *Op. cit.* For Lemma 3 see pp. 86-92, for Lemma 4 see p. 75.

tinuous, rectifiable curves ϵ-ergodic to M is the *ergodic function of* M and is denoted by $\Lambda(\epsilon)$.

We now prove

THEOREM 1. *The ergodic function* $\Lambda(\epsilon)$ *of* M *is finite and non-negative, being equal to zero if, and only if* $\epsilon \geqq \rho$, *where* ρ *is the radius of the circular region* Γ *in Lemma* 1. *There exists at least one well-defined continuous, rectifiable curve of length* $\Lambda(\epsilon)$ *which is* ϵ-*ergodic to* M.

To prove that $\Lambda(\epsilon)$ is finite, take r, θ as polar coördinates with the pole at the center of the circular region Γ in Lemma 1 and draw the curve $r = \epsilon\theta/2\pi$ $(0 \leqq \theta \leqq 2\pi\rho/\epsilon)$. This curve is ϵ-*ergodic* to M, and since it has finite length, $\Lambda(\epsilon)$ is finite.

Obviously $\Lambda(\epsilon)$ cannot be negative. That $\Lambda(\epsilon) = 0$ for $\epsilon \geqq \rho$ is trivial in view of the significance ascribed to ρ in Lemma 1. The proof that $\Lambda(\epsilon) > 0$ for $\epsilon < \rho$ is deferred until the end of the proof of this theorem.

Suppose $\epsilon < \rho$. We show that there exists a well-defined, continuous, rectifiable curve, ϵ-ergodic to M, whose length equals $\Lambda(\epsilon)$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ be a monotone decreasing sequence of positive numbers converging to $\Lambda(\epsilon)$. Consider the succession

$$(1) \qquad S_1, S_2, \cdots, S_n, \cdots,$$

of sets of continuous, rectifiable curves ϵ-ergodic to M, in which S_n comprises the totality of such curves whose lengths are less than λ_n, but not less than λ_{n+1}. A circular region concentric with the circular region Γ in Lemma 1 and having a radius $\rho + \lambda_1 + \epsilon$ contains all the curves in (1), inasmuch as the lengths of the curves in (1) are all less than λ_1. Hence (1) can be identified as the succession $S_1, S_2, \cdots, S_n, \cdots$ in Lemma 3. According to this lemma, the succession possesses at least one well-defined, continuous, rectifiable curve of accumulation C, part (c) of the lemma providing a succession

$$S'_1, S'_2, \cdots, S'_m, \cdots, \qquad S'_m \subset S_n \ (n \geqq m),$$

of sets of curves which converges uniformly to C. Since the curves in the succession are all ϵ-ergodic to M, Lemma 2 applies to show that C is ϵ-ergodic to M. Now consider the sets of curves

$$S'_m + S'_{n+1} + \cdots + S'_{m+p} + \cdots, \qquad (m = 1, 2, \cdots).$$

C is a curve of accumulation for each of these sets, and for any given value of m the lengths of the members of the set are all less than λ_m. From Lemma 4 it follows, therefore, that the length l of C cannot exceed λ_m, and hence,

since $\lim_{m\to\infty} \lambda_m = \Lambda(\epsilon)$, we have $l \leqq \Lambda(\epsilon)$. On the other hand C is ϵ-ergodic to M, so that $l \geqq \Lambda(\epsilon)$, i. e., $l = \Lambda(\epsilon)$.

We are now in a position to prove that $\Lambda(\epsilon) > 0$ for $\epsilon < \rho$. Suppose $\Lambda(\epsilon) = 0$. The curve C constructed above is ϵ-ergodic to M and has length zero, i. e., it is a point. A circular region of radius ϵ with this point as center would enclose M. Since $\epsilon < \rho$, this contradicts Lemma 1. Hence $\Lambda(\epsilon) > 0$.

When $\epsilon \geqq \rho$, we have $\Lambda(\epsilon) = 0$ and the above curve C degenerates to a point which is the center of Γ for $\epsilon = \rho$, and can be any point in a certain point set for $\epsilon > \rho$.

The theorem just established prompts

DEFINITION 2. A continuous, rectifiable curve of length $\Lambda(\epsilon)$ which is ϵ-ergodic to M will be termed an *ergodic curve of* M. When $\Lambda(\epsilon) = 0$, the ergodic curve of M is said to be *degenerate* or to be an *ergodic point of* M.

One readily sees that the set of ergodic points of M existing for a given value of ϵ $(\epsilon \geqq \rho)$ is closed. We complete this result by proving

THEOREM 2. *The set* S_ϵ *of ergodic curves of* M *existing for a given value of* ϵ $(\epsilon < \rho)$ *is closed.*

Let C be an arbitrary, continuous curve of accumulation of S_ϵ. Following Lemma 4, C is seen to be rectifiable and to have a length $l \leqq \Lambda(\epsilon)$.

Consider the succession

$$S'_1, S'_2, \cdots, S'_m, \cdots,$$

of sets of curves extracted from S_ϵ by taking S'_m to be the totality of those members C'_m of S_ϵ such that C'_m, C lie in the $1/m$-neighborhood of each other. No S'_m is vacuous. The succession converges uniformly to C, so that Lemma 2 applies to prove that C is ϵ-ergodic to M. Hence $l \geqq \Lambda(\epsilon)$, i. e., $l = \Lambda(\epsilon)$, and therefore C belongs to S_ϵ.

Further information as to the nature of $\Lambda(\epsilon)$ is given in

THEOREM 3. *The ergodic function* $\Lambda(\epsilon)$ *is a monotone non-increasing function of* ϵ *which is always continuous on the right.*

To prove that $\Lambda(\epsilon)$ is a monotone non-increasing function of ϵ, let $0 < \epsilon_1 < \epsilon_2$. According to Theorem 1, there is at least one ergodic curve for $\epsilon = \epsilon_1$. Since it is ϵ_1-ergodic to M, it is *a fortiori* ϵ_2-ergodic to M, and therefore $\Lambda(\epsilon_2) \leqq \Lambda(\epsilon_1)$.

We now establish that $\Lambda(\epsilon + 0) = \Lambda(\epsilon)$ for $\epsilon < \rho$. The equation obviously holds for $\epsilon \geqq \rho$. Since $\Lambda(\epsilon)$ is a monotone non-increasing function of ϵ, it is sufficient to show that the inequality

$$(2) \qquad \Lambda(\epsilon + 0) < \Lambda(\epsilon),$$

is impossible. Assuming (2) to hold, we let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ be a monotone decreasing sequence of ϵ-values lying in the open interval $(0, \rho)$ and converging to the value ϵ in (2). Consider the succession

$$(3) \qquad S_1, S_2, \cdots, S_n, \cdots$$

of sets of curves in which S_n comprises a single, well-defined curve, this curve being an ergodic curve of M for $\epsilon = \epsilon_n$ constructed according to the principles laid down in the proof of Theorem 1. The lengths of the curves in (3) are all less than λ; where λ is any number between $\Lambda(\epsilon + 0)$ and $\Lambda(\epsilon)$. A finite domain contains all the curves in (3) and therefore according to Lemma 3, there is at least one well-defined, continuous, rectifiable curve of accumulation C of the succession.

From Lemma 4 it is seen that the length of C cannot exceed λ and hence is less than $\Lambda(\epsilon)$.

Now Lemma 3 asserts that a succession

$$(4) \qquad S'_1, S'_2, \cdots, S'_m, \cdots, \qquad S'_m = S_n \ (n \geqq m),$$

of sets of curves can be found which converges uniformly to C. Consider the successions

$$(5) \qquad S'_m, S'_{m+1}, \cdots, S'_{m+p}, \cdots, \qquad (m = 1, 2, \cdots).$$

Each of these successions converges uniformly to C and for any given value of m the members of the succession are all ϵ_m-ergodic to M. Hence C is ϵ_m-ergodic to M and, since $\lim_{m \to \infty} \epsilon_m = \epsilon$, C is ϵ-ergodic to M.

This is a contradiction; for C cannot be ϵ-ergodic to M, since its length is less than $\Lambda(\epsilon)$. Hence $\Lambda(\epsilon + 0) = \Lambda(\epsilon)$.

Our last theorem deals with the structure of M and the behavior of $\Lambda(\epsilon)$ in the neighborhood of $\epsilon = 0$.

THEOREM 4. *In order that $\Lambda(+0)$ be finite, it is necessary and sufficient that M lie on a continuous, rectifiable curve.*

The sufficiency of the condition is obvious.

In order to prove the necessity of the condition, suppose $\Lambda(+0)$ to be finite and let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ be a monotone decreasing sequence of ϵ-values lying in the open interval $(0, \rho)$ and converging to 0. Following the proof of Theorem 3, construct the succession (3) of sets of curves. The lengths of the curves in this succession cannot exceed $\Lambda(+0)$ and are therefore less than λ, where λ denotes an arbitrary positive number greater than $\Lambda(+0)$ [$\Lambda(0)$ is as yet undefined]. On extracting the succession (4) from (3) and forming the successions (5), we see that the curve of accumulation C is ϵ_m-ergodic to M, with $\lim_{m\to\infty} \epsilon_m = 0$. Hence C contains all the points of M.

REMARKS. In Theorem 4 the length of C cannot exceed λ and therefore cannot exceed $\Lambda(+0)$. On the other hand, no continuous, rectifiable curve can contain M and have a length l less than $\Lambda(+0)$. For, if such a curve exists, $\Lambda(\epsilon_n) \leqq l$ and therefore $\Lambda(+0) \leqq l$, which is a contradiction. The length of C accordingly equals $\Lambda(+0)$.

If we now define $\Lambda(0)$ as the greatest lower bound of the lengths of the continuous, rectifiable curves which contain M, we see that Theorems 1 and 3 both hold for $0 \leqq \epsilon \leqq \rho$, provided that $\Lambda(+0)$ be finite.

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NOTE ON A RESULT OF M. H. MARTIN CONCERNING A PARTICULAR MASS RATIO IN THE RESTRICTED PROBLEM OF THREE BODIES.

By Natalie Rein.

In his paper, "On the libration points of the restricted problem of three bodies,"[1] M. H. Martin states the following theorem:

Theorem 3. *There exists in the interval* $0 < \mu < 1$ *one and only one value of* $\mu = \mu^*$ *for which the following relations hold between the functions* $\rho_1(\mu)$ *and* $\rho_2(\mu)$:

$$\rho_1(\mu) < \rho_2(\mu) \quad \textit{for} \quad 0 < \mu < \mu^*; \qquad \rho_1(\mu^*) = \rho_2(\mu^*);$$
$$\rho_1(\mu) > \rho_2(\mu) \quad \textit{for} \quad \mu^* < \mu < 1,$$

and here $\mu^* > \frac{1}{2}$.

Here μ is the mass of one of the two finite bodies (the unit of mass being taken as the sum of the masses of the two finite bodies), and $\rho_1(\mu)$ and $\rho_2(\mu)$ are the distances of the collinear libration points L_1 and L_2 from the mass μ (the constant distance between two finite bodies being taken as the unit of distance), which are functions of μ defined uniquely by the equations:

$$\text{(a)} \quad 1 - \rho_1 - \mu - \frac{1-\mu}{(1-\rho_1)^2} + \frac{\mu}{\rho_1^2} = 0, \quad 1 + \rho_2 - \mu - \frac{1-\mu}{(1+\rho_2)^2} - \frac{\mu}{\rho_2^2} = 0.$$

See the equations (14 a) and (14 b) of Martin's paper.

The above theorem is, however, erroneous and should be formulated in its correct form as follows:

Theorem. *In the interval* $0 < \mu < 1$ *the functions* $\rho_1(\mu)$ *and* $\rho_2(\mu)$ *satisfy the inequality* $\rho_1(\mu) < \rho_2(\mu)$.

That is, the particular mass ratio μ^* found by Martin does not exist.

Indeed, we can write the equations (a) in the form

$$\frac{1-\mu}{(1-\rho_1)^2} + \rho_1 = 1 - \mu + \frac{\mu}{\rho_1^2}, \qquad \frac{1-\mu}{(1+\rho_2)^2} - \rho_2 = 1 - \mu - \frac{\mu}{\rho_2^2}.$$

[1] *American Journal of Mathematics*, vol. 53 (1931), p. 167.

Then $\rho_1(\mu)$ will be defined as the positive abscissa of the intersection point of the curves

$$F_1(\rho)=\frac{1-\mu}{(1-\rho)^2}+\rho, \qquad f_1(\rho)=1-\mu+\frac{\mu}{\rho^2}$$

and $\rho_2(\mu)$ as the positive abscissa of the intersection point of the curves

$$F_2(\rho)=\frac{1-\mu}{(1+\rho)^2}-\rho, \qquad f_2(\rho)=1-\mu-\frac{\mu}{\rho^2}.$$

Thereby, $0<\rho_1<1$, $0<\rho_2<1$ for $0<\mu<1$ according to Martin's Theorem I.[2]

$$\text{But } \frac{dF_1}{d\rho}=\frac{2(1-\mu)}{(1-\rho)^3}+1>0, \qquad \frac{dF_2}{d\rho}=-\left[\frac{2(1-\mu)}{(1+\rho)^3}+1\right]<0$$

and

$$\left|\frac{dF_1}{d\rho}\right|>\left|\frac{dF_2}{d\rho}\right| \quad \text{for} \quad 0<\rho<1.$$

This means that the intersection point of the curves F_1 and f_1 always lies to the left of the intersection point of the curves F_2 and f_2 for both the curves f_1 and f_2 are symmetrical with respect to the straight line $f=1-\mu$ and $F_1=F_2=1-\mu$ for $\rho=0$. Hence, $\rho_1<\rho_2$ for all the values of μ within the interval $0<\mu<1$.

Dr. Martin's error entered when he derived his equation (27), which defines the value $\rho^*=\rho_1=\rho_2$. In fact, he writes:

$$\rho^4(\rho^5-6\rho^3-2\rho^2+6)=0.$$

Actually, the correct form of this equation is

$$\rho^4(\rho^5-3\rho^3-\rho^2+3)=0.$$

The examination of the equation $\rho^5-3\rho^3-\rho^2+3=0$ shows immediately that it has only two positive roots, $\rho=1$ and $\rho=\sqrt{3}$, both lying outside the interval of the values of ρ_1 and ρ_2 for $0<\mu<1$.

Miss Jenny E. Rosenthal in her paper "Note on the numerical value of a particular mass ratio in the restricted problem of three bodies," [3] has calculated from Martin's equation (27) the numerical values of ρ^* and μ^*. This equation being incorrect, the results of Miss Rosenthal's calculations could not be correct.

MOSCOW, U. S. S. R.

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[2] *Ibid.*

[3] *American Journal of Mathematics*, vol. 53 (1931), p. 258.

ON THE ADDITION OF CONVEX CURVES.

By Richard Kershner.

If C denotes a convex Jordan curve in the plane let $\Omega(C)$ denote the open, bounded domain surrounded by C, so that the closed region $\bar{\Omega}(C)$ is the logical sum $\Omega(C) + C$. If A and B are two point sets in the z-plane, where $z = x + y$, their vectorial sum [1] will be denoted by $A\ (+)\ B$.

In connection with his investigation of the distribution of the values of the Riemann zeta function,[2] Bohr [3] has studied the vectorial addition of convex curves from a geometrical point of view. He has proven,[4] among other things, that if $C_1, C_2, \cdots, C_n$ are convex curves, then either there exists a convex C_E such that

$$C_1(+)\,C_2(+)\cdots(+)\,C_n = \bar{\Omega}(C_E) \tag{1}$$

or there exist two convex curves C_E and C_I such that $\Omega(C_E) \supset \Omega(C_I)$ and

$$C_1(+)\,C_2(+)\cdots(+)\,C_n = \bar{\Omega}(C_E) - \Omega(C_I) \tag{2}$$

so that the vectorial sum is either a closed bounded region or a ring shaped region bounded by two convex curves.

Following a suggestion of Wintner, Haviland [5] applied the supporting function (Stützfunktion) of Brunn and Minkowski to the study of Bohr's problem and showed that if $h_1(\theta), h_2(\theta), \cdots, h_n(\theta)$ are the supporting functions of the curves $C_1, C_2, \cdots, C_n$ and if $h_E(\theta)$ is the supporting function of C_E, then

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[1] By the vectorial sum of two sets A and B is meant the set $z = z_a + z_b$, where $z_a \subset A$ and $z_b \subset B$. It is clear that this addition is associative and commutative.

[2] Cf. e. g. Titchmarsh, *The Zeta Function of Riemann.*

[3] H. Bohr, "Om Addition of unendelig mange konvekse Kurver," *Danske Videnskabernes Selskab* (Forhandlinger, 1913), pp. 325-366; also, for a further study cf. H. Bohr and B. Jessen, "Om Sandsynlighedsfordelinger ved Addition of konvekse Kurver," *Danske Videnskabernes Selskab, Skrifter*, (8), vol. 12, no. 3 (1929).

[4] Cf. H. Bohr, *loc. cit.*; also H. Bohr and B. Jessen, *loc. cit.* For a short presentation of the proof of this fact cf. B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), p. 69.

[5] E. K. Haviland, "On the addition of convex curves in Bohr's theory of Dirichlet series," *American Journal of Mathematics*, vol. 55 (1933), pp. 332-334.

$$h_E(\theta) = \sum_{j=1}^{n} h_j(\theta). \tag{3}$$

Using this explicit formula and Minkowski's results on mixed area, Haviland improved on the results of Bohr concerning relations between the areas of $\Omega(C_j)$ and $\Omega(C_E)$. He also derived results with regard to lengths and curvatures. Finally, Bohr and Jessen[6] have recently shown that, with the use of the supporting functions, it is possible to determine the set of those $\sigma > 1$ for which the closure of the values attained by the logarithm of the Riemann zeta function on the line σ is a convex region; for the remaining values of $\sigma > 1$ this closure is ring-shaped. They also pointed out a geometrical criterion for the existence of an inner curve in the case of arbitrary symmetric convex curves.

The object of the present note is an analytical discussion of the inner curve C_I similar to that given by Haviland for C_E. In particular there will be obtained a rule giving the supporting function $h_I(\theta)$ of C_I in terms of the supporting functions $h_j(\theta)$ of the added curves C_j in the same manner as the rule (2) gives $h_E(\theta)$. There will also be obtained results for the length of C_I, for the curvature along this curve and for the area of $\Omega(C_I)$. As an application of the analytical results concerning $h_I(\theta)$ there is given a simple mechanical construction of C_I whenever C_I exists.[7] In particular there is a geometrical criterion that C_I exist, i. e. that (2) hold. This criterion will imply the abovementioned criterion found by Bohr and Jessen in the case of symmetric curves.

We treat first the sum of two convex curves and prove

THEOREM I_0. *Let $n = 2$ so that either $C_1 \, (+) \, C_2 = \bar{\Omega}(C_E)$ or*

$$C_1 \, (+) \, C_2 = \bar{\Omega}(C_E) - \Omega(C_I).$$

It may be supposed that $l_1 \geqq l_2$ where l_j is the length of C_j. Let $\tilde{C}_2$ represent the curve obtained by rotating C_2 about the origin of the plane through an angle π. Then the curve C_I exists if and only if $\tilde{C}_2$ may be placed in $\Omega(C_1)$ by a translation.

Proof. Suppose that C_I exists so that $\Omega(C_I)$ is not empty. It is clear that a translation of C_2 effects only a translation on the region $\bar{\Omega}(C_E) - \Omega(C_I)$.

[6] H. Bohr and B. Jessen, "On the distribution of the values of the Riemann zeta function," *American Journal of Mathematics*, vol. 58 (1936), pp. 35-44.

[7] For further applications cf. R. Kershner and A. Wintner, "On the boundary of the range of values of $\zeta(s)$," *American Journal of Mathematics*, vol. 58 (1936), pp. 421-425.

Hence we may translate C_2 into a new position C_2^* in such a way that $\Omega(C_I^*)$ includes the origin of the plane, where, of course,

$$C_1(+)C_2^* = \bar{\Omega}(C_E^*) - \Omega(C_I^*).$$

Since the origin is in $\Omega(C_I^*)$, no point of C_2^* is symmetrical, with respect to the origin, to a point of C_1. Hence, the curve $\tilde{C}_2^*$, obtained by rotating C_2^* about the origin through an angle π, does not intersect C_1. Since $l_2 \leqq l_1$, either $\tilde{C}_2^*$ is in $\Omega(C_1)$ or $\bar{\Omega}(C_1)$ and $\bar{\Omega}(\tilde{C}_2^*)$ are disjoint. Now the latter case is impossible. For in this case the set $\bar{\Omega}(C_1)$ $(+)$ $\bar{\Omega}(C_2^*)$ would not contain the origin, whereas it is obvious [8] that

$$\bar{\Omega}(C_1)\ (+)\ \bar{\Omega}(C_2^*) = \bar{\Omega}(C_E^*)$$

and we have supposed that the origin was in $\Omega(C_I^*) \subset \Omega(C_E^*)$. Hence $\tilde{C}_2^*$ is in $\Omega(C_1)$ and the "only if" of Theorem I_0 is proven. The "if" may be shown by following the above proof in the reverse direction.

Next we prove

THEOREM II_0. *Let $n = 2$ and $l_1 \geqq l_2$. Let T_I be the set of all points (x, y) of the plane which satisfy*

$$x \cos\theta + y \sin\theta < h_1(\theta) - h_2(\theta + \pi) \text{ for all } \theta. \tag{4}$$

Then if T_I is not empty C_I exists and $T_I = \Omega(C_I)$; if T_I is empty C_I does not exist.

Proof. It is clear, here also, that a translation of C_2 effects a translation of T_I. Hence we may translate C_2 into a new position C_2^* in such a way that T_I^* contains the origin (where, of course, T_I^* is defined in terms of C_1 and C_2^* exactly as T_I was defined in terms of C_1 and C_2) if and only if T_I is not empty. But T_I^* contains the origin if and only if

$$0 < h_1(\theta) - h_2^*(\theta + \pi) \text{ for all } \theta \tag{5}$$

where $h_2^*(\theta)$ is the supporting function of C_2^*. Since the supporting function $h(\theta)$ of any convex curve is defined [9] as

$$h(\theta) = \max\,[x \cos\theta + y \sin\theta], \qquad (x, y) \text{ on } C, \tag{6}$$

it is clear that $h_2^*(\theta + \pi)$ is the supporting function of the curve $\tilde{C}_2^*$

[8] Cf. E. K. Haviland, *loc. cit.*, p. 332.

[9] Cf., e. g., G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I.* (Berlin, 1925), p. 106.

obtained by rotating C_2^* about the origin through an angle π. Hence the relation (5) implies that $\tilde{C}_2^*$ is within C_1. Thus T_I is not empty if and only if $\tilde{C}_2$ may be placed entirely within C_1 by means of a translation. Consequently, by Theorem I_0, T_I is empty if and only if C_I does not exist and the last part of Theorem II has been proven.

Suppose now that T_I is not empty.

First, $T_I \subset \bar{\Omega}(C_E)$. For, it is seen from (3) and (6) that $\bar{\Omega}(C_E)$ is the set of points (x, y) satisfying

$$(7) \qquad x \cos\theta + y \sin\theta \leqq h_1(\theta) + h_2(\theta) \quad \text{for all } \theta;$$

and (4) implies (7), since $-h_2(\theta + \pi) < h_2(\theta)$. In fact, $h_2(\theta) + h_2(\theta + \pi)$ (> 0) is precisely the width of the curve C_2 in the direction θ.

Next $T_I \subset \Omega(C_I)$. (It has already been shown that C_I exists). For let $z_j = x_j + iy_j$ be a point of C_j, where $j = 1, 2$, and let $\theta_1 = \theta(z_1)$ be a value of θ such that

$$(8) \qquad x_1 \cos\theta_1 + y_1 \sin\theta_1 = h_1(\theta_1).$$

Then, since z_2 is on C_2, the definition (6) of $h_2(\theta)$ gives

$$(9) \qquad x_2 \cos\theta + y_2 \sin\theta \leqq h_2(\theta) \quad \text{for all } \theta;$$

hence, in particular (9) holds for $\theta = \theta_1 + \pi$ and

$$(10) \qquad x_2 \cos\theta_1 + y_2 \sin\theta_1 \geqq -h_2(\theta_1 + \pi).$$

From (8) and (10)

$$(x_1 + x_2) \cos\theta_1 + (y_1 + y_2) \sin\theta_1 \geqq h_1(\theta_1) - h_2(\theta_1 + \pi),$$

so that $C_1 \, (+) \, C_2 = \bar{\Omega}(C_E) - \Omega(C_I) \subset \bar{\Omega}(C_E) - T_I$, i. e. $T_I \subset \Omega(C_I)$.

Finally, $T_I \supset \Omega(C_I)$. For, let B be the boundary of T_I and let $z_0 = x_0 + iy_0$ be any point of B. Then, by the definition of T_I,

$$(11) \qquad x_0 \cos\theta + y_0 \sin\theta \leqq h_1(\theta) - h_2(\theta + \pi) \quad \text{for all } \theta,$$

while, for at least one θ, say $\theta = \theta_0$,

$$(12) \qquad x_0 \cos\theta_0 + y_0 \sin\theta_0 = h_1(\theta_0) - h_2(\theta_0 + \pi).$$

Let $z_2 = z_2(\theta_0) = x_2(\theta_0) + iy_2(\theta_0)$ be a point of C_2 in the direction $\theta_0 + \pi$, i. e. satisfying

$$(13) \qquad x_2 \cos(\theta_0 + \pi) + y_2 \sin(\theta_0 + \pi) = h_2(\theta_0 + \pi).$$

From (6),

$$(14) \qquad x_2 \cos(\theta+\pi) + y_2 \sin(\theta+\pi) \leqq h_2(\theta+\pi) \text{ for all } \theta.$$

Let $z_0 - z_2 = z_1 = x_1 + iy_1$. Then, from (12), (13) and (11), (14)

$$x_1 \cos\theta_0 + y_1 \sin\theta_0 = h_1(\theta_0) \text{ and } x_1 \cos\theta + y_1 \sin\theta \leqq h_1(\theta) \text{ for all } \theta.$$

Thus z_1 is a point of C_1, as well as z_2 a point of C_2, and $z_1 + z_2 = z_0$ where z_0 was any point of B. Thus, $B \subset \bar{\Omega}(C_E) - \Omega(C_I)$. Since T_I is obviously convex, this implies $T_I \supset \Omega(C_I)$. But it has already been shown that $T_I \subset \Omega(C_I)$ so that the proof of Theorem II_0 is complete.

Theorem II_0 implies a simple mechanical way in which $\Omega(C_I)$ may be determined: $\Omega(C_I)$ is, up to a translation, the locus of a point which is considered as rigidly attached to C_2 when $\tilde{C}_2$ is translated in all possible manners in $\Omega(C_1)$. This fact, which is completely equivalent to Theorem II_0, merely expresses the amount of indeterminateness in the choice of the translation which occurred in the proof of Theorem I_0. It would be possible to give an entirely geometrical proof of Theorem II_0 based on this reasoning.

Since $B = C_I$, every point z_0 of C_I satisfies (11) and (12). From these inequalities it does not follow that the supporting function $h_1(\theta)$ of C_I is identically $h_1(\theta) - h_2(\theta+\pi)$. However (11) gives immediately

$$h_I(\theta) \leqq h_1(\theta) - h_2(\theta+\pi) \text{ for all } \theta.$$

Furthermore, if there is a unique supporting line through a given point z_0 of C_I, i. e. if

$$x_0 \cos\theta + y_0 \sin\theta = h_I(\theta)$$

is satisfied for a unique $\theta = \theta_0$, then by (12)

$$h_I(\theta_0) = h_1(\theta_0) - h_2(\theta_0 + \pi).$$

The curve C_I is said to have a corner at $z = z_0$ if the supporting line through z_0 is not unique. The above remarks imply

THEOREM III_0. *The supporting function $h_I(\theta)$ of C_I satisfies*

$$(15) \qquad h_I(\theta) \leqq h_1(\theta) - h_2(\theta+\pi) \textit{ for all } \theta$$

and the equality sign holds save at most for the θ-intervals corresponding to corners of C_I.

We now give the corresponding theorems for the sum of n convex curves.

THEOREM I. *Let $C_1, C_2, \cdots, C_n$ be n convex curves of lengths $l_1, l_2, \cdots, l_n$ respectively and call $C_1 (+) C_2 (+) \cdots (+) C_n = \bar{\Omega}(C_E) - \Omega(C_I)$ where $\Omega(C_I)$ represents the empty set if C_I does not exist. Suppose that $l_1 \geqq l_2 \cdots \geqq l_n$. Call $C_2 (+) C_3 (+) \cdots (+) C_n = \bar{\Omega}(D_E) - \Omega(D_I)$ where $\Omega(D_I)$ is the empty set if D_I does not exist. Then C_I exists if and only if $\tilde{D}_E$, the curve obtained by rotating D_E about the origin through an angle π, can be placed in $\Omega(C_1)$ by a translation.*

THEOREM II. *Let T_I be the set of all points z satisfying*

$$x \cos \theta + y \sin \theta < h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi) \quad \text{for all } \theta, \tag{16}$$

where $h_j(\theta)$ is the supporting function of C_j $(j = 1, 2, \cdots, n)$. Then if T_I is not empty C_I exists and $T_I = \Omega(C_I)$; if T_I is empty C_I does not exist.

Proof. Since the supporting function $g_E(\theta)$ of D_E satisfies, by (3), the relation

$$g_E(\theta) = \sum_{j=2}^{n} h_j(\theta), \tag{17}$$

it may be shown, as it was in the case of two curves, that Theorem I is implied by Theorem II; so it is sufficient to prove the latter.

Now Theorem II is true for the case of two curves. Suppose Theorem II has been proven for the case of $n - 1$ curves. Then, using the notations of Theorem I and calling $g_I(\theta)$ the supporting function of D_I, we have

$$g_I(\theta) \leqq h_2(\theta) - \sum_{j=3}^{n} h_j(\theta + \pi). \tag{18}$$

Now it is well known that if $h(\theta)$ is the supporting function and l the length of a convex curve,

$$l = \int_0^{2\pi} h(\phi) d(\phi). \tag{19}$$

Let λ_I be the length of D_I. Then, by (18) and (19),

$$\lambda_I \leqq l_2 - \sum_{j=3}^{n} l_j. \tag{20}$$

Suppose first that T_I is empty. Then, as in Theorem II_0, the curve $\tilde{D}_E$ cannot be placed within C_1 by a translation. On the other hand, the length λ_I of D_I is, by (20), less than l_2, and, since $l_1 \geqq l_2$, less than l_1. Hence, there exists a

convex curve E between D_I and D_E, i. e. in $\bar{\Omega}(D_E) - \Omega(D_I)$, of length less than or equal to l_1 such that $\bar{E}$ cannot be placed within C_1 by a translation. The region C_1 $(+)$ E is, by Theorem I_0, a closed convex region, and a fortiori, C_I does not exist.

Suppose now that T_I is not empty. Then C_1 $(+)$ $D_E = \bar{\Omega}(C_E) - T_I$ by (17) and Theorem II_0. Let F be any convex curve in $\bar{\Omega}(D_E) - \Omega(D_I)$ and let $k(\theta)$ be its supporting function. Let C_1 $(+)$ $F = \bar{\Omega}(G_E) - \Omega(G_I)$ where $\Omega(G_I)$ represents the empty set if G_I does not exist. Then the supporting function of G_E is, by (3), equal to $h_1(\theta) + k(\theta)$ and the supporting function of G_I is, by Theorem II_0, equal to $h_1(\theta) - k(\theta + \pi)$, except, at most, at the possible corners of G_I.

Since $k(\theta) \leqq g_E(\theta) = \sum_{j=2}^{n} h_j(\theta)$ for all θ, we have

$$\begin{aligned} h_1(\theta) - g_E(\theta+\pi) &\leqq h_1(\theta) - k(\theta+\pi) \\ &\leqq h_1(\theta) + k(\theta) \leqq h_1(\theta) + g_E(\theta) \quad \text{for all } \theta. \end{aligned}$$

Hence $\bar{\Omega}(G_E) - \Omega(G_I) \subset \Omega(C_E) - T_I$. But F was any convex curve in $\bar{\Omega}(D_E) - \Omega(D_I)$. Consequently

$$C_1 \ (+)\ D_E = C_1 \ (+)\ C_2 \ (+) \cdots (+)\ C_n \tag{21}$$

and $T_I = \Omega(C_I)$.

Relations (17) and (21), together with Theorem III_0, give immediately

Theorem III. *The supporting function $h_I(\theta)$ of C_I satisfies*

$$h_I(\theta) \leqq h_1(\theta) - \sum_{j=2}^{n} h_j(\theta+\pi), \ \textit{for all } \theta, \tag{22}$$

and the equality sign holds save at most for the θ-intervals corresponding to corners of C_I. Thus, if $\{z_k\}$ is the sequence of corners (if any) of C_I and $\{\Theta_k\}$ is the sequence of the corresponding open θ-intervals and if Θ denotes the closed set $\Theta = [0, 2\pi] - \Sigma\Theta_k$ then

$$\begin{aligned} h_I(\theta) &= h_1(\theta) - \sum_{j=2}^{n} h_j(\theta+\pi), \ \textit{if } \theta \subset \Theta \\ h_I(\theta) &= x_j \cos\theta + y_j \sin\theta, \ \textit{if } \theta \subset \Theta_j;\ z_j = x_j + iy_j. \end{aligned}$$

It may be mentioned that all the above results are true, with appropriate changes in statement, for the vectorial sum of convex bodies of any number of dimensions.

We now proceed to the length, area and curvature relations implied by (22).

Formula (19), for the length of a convex curve, together with (22), gives immediately

$$l_I \leqq l_1 - \sum_{j=2}^{n} l_j \tag{23}$$

where l_I is the length of C_I.

It is well known that the area $A(C)$ of $\Omega(C)$, where C is a convex curve with supporting function $h(\theta)$, is represented by the formula

$$A(C) = \tfrac{1}{2}\int_0^{2\pi} [h^2(\phi) - (h'(\phi))^2] d\phi. \tag{24}$$

Let $M(C, D)$, where C and D are convex curves, represent the Minkowski mixed area [10] of the regions $\Omega(C)$ and $\Omega(D)$. Then by (22), (24), and (17)

$$A(C_I) \leqq \tfrac{1}{2}\int_0^{2\pi} [(h_1(\phi) - g_E(\phi+\pi))^2 - (h'_1(\phi) - g'_E(\phi+\pi))^2] d\phi$$

or

$$A(C_I) \leqq \tfrac{1}{2}\int_0^{2\pi} [h_1{}^2(\phi) - (h'_1(\phi))^2] d\phi + \tfrac{1}{2}\int_0^{2\pi} [g_E{}^2(\phi+\pi) - (g'_E(\phi+\pi))^2] d\phi$$
$$- \int_0^{2\pi} [h_1(\phi) g_E(\phi+\pi) - h'_1(\phi) g'_E(\phi+\pi)] d\phi$$

i. e.

$$A(C_I) \leqq A(C_1) + A(D_E) - 2M(C_1, D_E). \tag{25}$$

Since, according to Minkowski,

$$M(C, D) \geqq \sqrt{A(C)A(D)},$$

(25) becomes

$$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sqrt{A(D_E)}$$

or, since [11] $\sqrt{A(D_E)} \geqq \sum_{j=2}^{n} \sqrt{A(C_j)}$,

$$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

If (25) is compared with the corresponding formula of Haviland,

$$A(C_E) = A(C_1) + A(D_E) + 2M(C_1, D_E),$$

one finds

$$A = A(C_E) - A(C_I) \geqq 4M(C_1, D_E) = 4\sum_{j=2}^{n} M(C_1, C_j),$$

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[10] Cf. e. g., W. Blaschke, *Kreis und Kugel* (Leipzig, 1916), pp. 106-107. or T. Bonnesen, *Les Problèmes des Isoperimètres* (Paris, 1929), Ch. V.

[11] Cf. E. K. Haviland, *loc. cit.*, p. 334.

where A is the area of the ring-shaped region (2). Hence

$$A \geqq 4\sqrt{A(C_1)} \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

We summarize the above results on area and length relations in

THEOREM IV. *Let l_j be the length of C_j and $A(C_j)$ the area of $\Omega(C_j)$. Let l_I be the length of C_I and $A(C_I)$ the area of $\Omega(C_I)$. Let A be the area of $C_1 (+) C_2 (+) \cdots (+) C_n$. Then, if C_I exists,*

(i) $$l_I \leqq l_1 - \sum_{j=2}^{n} l_j$$

and

(ii) $$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

Whether C_I exists or not

(iii) $$A \geqq 4 \sum_{j=2}^{n} M(C_1, C_j),$$

where $M(C_1, C_j)$ is the mixed area of $\Omega(C_1)$ and $\Omega(C_j)$, and

(iv) $$A \geqq 4\sqrt{A(C_1)} \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

In (i) *and* (iii) *the equality sign holds if C_I has no corners.*

Suppose, now, that the supporting functions $h_j(\theta)$ have continuous second derivatives, and that the radius of curvature $\rho_j(\theta)$ of C_j is defined, so that

(26) $$\rho_j(\theta) = h_j(\theta) - h_j''(\theta).$$

Suppose that the point in the direction θ_1 on C_I is not a corner or a cluster point of corners, so that

$$h_I(\theta) = h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi), \text{ if } \theta_1 - \epsilon \leqq \theta \leqq \theta_1 + \epsilon$$

for some $\epsilon > 0$. Then, by (26),

$$\rho_I(\theta_1) = \rho_1(\theta_1) - \sum_{j=2}^{n} \rho_j(\theta_1 + \pi)$$

where $\rho_I(\theta_1)$ is the radius of curvature of C_I at the point in the direction θ_1. In particular, if C_I has no corners then $\rho_I(\theta) \geqq \sum_{j=2}^{n} \rho_j(\theta + \pi)$ for all θ where

the equality sign holds at most for discrete θ-values. It is easy to see, in view of Theorem II, that the converse of this last statement is also true. Thus we have

THEOREM V. *If $h_j(\theta)$ has a continuous second derivative for $j = 1, 2, \cdots, n$, then a necessary and sufficient condition that C_I exist and have no corners is that $\rho_1(\theta) \geqq \sum_{j=2}^{n} \rho_j(\theta + \pi)$ for all θ where the equality sign holds at most for discrete θ-values. In general, if the point in the direction θ_1 on C_I is not a corner or a cluster point of corners of C_I then the radius of curvature $\rho_I(\theta)$ of C_I is defined for $\theta = \theta_1$, and*

$$\rho_I(\theta_1) = \rho_1(\theta_1) - \sum_{j=2}^{n} \rho_j(\theta_1 + \pi).$$

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THE IDEMPOTENT AND NILPOTENT ELEMENTS OF A MATRIX.

By John Williamson.

Introduction. Associated with each latent root or characteristic number of a square matrix A are two matrices, the principal idempotent and nilpotent elements. These two matrices are polynomials in A and are uniquely determined by certain simple conditions.[1] In the following pages we show how analogous results hold, when all operations are restricted to the field K, in which the elements of A lie. Since, as a rule, the characteristic numbers of A do not lie in the field K, we replace them, or more exactly, their corresponding linear factors in the characteristic equation of A, by the irreducible factors $p_i(x)$ of the characteristic equation of A. We prove that, associated with each *characteristic divisor* $p_i(x)$, there are two matrices which are polynomials in A with coefficients in K and which may justifiably be called the principal idempotent and nilpotent elements of A associated with $p_i(x)$. In case $p_i(x) = x - x_i$ is linear, these two matrices are the principal idempotent and nilpotent elements associated with the characteristic number x_i. These idempotent and nilpotent elements assume a very simple form, when the matrix A is reduced to a suitable canonical form. This canonical form is analogous to the classical canonical form of a matrix.[2] In the final section we use this canonical form to deduce theorems on matrices with elements in a field K from known theorems on matrices with elements in an algebraically closed field.

1. *Preliminary lemmas.* Let K be a commutative field of characteristic zero and let $p(x)$ be a polynomial in the ring $K[x]$ of degree ν, with leading coefficient unity, and irreducible in $K[x]$.

Lemma 1. *For every positive integer m there exists a unique polynomial $g(x)$ in $K[x]$ of degree less than $m\nu$ and such that*

$$(x - g(x))^m \equiv 0 \mod [p(x)]^m, \tag{1}$$

$$(x - g(x))^i \not\equiv 0 \mod [p(x)]^m, \quad 0 \leqq i < m, \tag{2}$$

and

$$p[g(x)] \equiv 0 \mod [p(x)]^m. \tag{3}$$

[1] J. H. Wedderburn, "Lectures on matrices," *American Mathematical Colloquium Publications* (1934), pp. 29 and 30.

[2] Wedderburn, *op. cit.*, p. 123.

We shall prove this lemma by induction on m. If $g(x)$ satisfies (1),

$$(4)\qquad g(x) \equiv x \bmod p(x),$$

and, since the only polynomial of degree less than ν satisfying (4) is x itself, our lemma is true, when $m = 1$, with $g(x) = x$. Since $p(x)$ is irreducible, $p(x)$ and its derivative $p'(x)$ are relatively prime. Accordingly there exist two uniquely determined polynomials in $K[x]$, $h(x)$ of degree less than ν and $r(x)$ of degree less than $\nu - 1$, such that

$$(5)\qquad h(x)p'(x) + r(x)p(x) = 1.$$

If,

$$(6)\qquad g_m(x) = x + h_1 p + h_2 p^2 + \cdots + h_{m-1}p^{m-1}, \quad m \geqq 2,$$

where $p = p(x)$ and each $h_i = h_i(x)$ is a polynomial in $K[x]$ of degree less than ν, then $g_m(x)$ is the most general polynomial in $K[x]$ of degree less than $m\nu$, which satisfies (4). Further,

$$(7)\qquad \begin{aligned} p(g_m) = p + p'[h_1 p + \cdots + h_{m-1}p^{m-1}] \\ + \cdots + p^{(\nu)}/\nu!\,[h_1 p + h_2 p^2 + \cdots + h_{m-1}p^{m-1}]^{\nu}. \end{aligned}$$

When the right hand of (7) is expanded in powers of p, it is apparent that h_{m-1} occurs only in terms containing a factor p^s where $s \geqq m - 1$. Hence, if we consider the value of $p(g_m)$ modulo p^{m-1}, all terms involving h_{m-1} disappear. Accordingly,

$$(8)\qquad p(g_m) \equiv p(g_{m-1}) \bmod p^{m-1},$$

where $g_{m-1}(x)$ is obtained from $g_m(x)$ in (6) by putting $h_{m-1} = 0$. It is therefore an immediate consequence of (8) that, if

$$p(g_m) \equiv 0 \bmod p^m,$$
$$p(g_{m-1}) \equiv 0 \bmod p^{m-1}.$$

Let us now make the induction assumption, that $g_{m-1}(x)$ is the unique polynomial of Lemma 1, satisfying (1), (2) and (3), when m is replaced by $m - 1$. Hence

$$(9)\qquad p(g_{m-1}) \equiv k_{m-1}p^{m-1} \bmod p^m,$$

where k_{m-1} is a uniquely determined polynomial of degree less than ν. From (7) and (8) we see that

$$\begin{aligned} p(g_m) &\equiv p(g_{m-1}) + p'h_{m-1}p^{m-1} \bmod p^m, \\ &\equiv (k_{m-1} + p'h_{m-1})p^{m-1} \bmod p^m \text{ by } (9). \end{aligned}$$

Therefore,

$$p(g_m) \equiv 0 \bmod p^m,$$

if and only if,

$$k_{m-1} + p'h_{m-1} \equiv 0 \bmod p. \tag{10}$$

But, it follows from (5) that

$$k_{m-1} + p'h_{m-1} \equiv (hk_{m-1} + k_{m-1})p' \bmod p.$$

Since p' is relatively prime to p, (10) is true, if and only if,

$$h_{m-1} + hk_{m-1} \equiv 0 \bmod p.$$

This last congruence determines k_{m-1} and therefore $g_m(x)$ uniquely. Since, when $m = 2$, $k_{m-1} = 1$, $h_1 = -h \neq 0$. Hence $g_m(x)$ satisfies (2) and $g(x) = g_m(x)$ is the unique polynomial of Lemma 1 satisfying (1), (2) and (3).

If $\nu = 1$, so that $p(x) = x - a$, $g_m(x)$ is the same for all values of $m \geqq 2$. In fact $g_m(x) = a$; for $a = x - (x - a)$ and $a - a = 0 \equiv 0 \bmod (x-a)^m$ for all positive integral values of m. This however is not true in general.

Let P be a square matrix of order ν with elements in K, whose characteristic polynomial is the irreducible polynomial $p(x)$. For definiteness we may take P to be the companion matrix of $p(x)$.[3] There is a one to one correspondence between the matrices $f(P)$, where $f(x)$ is a polynomial of $K[x]$, and the polynomials $f(\theta)$, where θ is a zero of $p(x)$. Since the correspondence is preserved under addition and multiplication it is apparent that the totality of matrices $f(P)$ forms a field simply isomorphic with the algebraic extension field $K(\theta)$. Consequently the minimal or reduced characteristic polynomial of $f(P)$ is an irreducible polynomial $q(x)$ of degree μ where $\nu/\mu = \sigma$, an integer.[4] Accordingly the characteristic polynomial of $f(P)$ is $[q(x)]^\sigma$ and the elementary factors of $f(P)$ are $q(x)$ counted σ times.[5] If Q is the companion matrix of $q(x)$, $f(P)$ is therefore similar in K to the *diagonal block matrix*

$$Q_\sigma = [Q, Q, \cdots, Q],$$

consisting of the matrix Q repeated σ times in the leading diagonal; that is, there exists a non-singular matrix C with elements in K, such that

$$Cf(P)C^{-1} = Q_\sigma. \tag{11}$$

Let E_i denote the unit matrix of order i and U_i the auxiliary unit matrix of order i, that is the matrix all of whose elements are zero except those in the

[3] C. C. MacDuffee, *The Theory of Matrices*, Berlin, 1933, p. 20.

[4] B. L. Van der Waerden, *Moderne Algebra*, vol. 1, p. 98.

[5] The powers of the distinct irreducible factors of the invariant factors of $A - xE$ are called the elementary factors of A.

diagonal to the right of the leading one, each of which is unity. Then the matrix W of order k

$$(12)\qquad W=\begin{bmatrix} f(P) & E_\nu & 0 & \cdots & 0 \\ 0 & f(P) & E_\nu & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & E_\nu \\ 0 & 0 & 0 & \cdots & f(P) \end{bmatrix}$$

may be written in the convenient form

$$(13)\qquad W=f(P)\cdot E_k+E_\nu\cdot U_k.$$

It now follows from (11) that

$$C\cdot E_kW(C\cdot E_k)^{-1}=Q_\sigma\cdot E_k+E_\nu\cdot U_k.$$

By a simple rearrangement of the rows and the same rearrangement of the columns, this last matrix may be reduced to a diagonal block matrix of σ blocks each block being the matrix $Q\cdot E_k+E_\mu\cdot U_k$. Since the minimal equation of $Q\cdot E_k+E_\mu\cdot U_k$ is $[q(x)]^k=0$,[6] we have proved;

LEMMA 2. *The elementary factors of the matrix W defined by* (12) *are $[q(x)]^k$ repeated σ times, where $q(x)$ is the minimal polynomial of $f(P)$ and is of degree $\mu=\nu/\sigma$.*

2. *Matrices with a single characteristic divisor.* Let A be a square matrix with elements in K, whose minimal equation is $\phi(x)=[p(x)]^m=0$, where $p(x)$ is the irreducible polynomial of the previous section, and let $g(x)$ be the unique polynomial of Lemma 1 satisfying (1), (2) and (3). If $g(x)=y$, as a consequence of (1) and (2),

$$(14)\qquad (x-y)^m\equiv 0,\quad (x-y)^i\not\equiv 0 \bmod [p(x)]^m,\quad 0\leqq i<m.$$

Hence, if $f(x)$ is any polynomial of $K[x]$,

$$(15)\qquad f(x)\equiv f(y)+f'(y)(x-y)+\cdots+1/(m-1)!\,f^{(m-1)}(y)(x-y)^{m-1} \bmod [p(x)]^m.$$

If we substitute the matrix A for the indeterminate x, we deduce from (3) that, if $g(A)=A_1$,

$$(16)\qquad p(A_1)=0$$

and, from (14), that the matrix $\eta_1=A-A_1$ satisfies

$$\eta_1^m=0,\quad \eta_1^i\neq 0,\quad 0\leqq i<m.$$

[6] Wedderburn, *op. cit.*, p. 124.

Further, from (15), we have

(17) $f(A) = f(A_1) + f'(A_1)\eta_1 + \cdots + 1/(m-1)!\, f^{(m-1)}(A_1)\eta_1^{m-1}$.

In (17) the polynomials $f(A_1)$, $f'(A_1)$ etc. may all be reduced modulo $p(A_1)$ by virtue of (16).

We now prove the theorem.

THEOREM I. *If $q(x) = 0$ is the minimal equation of $f(A_1)$,*

$$\textit{and } f'(A_1) = f''(A_1) = \cdots = f^{(s-1)}(A_1) = 0, \quad f^{(s)}(A_1) \neq 0,$$

then the minimal equation of $f(A)$ is $[q(x)]^\kappa = 0$, where κ is a positive integer uniquely defined by the inequalities,

$$(18) \qquad (\kappa - 1)s < m \leqq \kappa s.$$

Proof. Since $q[f(A_1)] = 0$, $q[f(A)]^\kappa = 0$, while, since $f^{(s)}(A_1) \neq 0$, $q[f(A)]^{\kappa-1} \neq 0$. But $q(x)$ is irreducible [7] and so the minimal equation of $f(A)$ is $[q(x)]^\kappa = 0$.

Moreover

$$q[f(A)] \equiv q[f(A_1)] + q'[f(A_1)]\, \frac{f^{(s)}(A_1)}{s!}\, \eta_1^s \mod \eta_1^{s+1}$$

and

$q'[f(A_1)]\, \dfrac{f^{(s)}(A_1)}{s!} \neq 0$. Hence, as a consequence of the definition of η_1,

$$(19) \qquad q[f(x)] \equiv 0 \mod [p(x)]^s, \quad q[f(x)] \not\equiv 0 \mod [p(x)]^{s+1}.$$

We have therefore the corollary. *The integer s may be determined by the congruences* (19).

3. *Matrices with more than one characteristic divisor.* Let A be a square matrix with elements in K, whose reduced characteristic equation is

$$\phi(x) = \prod_{i=1}^{t} [p_i(x)]^{m_i} = 0$$

where $p_i(x)$ $(i = 1, 2, \cdots, t)$, are t distinct irreducible polynomials in $K[x]$ of degrees ν_i respectively each with leading coefficient unity. The degree of $\phi(x)$ is accordingly given by the equation,

$$\nu = \sum_{i=1}^{t} \nu_i m_i.$$

If

$$h_i(x) = \phi(x)/[p_i(x)]^{m_i},$$

[7] The totality of matrices $f(A_1)$ is simply isomorphic to the field $K(\theta)$.

$h_i(x)$ and $[p_i(x)]^{m_i}$ are relatively prime, so that there exist two polynomials $M_i(x)$ and $N_i(x)$ in $K[x]$, of degrees less than $m_i\nu_i$ and $\nu - m_i\nu_i$ respectively, such that

$$M_ih_i + p_i^{m_i}N_i = 1.$$

Since, when $i \neq j$, $h_i \equiv 0 \bmod p_j^{m_j}$, it follows that

$$\sum_{i=1}^{t} M_ih_i \equiv 1 \bmod p_j^{m_j} \qquad (j = 1, 2, \cdots, t),$$

and accordingly that

$$\sum_{i=1}^{t} M_ih_i \equiv 1 \bmod \phi(x). \tag{20}$$

Since the congruence (20) is of degree less than ν in x, it must be an identity and on writing $M_ih_i = \phi_i$ we have

$$\sum_{i=1}^{t} \phi_i = 1. \tag{21}$$

It is an immediate consequence of the definition of ϕ_i that

$$\phi_i\phi_j \equiv 0 \bmod \phi(x) \qquad (i \neq j), \tag{22}$$

and of (21) and (22) that

$$\phi_i^2 \equiv \phi_i \not\equiv 0 \bmod \phi(x). \tag{23}$$

Let $g_i(x) = y_i$ be the polynomial of Lemma 1, when $p(x)$, ν and m are replaced by $p_i(x)$, ν_i and m_i respectively. Then

$$[(x - y_i)\phi_i]^{m_i} \equiv (x - y_i)^{m_i}\phi_i \equiv 0 \bmod \phi(x),$$
$$[(x - y_i)\phi_i]^{j} \not\equiv 0 \bmod \phi(x), \quad 0 \leqq j < m_i.$$

Consequently, if $f(x)$ is any polynomial of $K[x]$,

$$\begin{aligned} f(x) = f(x) \sum_{i=1}^{t} \phi_i = \sum_{i=1}^{t} f(x)\phi_i \\ \equiv \sum_{i=1}^{t} \{f(y_i)\phi_i + f'(y_i)(x - y_i)\phi_i \\ + \cdots + 1/(m_i - 1)!\, f(y_i)^{(m_i-1)}(x - y_i)^{m_i-1}\phi_i\} \bmod \phi(x), \\ \equiv \sum_{i=1}^{t} \{f(y_i\phi_i)\phi_i + f'(y_i\phi_i)(x - y_i)\phi_i \\ + \cdots + 1/(m_i - 1)!\, f(y_i\phi_i)^{(m_i-1)}(x - y_i)^{m_i-1}\phi_i\} \bmod \phi(x). \end{aligned}$$

We now replace the indeterminate x by the matrix A and write

$$\phi_i(A) = \phi_i, \quad A_i = g_i(A)\phi_i(A), \quad (A - A_i)\phi_i = \eta_i,$$

so that

(24) $\sum_{i=1}^{t}\phi_i = E$, the unit matrix, $\phi_i^2 = \phi_i \neq 0$, $\phi_i\phi_j = 0$, $i \neq j$, $\eta_i^{m_i} = 0$,

$\eta_i^j \neq 0$, $0 \leqq j < m_i$ $(i = 1, 2, \cdots, t)$.

Further, since

$$p_i(y_i) \equiv 0 \bmod p_i^{m_i},$$

$$p_i(y_i)\phi_i \equiv p_i(y_i\phi_i)\phi_i \equiv 0 \bmod \phi(x),$$

so that

(25) $$p_i(A_i)\phi_i = 0 \qquad (i = 1, 2, \cdots, t).$$

If $\phi_i = E$, that is, if $t = 1$, the minimal equation of A_i is $p_i(x) = 0$. Otherwise the minimal equation of A_i is $xp(x) = 0$, since, by (25), $A_ip_i(A_i) = 0$ and $p_i(x)$ is irreducible. If $p_i(x)$ is linear, so that $p_i(x) = x - a_i$, $A_i = A_i\phi_i = a_i\phi_i$; but as a rule it is simpler in this case to take $A_i = a_iE$.

It is apparent that the matrices ϕ_i and η_i, which satisfy (24), are respectively idempotent and nilpotent and that

$$A = A_1\phi_1 + \eta_1 + A_2\phi_2 + \eta_2 + \cdots + A_t\phi_t + \eta_t.$$

More generally any polynomial $f(A)$ may be written in the form

(26) $$f(A) = \sum_{i=1}^{t}\{f(A_i)\phi_i + f'(A_i)\eta_i + \cdots + 1/(m_i - 1)!\, f^{(m_i-1)}(A_i)\eta_i^{m_i-1}\}.$$

In (26), since η_i contains the factor ϕ_i, by virtue of (25) each polynomial $f^{(k)}(A_i)$ may be reduced modulo $p_i(A_i)$. The minimal equation of $f(A)$ may now be determined. Let $f^{(s_i)}(A_i)$ be the first of the derivatives of $f(A_i)$, which does not vanish, and let κ_i be determined by the inequalities

(27) $$(\kappa_i - 1)s_i < m_i \leqq \kappa_i s_i.$$

If $q_i(x) = 0$, is the minimal equation of $f(A_i)$, it may happen that, for two or more distinct values of i, the polynomials $q_i(x)$ coincide. If so, we write k_i for the largest of the corresponding integers κ_i, while, if $q_i(x)$ arises from only one value of i, we write $k_i = \kappa_i$. Then, *the minimal equation of* $f(A)$ *is*

(28) $$\Pi[q_i(x)]^{k_i} = 0,$$

where the product extends over all distinct polynomials $q_i(x)$. As in the corollary to Theorem 1 we see that the integers s_i in (27) may be determined from the congruences

$$q_i[f(x)] \equiv 0 \bmod p_i^{s_i}, \qquad q_i[f(x)] \not\equiv 0 \bmod p_i^{s_i+1}.$$

We shall call the matrices ϕ_i and η_i *the principal idempotent* and *nilpotent elements* of A associated with the characteristic divisor $p_i(x)$. The matrices ϕ_i, η_i and A_i are uniquely defined by (24) and (25) as is shown by the following theorem.

THEOREM 2. *If ψ_i and B_i are $2t$ matrices with elements in K all commutative with A, which satisfy the conditions,*

(i) $B_i = B_i\psi_i = \psi_i B_i$, (ii) $p_i(B_i)\psi_i = 0$,

(iii) $\sum_{i=1}^{t}\psi_i = E, \quad \psi_i^2 = \psi_i \neq 0$, (iv) $(A - B_i)\psi_i$ is nilpotent,

then $\psi_i = \phi_i$ and $B_i = A_i$ $(i = 1, 2, \cdots, t)$.

The proof of this theorem is similar to that of Wedderburn for the simpler case in which K is algebraically closed.[8] Let

$$\theta_{ij} = \phi_i\psi_j \qquad (i, j = 1, 2, \cdots, t).$$

Then, since ϕ_i is a polynomial in A and ψ_j is commutative with A,

$$\theta_{ij} = \psi_j\phi_i.$$

Further, the matrices

$$\eta_i = (A - A_i)\phi_i \quad \text{and} \quad \xi_j = (A - B_j)\psi_j.$$

are both nilpotent and are commutative. Since

$$\begin{aligned} A\theta_{ij} &= [A_i + (A - A_i)]\phi_i\psi_j = A_i\theta_{ij} + \eta_i\psi_j, \\ &= [B_j + (A - B_j)]\psi_j\phi_i = B_j\theta_{ij} + \xi_j\phi_i, \end{aligned}$$

$$(A_i - B_j)\theta_{ij} = \xi_j\phi_i - \eta_i\psi_j. \tag{29}$$

Since ξ_j and η_i are both nilpotent and all matrices on the right of (29) are commutative, $(A_i - B_j)\theta_{ij}$ is nilpotent. Consequently $[p_i(A_i) - p_i(B_j)]\theta_{ij}$ is nilpotent and by (25) $p_i(B_j)\theta_{ij}$ is nilpotent, so that for some integer k,

$$[p_i(B_j)]^k\theta_{ij} = 0. \tag{30}$$

Further as a consequence of (ii)

$$p_j(B_j)\theta_{ij} = 0. \tag{31}$$

[8] Wedderburn, *op. cit.*, p. 29. If K is algebraically closed the conditions are simplified since $B_i = a_i\psi_i$ and (i) and (ii) are automatically satisfied.

Since, when $i \neq j$, $[p_i(x)]^k$ and $p_j(x)$ are relatively prime there exist two polynomials $h(x)$ and $r(x)$ in $K[x]$ such that

$$h(x)[p_i(x)]^k + r(x)p_j(x) = 1.$$

Accordingly from (30) and (31) we have

$$[h(B_j)[p_i(B_j)]^k + r(B_j)p_j(B_j)]\theta_{ij} = \theta_{ij} = 0, \quad i \neq j.$$

Hence

$$\phi_i\psi_j = \psi_j\phi_i = 0, \quad i \neq j.$$

But

$$\phi_i = \phi_i E = \phi_i \sum_{j=1}^{t} \psi_j = \sum_{j=1}^{t} \phi_i\psi_j = \phi_i\psi_i = \sum_{j=1}^{t} \phi_j\psi_i = E\psi_i = \psi_i,$$

so that the first part of the theorem is proved.

As $(A - A_i)\phi_i$ and $(A - B_i)\psi_i$ are nilpotent, $(B_i - A_i)\phi_i = \rho$ is nilpotent. If $\rho \neq 0$, let $\rho^{k+1} = 0$, $\rho^k \neq 0$. Since,

$$B_i\phi_i = A_i\phi_i + \rho,$$
$$p_i(B_i)\phi_i \equiv p_i(A_i)\phi_i + p'_i(A_i)\phi_i\rho \mod \rho^2,$$

and by (ii) and (25)

(32) $$p'_i(A_i)\phi_i\rho \equiv 0 \mod \rho^2.$$

If $r_i(x)$ is the polynomial $r(x)$ of (5), when $p(x)$ is replaced by $p_i(x)$,

$$r_i(A_i)p'_i(A_i)\phi_i = \phi_i,$$

so that by (32)

$$\phi_i\rho = \rho \equiv 0 \mod \rho^2.$$

Therefore $\rho^k \equiv 0 \mod \rho^{k+1}$ and hence $\rho^k = 0$, contrary to hypothesis. Accordingly $\rho = 0$, so that

$$A_i\phi_i = B_i\phi_i = B_i\psi_i$$

and our theorem is proved.

4. *The canonical form.* The principal idempotent and nilpotent elements assume a very simple form, if the matrix A is taken in the canonical form described below.[9] Let the elementary factors of $A - xE$ be

(33) $$[p_i(x)]^{e_{ij}} \quad (i = 1, 2, \cdots, t;\ j = 1, 2, \cdots, r_i;\ e_{i1} \geqq e_{i2} \geqq \cdots \geqq e_{ir_i}).$$

Further, let P_i denote the companion matrix of $p_i(x)$ and E_{ij} and U_{ij} the

[9] Wedderburn, *op. cit.*, p. 124.

unit matrix and the auxiliary unit matrix of order e_{ij}. Then with the notation of (12) and (13) the matrix

$$N_{ij} = P_i \cdot E_{ij} + E_{v_i} \cdot U_{ij} \tag{34}$$

has the single elementary factor $[p_i(x)]^{e_{ij}}$; the diagonal block matrix

$$M_i = [N_{i1}, N_{i2}, \cdots, N_{ir_i}]$$

has the elementary factors $[p_i(x)]^{e_{ij}}$ $(j = 1, 2, \cdots, r_i)$, and the matrix

$$M = [M_1, M_2, \cdots, M_t] = [F_1, M_i, F_2] \tag{35}$$

the elementary factors (33). Accordingly M is similar to A in K and may be taken as the canonical form of A. If A is in the canonical form (35), it follows from Theorem 2 that the matrices ϕ_i, η_i and A_i are given by

$$\begin{aligned} \phi_i &= [0_1, E_{v_i} \cdot E_{i1}, E_{v_i} \cdot E_{i2}, \cdots, E_{v_i} \cdot E_{ir_i}, 0_2], \\ \eta_i &= [0_1, E_{vi} \cdot U_{i1}, E_{v_i} \cdot U_{i2}, \cdots, E_{v_i} \cdot U_{ir_i}, 0_2], \\ A_i &= [0_1, P_i \cdot E_{i1}, P_i \cdot E_{i2}, \cdots, P_i \cdot E_{ir_i}, 0_2], \end{aligned}$$

where 0_1 and 0_2 are the zero matrices of the same orders respectively as F_1 and F_2 in (35). It is apparent that η_i is nilpotent of index e_{i1} so that $e_{i1} = m_i$.

Corresponding to each particular elementary factor $[p_i(x)]^{e_{ij}}$ is the idempotent element ϕ_{ij}, obtained from ϕ_i by replacing each block except $E_{v_i} \cdot E_{ij}$ by the zero matrix, and the nilpotent element η_{ij}, obtained from η_i in the same manner. These idempotent and nilpotent elements reduce to the partial idempotent and nilpotent elements associated with a characteristic number a_i is case $p_1(x)$ is linear.[10]

5. *Applications of the canonical form.* We now proceed to consider the general form of a matrix X commutative with the matrix A, whose elementary factors are given by (33). If $AX = XA$, since ϕ_i is a polynomial in A,

$$\phi_i X = X \phi_i = X_i \qquad (i = 1, 2, \cdots, t), \tag{36}$$

and

$$X = XE = X \sum_{i=1}^{t} \phi_i = \sum_{i=1}^{t} X_i.$$

Further, if

$$A\phi_i X_i = X_i A \phi_i \qquad (i = 1, 2, \cdots, t), \tag{37}$$

$$\begin{aligned} AX &= A \sum_{j=1}^{t} \phi_j \sum_{i=1}^{t} X_i = \sum_{i=1}^{t} A\phi_i X_i, \\ &= \sum_{i=1}^{t} X_i A \phi_i = \sum_{i=1}^{t} X_i \phi_i A = XA. \end{aligned}$$

[10] Cf. Wedderburn, *op. cit.*, p. 42.

Hence, a necessary and sufficient condition, that X be commutative with A, is that (37) be true. If A is in the canonical form (35), it follows from (37) that X is a diagonal block matrix $[X_1, X_2, \cdots, X_t]$, where X_i is a square matrix of the same order as M_i and is commutative with M_i. Since M_i has the single characteristic divisor $p_i(x)$, it will, therefore, be sufficient to consider the case in which A has a single characteristic divisor. We take A in the canonical form

$$A = [N_1, N_2, \cdots, N_r],$$

where N_j is obtained from N_{ij} in (34) by replacing P_i by P, ν_i by ν, e_{ij} by e_j etc. Since $AX = XA$ and A_1 is a polynomial in A, $A_1X = XA_1$. If A_1 and X are considered as matrices with elements, which are matrices of order ν, A_1 is a diagonal matrix, each element being P, and accordingly each element of X is commutative with P and is therefore a polynomial in P. Hence, if θ is a zero of $p(x)$ and $A\{\theta\}$ is the matrix obtained from A by replacing P by θ and E_ν by 1, there is a one to one correspondence between the matrices with elements in K commutative with A and the matrices with elements in $K(\theta)$ commutative with $A\{\theta\}$. The form of a matrix $X\{\theta\}$ commutative with $A\{\theta\}$ is known,[11] and from it we deduce the following result. Let

$$X = (X_{ij}) \qquad (i, j = 1, 2, \cdots, r),$$

where X_i is a matrix with the same number of rows e_i as N_i and the same number of columns e_j as N_j, be a matrix with elements in K commutative with A. Then, if $e_i \geqq e_j = e$,

$$X_{ij} = \begin{pmatrix} G_{ij} \\ 0 \end{pmatrix}, \quad X_{ji} = (0\ G_{ji}), \quad \text{where } G_{ij} \text{ and } G_{ji}$$

are square matrices of order $e\nu$ and are of the form

$$h_1(P) \cdot E_e + h_2(P) \cdot U_e + h_3(P) \cdot U_e^2 + \cdots + h_e(P) \cdot U_e^{e-1},$$

where $h_i(P)$ is a polynomial in P with coefficients in K.[12]

In conclusion we determine the elementary factors of a matrix polynomial $f(A)$ from the corresponding theorem in the case that K is algebraically closed.[13] It is apparent from the canonical form (35) that the elementary

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[11] Wedderburn, *op. cit.*, p. 104.

[12] R. C. Trott, *Bulletin of the American Mathematical Society*, January, 1935, abstract no. 95, p. 42.

[13] These results were determined by N. H. McCoy, "On the rational canonical form of a function of a matrix," *American Journal of Mathematics*, vol. 57 (1935), pp. 491-502. The following shows the close connection between his results and the earlier ones.

factors of $f(A)$ are the elementary factors of the matrices $f(N_{ij})$. Hence it will be sufficient to consider the case in which A has the single elementary factor $[p(x)]^m$. We may assume then that A is in the normal form (cf. (13)).

$$A = P \cdot E_m + E_\nu \cdot U_m,$$

so that $A_1 = P \cdot E_m$. Let $f^s(P)$ be the first of the derivatives of $f(P)$ different from zero and let κ be determined by (18), and let

$$l = m - (\kappa - 1)s, \tag{38}$$

so that $1 \leqq l \leqq s$. If θ is a zero of $p(x)$ and $A\{\theta\} = \theta E_m + U_m$, $f^{(s)}(\theta)$ is the first of the derivatives of $f(\theta)$ different from zero, and therefore $f(A\{\theta\})$ has the elementary divisors $[x - f(\theta)]^\kappa$ counted l times and $[x - f(\theta)]^{\kappa-1}$ counted $s - l$ times.[14] Hence there exists a non-singular matrix $C\{\theta\}$ with elements in $K(\theta)$, such that

$$C\{\theta\}A\{\theta\}C^{-1}\{\theta\} = H\{\theta\}, \tag{39}$$

where $H\{\theta\}$ is a diagonal block matrix, consisting of l blocks $f(\theta)E_\kappa + U_\kappa$ and $s - l$ blocks $f(\theta)E_{\kappa-1} + U_{\kappa-1}$. If in (39) θ is replaced by the matrix P,

$$CAC^{-1} = H,\text{[15]}$$

where H consists of l blocks $f(P) \cdot E_\kappa + E_\nu \cdot U_\kappa$ and $s - l$ blocks

$$f(P) \cdot E_{\kappa-1} + E_\nu \cdot U_{\kappa-1}.$$

But, with the notation of Lemma 2, the elementary factors of $f(P) \cdot E_\kappa + E_\nu \cdot U_\kappa$ are $[q(x)]^\kappa$ counted σ times and those of $f(P) \cdot E_{\kappa-1} + E_\nu \cdot U_{\kappa-1}$ are $[q(x)]^{\kappa-1}$ counted σ times. Hence, *if $q(x)$, of degree μ, is the minimal polynomial of $f(P)$, i. e. of $f(A_1)$, the elementary factors of $f(A)$ are $[q(x)]^\kappa$ counted σl times and $[q(x)]^{\kappa-1}$ counted $\sigma(s - l)$ times, where $\sigma = \nu/\mu$.*

Since the integer s in (38) may be defined by the congruences (19), this last result is McCoy's Theorem 1.

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[14] D. E. Rutherford, "On the canonical form of a rational integral function of a matrix," *Proceedings of the Edinburgh Mathematical Society*, ser. 2, vol. 3 (1932), pp. 135-143; C. C. MacDuffee, "On a fundamental theorem in matric theory," *American Journal of Mathematics*, vol. 58 (1936), pp. 504-506.

[15] Since $C\{\theta\}$ is non-singular, C is non-singular. See J. Williamson, "Latent roots of a matrix of special type," *Bulletin of the American Mathematical Society*, vol. 37 (1931), p. 587, Theorem 1.

THE ARITHMETICAL FUNCTION $M(n, f, g)$ AND ITS ASSOCIATES CONNECTED WITH ELLIPTIC POWER SERIES.

By E. T. Bell.

1. *Introduction.* Let $M(n, f, g)$ denote the number of those representations of the integer n as a sum of f squares, precisely g of which are odd and occupy the first g places in the representations, both the arrangement of the squares and the signs of their square roots being relevant in enumerating the representations; also let $N(n, f, g)$ denote the similarly defined function in which the square roots of the odd squares are positive. Then

$$(1.1) \qquad M(n, f, g) = 2^g N(n, f, g).$$

In a former paper [1] it was shown that the calculation of $N(n, f, g)$ is an essential step in obtaining the explicit forms of the numerical coefficients occurring in the power series for elliptic functions (in either the Jacobian or the Weierstrassian form). Here we shall give several linear recurrences by which the calculations can be performed expeditiously. Certain sets of these (§§ 9, 10, 11) are complete—no more of the kind given exist. The recurrences introduce functions of divisors, and although these may be evaluated with ease directly, we have also reduced their calculation to linear recurrences. Incidentally some curious results (§ 12) on numbers of representations are noted.

The initial values must be noticed. Refer to (2.1) for ϵ.

$$(1.2) \qquad M(s, 1, 0) = 0 \quad \text{if} \quad s \not\equiv 0 \bmod 4,$$
$$M(s, 1, 0) = 2\epsilon(s/4) \quad \text{if} \quad s \equiv 0 \bmod 4;$$

$$(1.3) \qquad M(s, g, g) = 0 \quad \text{if} \quad s \not\equiv g \bmod 8,$$
$$M(8s + 1, 1, 1) = 2\epsilon(8s + 1);$$

$$(1.4) \qquad M(s, f, g) = 0 \quad \text{if} \quad g > f; \quad \text{in all of which } s \geqq 0;$$

$$(1.5) \qquad M(0, f, 0) = 1, \quad f \geqq 0; \quad M(n, 0, 0) = 0, \quad n > 0.$$

If $R_r(n)$ denotes the total number of representations of n as a sum of r squares,

[1] *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 841-852. The functions M, N are also useful in the theory of numbers; see R. D. James, *American Journal of Mathematics*, vol. 58 (1936), pp. 536-544, where further references are given.

(1.6) $M(4n, r, 0) = R_r(n)$.

2. *Arithmetical functions.* For convenient reference we collect here the definitions of the arithmetical functions occurring in the sequel.

In the following definitions n, m, α, r, a, b denote integers; $n > 0$; $\alpha \geqq 0$; $m > 0$ is odd; $n = 2^\alpha m$; $r, a, b \geqq 0$.

(2.1) $\epsilon(n) = 1$ or 0 according as n is or is not a square; $\epsilon(0) = \frac{1}{2}$.

(2.2) $(-1|m) \equiv (-1)^{(m-1)/2}$.

(2.3) $\zeta_r(n) =$ the sum of the r-th powers of all the divisors of n.

(2.4) $\zeta'_r(n) \equiv \zeta_r(m)$.

(2.5) $\xi_r(n) =$ the excess of the sum of the r-th powers of all those divisors of n that are of the form $4k+1$ over the like sum for the divisors of the form $4k+3$; $\xi(n) \equiv \xi_0(n)$.

(2.6) $\xi'_r(n) =$ the excess of the sum of the r-th powers of all those divisors of n whose conjugates are of the form $4k+1$ over the like sum in which the conjugates are of the form $4k+3$. (If $n = d\delta$, d, δ integers > 0, d, δ are called conjugate divisors of n.)

(2.7) $\xi''_r(n) \equiv \xi_r(n) - \xi'_r(n)$.

(2.8) $\alpha_r(n) \equiv n^r \zeta'_{-r}(n)$, $=$ the sum of the r-th powers of all those divisors of n whose conjugates are odd.

(2.9) $\lambda_r(n) \equiv [1 + 2(-1)^n]\zeta'_r(n)$.

(2.10) $\beta_r(n) \equiv 4\xi'_r(n) - \xi_r(n)$.

(2.11) $\zeta''_r(n) =$ the sum of the r-th powers of all the even divisors of n.

(2.12) $\rho_r(n) \equiv \zeta'_r(n) - \zeta''_r(n)$.

(2.13) $\omega(n) \equiv 2^\alpha \zeta_1(m)$.

(2.14) $\theta_{a,b}(n)$

$$\equiv [3a + b + (-1)^n(3a - b)]\zeta'_1(n) - (a+b)[1 + (-1)^n]\omega(n).$$

(2.15) $\gamma(n) \equiv 2\zeta'_1(n) - \zeta_1(n) = (3 - 2^{\alpha+1})\zeta_1(m)$.

As immediate consequences of the definitions we have

(2.16) $\xi_r(n) = \xi_r(m)$, $\xi'_r(m) = (-1|m)\xi_r(m)$;
$\xi'_r(n) = (-1|m)2^{\alpha r}\xi_r(m)$, $\xi''_r(n) = [1 - (-1|m)2^{\alpha r}]\xi_r(m)$;
$\beta_r(n) = [2^{\alpha r+2}(-1|m) - 1]\xi_r(m)$;
$\xi(4n-1) = 0$; $\omega(2n) = 2\omega(n)$.

(2.17) $\theta_{a,b}(2n) = 6a\zeta'_1(n) - 4(a+b)\omega(n)$,
$\theta_{a,b}(m) = 2b\zeta_1(m)$.

3. *Connection of M, N with elliptic power series.* By (1.1) it is suffi-

cient to show the connection for either M or N. For comparison with the paper cited [2] we choose N. As before, m is odd.

$$(3.1)\qquad \operatorname{cn} x = 1 + \sum_{s=1}^{\infty} (-1)^s Q_s(k^2)[x^{2s}/(2s)!], \quad Q_s(k^2) \equiv \sum_{r=0}^{s-1} q_r(s) k^{2r};$$

$$\sum_{r=0}^{(m-1)/2} 2^{4r} N(2m, 4s+2, 4r+2) q_r(s) = \xi_{2s}(m).$$

$$(3.2)\qquad \operatorname{sn} x = \sum_{s=0}^{\infty} \frac{(-1)^s P_s(k^2)}{(2s+1)!} x^{2s+1}, \quad P_s(k^2) \equiv \sum_{r=0}^{s} p_r(s) k^{2r};$$

$$\sum_{r=0}^{s} 2^{4r} p_r(s) N(2m, 4s+4, 4r+2) = \zeta_{2s+1}(m).$$

$$(3.3)\qquad \operatorname{dn} x = 1 + \sum_{s=1}^{\infty} (-1)^s R^s(k^2)[x^{2s}/(2s)!], \quad R_s(k^2) \equiv \sum_{j=0}^{s-1} r_j(s) k^{2j+2};$$

$$\sum_{j=0}^{n-1} 2^{4j} N(4n, 4s+2, 4j+4) r_j(s) = 2^{2s-2} \xi_{2s}(n).$$

There are similar results for $x/\operatorname{sn} x$, $x^2/\operatorname{sn}^2 x$, $\wp(x)$, or any elliptic function or quotient of theta functions. From the recurrences for the $q_r(s)$, $p_r(s)$, $r_j(s)$ the explicit forms of the $Q_s(k^2)$, $P_s(k^2)$, $R_s(k^2)$ are calculated in terms of functions N, ξ_{2s}, ζ_{2s+1}. Thus (see the paper cited)

$$q_0(s) = 1,\ 2^4 q_1(s) = 3^{2s} - 8s - 1,$$
$$2^8 q_2(s) = 5^{2s} - 8(s-1)3^{2s} + 32s^2 - 48s - 9,$$

and so on. The general form of $q_r(s)$ (also of the other coefficients) is readily inferred (*loc. cit.*) from the recurrences involving N. For the first few coefficients $q_r(s)$, etc., the necessary N's are easily calculated directly; for a systematic evaluation this more or less tentative method is impracticable.

It may be noted that an interesting sequence of successive approximations to the values of elliptic functions is obtained by retaining only powers of k^2 not exceeding the t-th, for $t = 0, 1, 2, \cdots$. The series so truncated at the t-th power of k^2 can be summed in finite form.

4. *Generating functions.* In the usual notation for the elliptic theta functions we have (4.1)–(4.3), in which sums and products refer to all integers $n > 0$, to all integers $\nu \gtreqless 0$, to all odd integers $m > 0$, or to all odd integers $\mu \gtrless 0$ respectively.

$$(4.1)\qquad Q_r \equiv Q_r(q):$$
$$Q_0 = \Pi(1 - q^{2n}), \qquad Q_1 = \Pi(1 + q^{2n}),$$
$$Q_2 = \Pi(1 + q^m), \qquad Q_3 = \Pi(1 - q^m).$$

[2] The following misprints occur: p. 842 (1), for x^2 read x^{2s}; p. 843, last line, for 2^{2s} read 3^{2s}.

$$(4.2)\qquad \vartheta_r \equiv \vartheta_r(q):$$
$$\vartheta_0 = 1 + 2\Sigma(-1)^n q^{n^2} = \Sigma(-1)^\nu q^{\nu^2};$$
$$\vartheta'_1 = 2\Sigma(-1|m) m q^{m^2/4} = \Sigma(-1|\mu)\mu q^{\mu^2/4};$$
$$\vartheta_2 = 2\Sigma q^{m^2/4} = \Sigma q^{\mu^2/4};$$
$$\vartheta_3 = 1 + 2\Sigma q^{n^2} = \Sigma q^{\nu^2}.$$

$$(4.3)\qquad Q_1Q_2Q_3 = 1, \qquad \vartheta'_1 = \vartheta_0\vartheta_2\vartheta_3;$$
$$\vartheta_0 = Q_0Q_3^2, \qquad \vartheta'_1 = 2q^{1/4}Q_0^3,$$
$$\vartheta_2 = 2q^{1/4}Q_0Q_1^2, \qquad \vartheta_3 = Q_0Q_2^2.$$

The sum of a odd squares is of the form $8s + a$; the sum of b even squares is of the form $4s$. Thus, from the definition (§ 1) of M,

$$(4.4)\qquad \vartheta_2^a(q^4)\vartheta_3^b(q^4) = \sum_{s=0}^{\infty} q^{4s+a} M(4s + a, a + b, a),$$
$$(a, b \text{ integers} \geqq 0).$$

Hence, from (4.3),

$$(4.5)\qquad 2^a Q_0^{a+b} Q_1^{2a} Q_2^{2b} = \sum_{s=0}^{\infty} q^s M(4s + a, a + b, a).$$

Define the operation Δ by

$$(4.6)\qquad \Delta F(q) = q(d/dq)\log F(q).$$

Then $\Delta[F(q)\cdots G(q)] = \Delta F(q) + \cdots + \Delta G(q)$. A short calculation gives (4.7)–(4.9), in which Σ refers to all integers $n > 0$, (see the definitions in § 2).

$$(4.7)\qquad \Delta Q_0 = -2\Sigma q^{2n}\zeta_1(n), \qquad \Delta Q_1 = 2\Sigma q^{2n}\zeta'_1(n),$$
$$\Delta Q_2 = -\Sigma q^n(-1)^n\zeta'_1(n), \qquad \Delta Q_3 = -\Sigma q^n\zeta'_1(n);$$

hence, by (4.6), (4.3),

$$(4.8)\qquad \Delta(Q_0^{a+b}Q_1^{2a}Q_2^{2b}) = \Sigma q^n\theta_{a,b}(n);$$

$$(4.9)\qquad \Delta\vartheta_0 = -2\Sigma q^n\omega(n), \qquad 4\Delta\vartheta'_1 = 1 - 24\Sigma q^{2n}\zeta_1(n),$$
$$4\Delta\vartheta_2 = 1 + 8\Sigma q^{2n}\gamma(n), \qquad \Delta\vartheta_3 = -2\Sigma q^n(-1)^n\omega(n).$$

The next are equivalent to the classical theorems on numbers of representations as sums of 2, 4, 6, 8 squares.[3] By (4.4) the two statements in each of (4.10)–(4.23) express the same theorem. The summations refer to all integers $n > 0$ or to all odd integers $m > 0$, and s is an integer $\geqq 0$.

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[3] All are collected in my paper, *American Journal of Mathematics*, vol. 42 (1920), pp. 168-188. They follow at once from theorems of Jacobi in the *Fundamenta Nova*.

(4.10) $\vartheta_2^2 = 4\Sigma q^{m/2}\xi(m)$, $M(8s+2, 2, 2) = 4\xi(4s+1)$.

(4.11) $\vartheta_2^4 = 16\Sigma q^m\zeta_1(m)$, $M(8s+4, 4, 4) = 16\zeta_1(2s+1)$.

(4.12) $\vartheta_2^6 = -4\Sigma q^{m/2}\xi_2''(m)$, $M(8s+6, 6, 6) = -4\xi_2''(4s+3)$.

(4.13) $\vartheta_2^8 = 256\Sigma q^{2n}\alpha_3(n)$, $M(8s+8, 8, 8) = 256\alpha_3(4s+4)$.

(4.14) $\vartheta_3^2 = 1 + 4\Sigma q^n\xi(n)$, $M(4n, 2, 0) = 4\xi(n)$.

(4.15) $\vartheta_3^4 = 1 + 8\Sigma q^n(-1)^n\lambda_1(n)$, $M(4n, 4, 0) = 8(-1)^n\lambda_1(n)$.

(4.16) $\vartheta_3^6 = 1 + 4\Sigma q^n\beta_2(n)$, $M(4n, 6, 0) = 4\beta_2(n)$.

(4.17) $\vartheta_3^8 = 1 - 16\Sigma q^n(-1)^n\rho_3(n)$, $M(4n, 8, 0) = -16(-1)^n\rho_3(n)$.

(4.18) $\vartheta_2\vartheta_3 = 2\Sigma q^{m/4}\xi(m)$, $M(4s+1, 2, 1) = 2\xi(4s+1)$.

(4.19) $\vartheta_2^2\vartheta_3^2 = 4\Sigma q^{m/2}\zeta_1(m)$, $M(4s+2, 4, 2) = 4\zeta_1(2s+1)$.

(4.20) $\vartheta_2^3\vartheta_3^3 = -\frac{1}{2}\Sigma q^{m/4}\xi_2''(m)$, $M(4s+3, 6, 3) = -\frac{1}{2}\xi_2''(4s+3)$.

(4.21) $\vartheta_2^4\vartheta_3^4 = 16\Sigma q^n\alpha_3(n)$, $M(4s+4, 8, 4) = 16\alpha_3(s+1)$.

(4.22) $\vartheta_2^2\vartheta_3^4 = 4\Sigma q^{m/2}\xi'_2(m)$, $M(4s+2, 6, 2) = 4\xi'_2(2s+1)$.

(4.23) $\vartheta_2^4\vartheta_3^2 = 16\Sigma q^n\xi'_2(n)$, $M(4s+4, 6, 4) = 16\xi'_2(s+1)$.

5. *Recurrence for $M(n, f, g)$ with f, g constant throughout.* Operating with Δ on both sides of (4.5), using (4.8), and comparing coefficients of like powers of q in the result, we get

$$(5.1)\quad nM(4n+a, a+b, a) = \sum_{s=1}^{n} \theta_{a,b}(s)M(4n+a-4s, a+b, a).$$

With the initial value $M(a, a+b, a) = 2^a$, (5.1) suffices to calculate all $M(n, f, g)$ with f, g constant throughout; necessarily $n \equiv g \bmod 4$:

$$(5.2)\quad nM(4n+g, f, g) = \sum_{s=1}^{n} \theta_{g,f-g}(s)M(4n+g-4s, f, g).$$

From (2.14), (2.17) we have

$$(5.3)\quad \begin{aligned} \theta_{g,f-g}(n) &= [2g+f+(-1)^n(4g-f)]\zeta'_1(n) - f[1+(-1)^n]\omega(n), \\ \theta_{g,f-g}(2n) &= 6g\zeta'_1(n) - 4f\omega(n), \quad \theta_{g,f-g}(m) = 2(f-g)\zeta_1(m). \end{aligned}$$

Recurrences for the associated functions ζ'_1 (by means of λ_1) and ω are given in §§ 7, 8. This type is continued in § 10.

6. *Arguments n in $N(n, f, g)$ decreasing by squares.* Multiply the left of (4.4) by the second form of $\vartheta_2(q^4)$ in (4.2), and the right by the first form; do the like with $\vartheta_3(q^4)$. Then, for $n \geqq 0$,

$$(6.1)\quad \begin{aligned} &M(4n+a+1, a+b+1, a+1) \\ &\qquad = 2\Sigma M(4n+a+1-m^2, a+b, a), \end{aligned}$$

$$(6.2)\quad \begin{aligned} &M(4n+a, a+b+1, a) \\ &\qquad = M(4n+a, a+b, a) + 2\Sigma M(4n+a-4s^2, a+b, a), \end{aligned}$$

where the summations refer to $m=1,3,5,\cdots$, and $s=1,2,3,\cdots$ respectively, continuing so long as the arguments are $\geqq 0$. Hence

$$(6.3)\quad M(4n+g,f,g)=2\Sigma M(4n+g-m^2,f-1,g-1),$$

$$(6.4)\quad M(4n+g,f,g)=M(4n+g,f-1,g)+2\Sigma M(4n+g-4s^2,f-1,g).$$

The sum of u odd squares is of the form $8t+u$ $(t\geqq 0)$; the sum of v even squares is of the form $4t$. Hence

$$(6.41)\quad \vartheta_2{}^u(q^4)=\sum_{t=0}^{\infty} q^{8t+u}M(8t+u,u,u),\quad \vartheta_3{}^v=\sum_{t=0}^{\infty} q^{4t}M(4t,v,0).$$

Operating on these with Δ and proceeding as in § 5 we get

$$(6.5)\quad \Sigma[8n-(u+1)(m^2-1)]M(8n+u+1-m^2,u,u)=0,$$

$$(6.6)\quad nM(4n,v,0)+2\Sigma[n-(v+1)s^2]M(4n-4s^2,v,0)=2v\epsilon(n)n,$$

the summations with respect to m,s being as in (6.3), (6.4), with $s^2<n$ in (6.6). We shall generalize (6.5), (6.6) in § 10.

7. *Recurrences for the functions in* § 2. Applying (4.10)–(4.13) to (6.5) we find the following, in which $n>0$, and the summations refer to $m=1,3,5,\cdots$.

$$(7.1)\quad \Sigma[8n-3(m^2-1)]\xi\ (4n+1-\tfrac{1}{2}(m^2-1))=0,$$
$$(7.2)\quad \Sigma[8n-5(m^2-1)]\zeta_1\ (2n+1-\tfrac{1}{4}(m^2-1))=0,$$
$$(7.3)\quad \Sigma[8n-7(m^2-1)]\xi_2''(4n+3-\tfrac{1}{2}(m^2-1))=0,$$
$$(7.4)\quad \Sigma[8n-9(m^2-1)]\alpha_3\ (4n+4-\tfrac{1}{2}(m^2-1))=0.$$

In the same way from (4.14)–(4.17) and (6.6) we get the following recurrences with $n>0$, and Σ referring to $t=1,2,3,\cdots$, with $t^2<n$.

$$(7.5)\quad n\xi(n)+2\Sigma(n-3t^2)\xi(n-t^2)=\epsilon(n)n,$$
$$(7.6)\quad n\lambda_1(n)+2\Sigma(-1)^t(n-5t^2)\lambda_1(n-t^2)=(-1)^n\epsilon(n)n,$$
$$(7.7)\quad n\beta_2(n)+2\Sigma(n-7t^2)\beta_2(n-t^2)=3\epsilon(n)n,$$
$$(7.8)\quad n\rho_3(n)+2\Sigma(-1)^t(n-9t^2)\rho_3(n-t^2)=(-1)^{n-1}\epsilon(n)n.$$

8. *Continuation of* § 7. Operating with Δ on the first forms of the thetas in (4.2) and comparing with (4.9) we find

$$(8.1)\quad \omega(n)+2\Sigma(-1)^t\omega(n-t^2)=\epsilon(n)(-1)^{n-1}n;$$

$$(8.2)\quad \zeta_1(n)+\Sigma(-1)^t(2t+1)\zeta_1(n-\tfrac{1}{2}t(t+1))$$
$$=\begin{cases} 0 \ \text{ if } \ n\neq\tfrac{1}{2}h(h+1)\quad (h>0),\\ -\tfrac{1}{6}(-1)^h h(h+1)(2h+1)\ \text{ if }\ n=\tfrac{1}{2}h(h+1);\end{cases}$$

$$(8.3) \qquad \gamma(n) + \Sigma\gamma(n - \tfrac{1}{2}t(t+1)) = \begin{cases} 0, & n \neq \tfrac{1}{2}h(h+1), \\ n, & n = \tfrac{1}{2}h(h+1). \end{cases}$$

The summations refer to $t = 1, 2, 3, \cdots$; and $n > 0$. This is continued in § 11.

9. *Recurrences with arguments n of $M(n, f, g)$ in arithmetic progression.* Applying (4.4), (6.41) to $\vartheta_2^{g+u}\vartheta_3^b = \vartheta_2^u \times \vartheta_2^g\vartheta_3^b$ we get

$$M(4n + g + u, f + u, g + u) = \Sigma M(8s + u, u, u) M(4n + g - 8s, f, g),$$

with $n \geqq 0$ and Σ referring to $s = 0, 1, \cdots, [n/2]$.

Hence, by (4.10)–(4.13), for the same n, s,

$$(9.1) \quad M(4n + g + 2, f + 2, g + 2) = 4\Sigma\xi\,(4s + 1) M(4n + g - 8s, f, g);$$
$$(9.2) \quad M(4n + g + 4, f + 4, g + 4) = 16\Sigma\zeta_1\,(2s + 1) M(4n + g - 8s, f, g);$$
$$(9.3) \quad M(4n + g + 6, f + 6, g + 6) = -4\Sigma\xi_2''(4s + 3) M(4n + g - 8s, f, g);$$
$$(9.4) \quad M(4n + g + 8, f + 8, g + 8) = 256\Sigma\alpha_3\,(4s + 4) M(4n + g - 8s, f, g).$$

Similarly, $\vartheta_2^a\vartheta_3^{b+v} = \vartheta_2^a\vartheta_3^b \times \vartheta_3^v$;

$$M(4n + g, f + 2, g) = M(4n + g, f, g) + 4\Sigma\xi(s) M(4n + g - 4s, f, g),$$

for $n \geqq 0$, $s = 1, \cdots, n$, *with the convention that a sum in which the lower limit exceeds the upper is vacuous.* Hence, from (4.14)–(4.17), for the same n, s,

$$(9.5) \quad M(4n + g, f + 2, g) = M(4n + g, f, g) + 4\Sigma\xi(s) M(4n + g - 4s, f, g);$$
$$(9.6) \quad M(4n + g, f + 4, g) = M(4n + g, f, g) + 8\Sigma(-1)^s\lambda_1(s) M(4n + g - 4s, f, g);$$
$$(9.7) \quad M(4n + g, f + 6, g) = M(4n + g, f, g) + 4\Sigma\beta_2(s) M(4n + g - 4s, f, g);$$
$$(9.8) \quad M(4n + g, f + 8, g) = M(4n + g, f, g) - 16\Sigma(-1)^s\rho_3(s) M(4n + g - 4s, f, g).$$

From $\vartheta_2^{a+c}\vartheta_3^{b+c} = \vartheta_2^a\vartheta_3^b \times \vartheta_2^c\vartheta_3^c$,

$$M(4n + g + c, f + 2c, g + c) = \Sigma M(4s + c, 2c, c) M(4n + g - 4s, f, g),$$

for $n \geqq 0$, $s = 0, \cdots, n$. Hence for the same n, s, (4.18)–(4.21) give

$$(9.9) \quad M(4n + g + 1, f + 2, g + 1) = 2\Sigma\xi\,(4s + 1) M(4n + g - 4s, f, g);$$
$$(9.10) \quad M(4n + g + 2, f + 4, g + 2) = 4\Sigma\zeta_1\,(2s + 1) M(4n + g - 4s, f, g);$$
$$(9.11) \quad 2M(4n + g + 3, f + 6, g + 3) = -\Sigma\xi_2''(4s + 3) M(4n + g - 4s, f, g);$$
$$(9.12) \quad M(4n + g + 4, f + 8, g + 4) = 16\Sigma\alpha_3\,(s + 1) M(4n + g - 4s, f, g).$$

Finally, $\vartheta_2^{a+u}\vartheta_3^{b+v} = \vartheta_2^a\vartheta_3^b \times \vartheta_2^u\vartheta_3^v$;

$$M(4n+g+u, f+u+v, g+u) = \Sigma M(4s+u, u+v, u) M(4n+g-4s, f, g),$$

for $n \geqq 0$, and $s = 0, \cdots, n$. For the same n, s, we get from (4.22), (4.23),

$$(9.13)\quad M(4n+g+2, f+6, g+2) = 4\Sigma\xi'_2(2s+1) M(4n+g-4s, f, g);$$
$$(9.14)\quad M(4n+g+4, f+6, g+4) = 16\Sigma\xi'_2(s+1) M(4n+g-4s, f, g).$$

10. *Continuation of* § 5. By (4.4),

$$\sum_{s=0}^{\infty} q^{4s+gr} M(4s+gr, fr, gr) = [\sum_{t=0}^{\infty} q^{4t+g} M(4t+g, f, g)]^r$$

is equivalent to the identity

$$\vartheta_2^{gr}\vartheta_3^{(f-g)r} = (\vartheta_2^g\vartheta_3^{f-g})^r;$$

hence, operating throughout the first with Δ, we find

$$(10.1)\quad \Sigma[n-(r+1)s] M(4s+g, f, g) M(4n+gr-4s, fr, gr) = 0,$$

valid for $n \geqq 0$, the sum referring to $s = 0, \cdots, n$. This contains as special cases (6.5), (6.6). To obtain (6.5) from (10.1) take $r = u$, $f = g = 1$, and note that s, n must then be even; to obtain (6.6) take $r = v$, $f = 1$, $g = 0$, and apply (1.2). In (10.1) take $f = g$, and apply (1.3),

$$(10.2)\quad \Sigma[n-(r+1)s] M(8s+g, g, g) M(8n+gr-8s, gr, gr) = 0.$$

For $g = 0$, (10.1) becomes, by (1.5),

$$(10.3)\quad nM(4n, fr, 0) + 4\Sigma[n-(r+1)s] M(4s, f, 0) M(4n-4s, fr, 0) = rnM(4n, f, 0),$$

in which $n \geqq 0$ and Σ refers to $s = 1, \cdots, n-1$.

From (10.2) and (4.10)-(4.13) we have the following, n and Σ being as in (10.1),

$$(10.4)\quad \Sigma[n-(r+1)s]\xi\ (4s+1) M(8n+2r-8s, 2r, 2r) = 0,$$
$$(10.5)\quad \Sigma[n-(r+1)s]\zeta_1\ (2s+1) M(8n+4r-8s, 4r, 4r) = 0,$$
$$(10.6)\quad \Sigma[n-(r+1)s]\xi_2''(4s+3) M(8n+6r-8s, 6r, 6r) = 0,$$
$$(10.7)\quad \Sigma[n-(r+1)s]\alpha_3\ (4s+4) M(8n+8r-8s, 8r, 8r) = 0.$$

Taking $f = 2, 4, 6, 8$ in (10.3) and referring to (4.14)-(4.17), we find (10.8)-(10.11), in which n, Σ are as in (10.3),

(10. 8) $nM(4n, 2r, 0) + 4\Sigma[n - (r+1)s]\xi(s)M(4n - 4s, 2r, 0) = 4rn\xi(n)$,

(10. 9) $nM(4n, 4r, 0) + 8\Sigma(-1)^s[n - (r+1)s]\lambda_1(s)M(4n - 4s, 4r, 0)$
$= 8(-1)^n rn\lambda_1(n)$,

(10. 10) $nM(4n, 6r, 0) + 4\Sigma[n - (r+1)s]\beta_2(s)M(4n - 4s, 6r, 0)$
$= 4rn\beta_2(n)$,

(10. 11) $nM(4n, 8r, 0) - 16\Sigma(-1)^s[n - (r+1)s]\rho_3(s)M(4n - 4s, 8r, 0)$
$= -16(-1)^n rn\rho_3(n)$.

From (10. 1) and (4. 18)-(4. 23),

(10. 12) $\Sigma[n - (r+1)s]\xi\ (4s+1)M(4n + r - 4s, 2r, r) = 0$,
(10. 13) $\Sigma[n - (r+1)s]\zeta_1\ (2s+1)M(4n + 2r - 4s, 4r, 2r) = 0$,
(10. 14) $\Sigma[n - (r+1)s]\xi_2''(4s+3)M(4n + 3r - 4s, 6r, 3r) = 0$,
(10. 15) $\Sigma[n - (r+1)s]\alpha_3\ (s+1)M(4n + 4r - 4s, 8r, 4r) = 0$,
(10. 16) $\Sigma[n - (r+1)s]\xi'_2\ (2s+1)M(4n + 2r - 4s, 6r, 2r) = 0$,
(10. 17) $\Sigma[n - (r+1)s]\xi'_2\ (s+1)M(4n + 4r - 4s, 6r, 4r) = 0$,

with Σ, n as in (10. 1).

11. *Continuation of* § 8. In (10. 4) take $r = 2, 3, 4$; in (10. 5), $r = 2$; in (10. 6), $r = 1$, and apply (4. 10)-(4. 13). Then, for $n \geqq 0$ and the summation referring to $s = 0, 1, \cdots, n$, we have (11. 1)-(11. 5). The obvious omitted possibilities give relations which are trivially true, and likewise for subsequent omissions.

(11. 1) $\Sigma(n - 3s)\xi\ (4s+1)\zeta_1\ (2n + 1 - 2s) = 0$,
(11. 2) $\Sigma(n - 4s)\xi\ (4s+1)\xi_2''(4n + 3 - 4s) = 0$.
(11. 3) $\Sigma(n - 5s)\xi\ (4s+1)\alpha_3\ (4n + 4 - 4s) = 0$,
(11. 4) $\Sigma(n - 3s)\zeta_1\ (2s+1)\alpha_3\ (4n + 4 - 4s) = 0$,
(11. 5) $\Sigma(n - 7s)\xi_2''(4s+3)\xi_2''(4n + 3 - 4s) = 0$.

In (10. 8) take $r = 2, 3, 4$; in (10. 9) take $r = 2$, and apply (4. 14)-(4. 17). Then, for n, Σ as in (10. 3), we have

(11. 6) $n\lambda_1(n) + 4\Sigma(-1)^s(n - 3s)\xi(s)\lambda_1(n - s) = (-1)^n\xi(n)$,
(11. 7) $n\beta_2(n) + 4\Sigma(n - 4s)\xi(s)\beta_2(n - s) = 3n\xi(n)$,
(11. 8) $n\rho_3(n) + 4\Sigma(-1)^s(n - 5s)\xi(s)\rho_3(n - s) = -(-1)^n n\xi(n)$,
(11. 9) $n\rho_3(n) + 8\Sigma(n - 3s)\lambda_1(s)\rho_3(n - s) = -n\lambda_1(n)$.

The formulas (10. 12)-(10. 17) furnish no new results of the above kind.

12. *Polynomial forms of M.* Taking $n = 0, 1, 2, \cdots$ in (5.1) and referring to (2.17) we find

$$
\begin{aligned}
(12.1) \qquad M(a, a+b, a) &= 2^a, \\
M(4+a, a+b, a) &= 2^{a+1}b, \\
M(8+a, a+b, a) &= 2^a(2b^2 - 2b + a), \\
3M(12+a, a+b, a) &= 2^{a+1}b(2b^2 - 6b + 3a + 4), \\
6M(16+a, a+b, a)& \\
= 2[4b^4 - 24b^3 &+ 4(3a+11)b^2 - 12(a+1)b + 3a(a-1)],
\end{aligned}
$$

the general nature of $M(4n + a, a + b, a)$ (proved by mathematical induction) being evident. Take $a = 0$ and refer to (1.6); then

(12.2) *The number of representations of n as a sum of b squares is expressible as a polynomial of degree n in b with rational coefficients.*

Taking $b = 0$ we get

(12.3) *The number of representations of $8n + a$ as a sum of a odd squares is expressible as 2^a times a polynomial of degree n in a with rational coefficients.*

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REPRESENTATION AS SUMS OF MULTIPLES OF GENERALIZED POLYGONAL NUMBERS.

By L. W. Griffiths.

1. *Introduction.* The problem is a generalization of that of representation as sums of multiples of squares,[1] as sums of polygonal numbers,[2] of extended polygonal numbers,[3] or of generalized polygonal numbers.[4]

In my first two papers [5] there are summaries of all of these facts on representation, and illustrations.

The generalized polygonal numbers of order $m+2$ are the values of

$$(1) \qquad g(x) = x + m(x^2 - x)/2 \quad \text{for} \quad x = 0, \pm 1, \pm 2, \cdots$$

with m a fixed positive integer. The following notations are used: n and $a_1, \cdots, a_n$ are positive integers, $g_i = g(x_i)$, and

$$\begin{aligned} & f = a_1 g_1 + \cdots + a_n g_n = (a_1, \cdots, a_n), \quad 1 \leqq a_1 \leqq \cdots \leqq a_n, \\ (2) \qquad & f_4 = (a_5, \cdots, a_n), \\ & w_k = a_1 + \cdots + a_k \ (1 \leqq k \leqq n), \quad w = w_n. \end{aligned}$$

The positive integer N is represented by the function f when there are integers $x_1, \cdots, x_n$ such that $f = N$. In section 4 of II, conditions were found that f represent the integers $0, 1, \cdots, 34m - 16$. In section 5 of II, for certain functions f satisfying these necessary conditions, there was found a positive integer M, depending only on m and f, such that f represents every integer $N \geqq M$. For other functions satisfying the necessary conditions the methods there employed gave no conclusion. For the remaining functions satisfying the necessary conditions there was no generalized Cauchy lemma on simultaneous representation, and hence the methods there employed were inapplicable.

In this paper use is made of the new facts on simultaneous representation due to Dickson [6] and, for the case $(1, 1, 2, 4)$ with b divisible by four,

[1] Dickson, *Bulletin of the American Mathematical Society*, vol. 33 (1927), pp. 63-70.

[2] Cauchy, *Œuvres*, ser. 2, vol. 6, pp. 320-353.

[3] Dickson, *American Journal of Mathematics*, vol. 50 (1928), pp. 1-48.

[4] Dickson, *Journal de Mathématiques*, ser. 9, vol. 7 (1928), Theorems 11-15.

[5] *Annals of Mathematics*, ser. 2, vol. 31 (1930), pp. 1-12, and *American Journal of Mathematics*, vol. 55 (1933), pp. 102-110. These papers will be cited as I and II.

[6] *American Journal of Mathematics*, vol. 56 (1934), pp. 512-528.

here first treated in section 7. There are five fundamental cases, and the details of proof are usually omitted in the last four cases. Each case is summarized in a theorem: *if f satisfies the necessary conditions mentioned, then there is exhibited a positive integer M, depending only on m and f, such that f represents every integer $N \geqq M$.*

2. *The functions* $(1, 1, 2, 3, \cdots)$. By (1) and (2) the positive integer N is represented by f if there exist positive integers a, b, r such that $N = r + b + m(a - b)/2$, that r is represented by f_4, and that the equations $a = x^2 + y^2 + 2z^2 + 3w^2$, $b = x + y + 2z + 3w$ have a solution in integers $\geqq 0$. Sufficient conditions for the existence of these integers x, y, z, w are[7]

$$(3) \qquad a \equiv b \pmod 2, \quad b^2 \leqq ta, \quad (t-1)a < b^2 + 2b + t, \quad b \geqq 0,$$

$$(4) \qquad 6(7a - b^2) \neq 49^s(7n + e), \quad e = 3, 5, 6.$$

In this paper there is found an integer M, depending only on m and f, such that if N is an integer $\geqq M$ then there exist integers a, b, r satisfying (3) and (4), and such that $N = r + b + m(a - b)/2$ and f_4 represents r. Hence to prove the universality of the functions f, satisfying the necessary conditions mentioned in the introduction, it remains to prove that f represents integers $34m - 16 < N < M$.

Dickson proved that (4) is satisfied if either a or b is prime to $t = 7$. The first step in my new method is to obtain conditions which are equivalent to (4) when $a \equiv b \equiv 0 \pmod 7$. Write $b = 7^iB$ with $B \not\equiv 0 \pmod 7$, and $a = 7^{2h}A$ or $7^{2h-1}A$ with $A \not\equiv 0 \pmod 7$. Then it is easily seen that $6(7a - b^2) = 49^s(7n + e)$ with $e = 3, 5, 6$ if and only if one of (5) is satisfied:

$$(5) \qquad \begin{aligned} &a = 7^{2h-1}A, \quad 1 \leqq h < i, \quad A \equiv -e \pmod 7; \quad \text{or,} \\ &a = 7^{2h-1}A, \quad 1 \leqq h = i, \quad A \equiv B^2 - e \pmod 7). \end{aligned}$$

That is, (4) is satisfied if and only if one of (6) or (7) is satisfied:

$$(6) \qquad \begin{aligned} &a = 7^{2h-1}A, \quad 1 \leqq h < i, \quad A \equiv e \pmod 7; \quad \text{or} \\ &a = 7^{2h-1}A, \quad 1 \leqq h = i, \quad A \not\equiv B^2 - e \pmod 7; \quad \text{or} \\ &a = 7^{2h-1}A, \quad 1 \leqq i < h; \end{aligned}$$

$$(7) \qquad a = 7^{2h}A.$$

The next step leading to the desired integers a, b, r is a consideration

[7] Dickson, *American Journal of Mathematics*, vol. 56, loc. cit. Here, by definition, $t = a_1 + a_2 + a_3 + a_4$.

of the integers, between 0 and $m-2-t$, not represented by f_4. Among the necessary conditions which these functions f satisfy are $w=m-2$ and $a_k \leqq w_{k-1}+1$ $(5 \leqq k \leqq n)$. Hence, defining $w'_j = a_5 + \cdots + a_{j+4}$, $w'_j = w_{j-4} - t$ $(1 \leqq j \leqq n-4)$ and $a_{j+4} \leqq w'_{j-1} + t + 1$. Hence, by applying Lemma 1 of my first paper I (valid, as proved in II, page 107), we have

LEMMA 1. *If η is an integer between 0 and $m-2-t$ then f_4 represents at least one of the integers $\eta, \eta-1, \cdots, \eta-t$.*

Next I shall prove that if $N \geqq 28$ then there exist integers a, b, r satisfying all the conditions except perhaps (3_2) and (3_3).

LEMMA 2. *Let f satisfy the necessary conditions mentioned above. Let ξ be an integer not divisible by t, and $4t \leqq \xi$. Let $N \geqq \xi$. Then there are integers a, b, r, each $\geqq 0$, such that $N = r + b + m(a-b)/2$, $a \equiv b \pmod 2$, (4) is satisfied, and $f_4 = r \leqq m-2-t$.*

For there are integers g and ρ such that $N = mg + \xi + \rho$, $g \geqq 0$, $0 \leqq \rho \leqq m-1$. If $f_4 \neq \rho < m-2-t$, then by Lemma 1 we can write $N = mg + \xi + i + (\rho - i)$ where i is one of $0, 1, \cdots, t$ and $f_4 = \rho - i$. If $m-2-t < \rho \leqq m-2$, the same conclusion holds obviously, with $\rho - i = m-2-t$. The case $\rho = m-1$ is treated later. Hence if $\rho \neq m-1$, there exist integers ζ and σ such that $N = mg + \zeta + \sigma$, $0 \leqq \sigma \leqq m-2-t$, $f_4 = \sigma$, and ζ is one of $\xi, \cdots, \xi + t$. Define $a = 2g + \zeta$, and $b = \zeta$. Then if b and a satisfy (4) the proof is finished. But if b and a satisfy one of (5), then there are exhibited integers g', ζ', σ' such that $N = mg' + \zeta' + \sigma'$, and that $b' = \zeta'$, $a' = 2g' + \zeta'$ do satisfy (4). There are two cases, according as a_5 is not, or is, divisible by t.

First, if $a_5 = 3, 4, 5, 6$, or 8, and if a_5 appears explicitly in σ, then $N = mg + \zeta + a_5 + \sigma - a_5$; and $\zeta' = \zeta + a_5$ and $g' = g$ yield a', b' which do satisfy (4), because $\zeta' \not\equiv \zeta \equiv 0 \pmod t$. Again, if $a_5 = 3, 4, 5, 6$, or 8, and if a_5 does not appear explicitly in σ, then $N = mg + \zeta - a_5 + \sigma + a_5$; and $\zeta' = \zeta - a_5$ and $g' = g$ yield a', b' which do satisfy (4). In each case $r = \sigma'$. There remains the case $a_5 = 7$, for which the proof is long and intricate. Therefore the proof for $\rho = m-1$ is next presented. Here

$$N = mg + \xi + m - 1 = mg + \zeta + (m-2-t),$$

with $\zeta = \xi + t + 1$ and $\sigma = m-2-t$. Then if a, b satisfy (5), $N = m(g+1) + \zeta - t - 2$ yields $g' = g+1$, $\zeta' = \xi - 1$, $\sigma' = 0$; so that $b' = \xi - 1$ and $a' = 2g' + b'$ satisfy (4), since $\zeta' \not\equiv \zeta \equiv 0 \pmod t$.

To complete the proof of Lemma 2 for the case $a_5 = 7$ the following Lemmas 3, 4, 5 and 6 were proved.

LEMMA 3. *Let a, b satisfy* (5_1). *Then $a + 7$, $b + 7$ satisfy* (6) *or* (7), *and hence* (4), *if and only if $h > 1$. Also $a - 7$, $b - 7$ satisfy* (6) *or* (7) *if and only if $h > 1$, or $h = 1$ and $A \not\equiv -3 \pmod 7$.*

LEMMA 4. *Let a, b satisfy* (5_1), *with $h = 1$. Then $a + 14$, $b + 14$ satisfy* (6) *or* (7) *if and only if $A \not\equiv -3 \pmod 7$. If $A \equiv -3 \pmod 7$ then each of the pairs $a - 14$, $b - 14$, $a + 21$, $b + 21$, $a - 21$, $b - 21$ satisfies* (6) *or* (7).

LEMMA 5. *Let a, b satisfy* (5_2). *Then $a + 7$, $b + 7$ satisfy* (6) *or* (7) *except in the following six cases, when they satisfy* (5); *then the pairs involving $b + 14$, $b - 14$, $b + 21$, $b + 28$ satisfy* (6), (7) *or* (5) *as indicated:*

$h = 1$, $B \equiv 1$, $A \equiv B^2 - 3$, $b + 14$ *satisfy* (6) *or* (7),
$h = 1$, $B \equiv -1$, $A \equiv B^2 - 5$, $b - 7$, $b - 14$, $b + 28$ *satisfy* (6) *or* (7); $b + 14$, $b + 21$ *satisfy* (5),
$h = 1$, $B \equiv 2$, $A \equiv B^2 - 6$, $b + 14$ *satisfy* (6) *or* (7),
$h = 1$, $B \equiv -2$, $A \equiv B^2 - 3$, $b + 14$ *satisfy* (6) *or* (7),
$h = 1$, $B \equiv 3$, $A \equiv B^2 - 6$, $b + 21$, $b - 14$ *satisfy* (6) *or* (7); $b + 14$ *satisfy* (5),
$h = 1$, $B \equiv -3$, $A \equiv B^2 - 5$, $b + 14$ *satisfy* (6) *or* (7).

LEMMA 6. *Let a, b satisfy* (5_2). *Then $a - 7$, $b - 7$ satisfy* (6) *or* (7) *except in the following eight cases, when they satisfy* (5); *then the pairs involving $b + 14$, $b - 14$, $b + 21$, $b - 21$, $b + 7$, $b - 28$ satisfy* (6), (7) *or* (5) *as indicated:*

$h = 1$, $B \equiv 1$, $A \equiv B^2 - 3$, $b + 14$, $b \pm 21$ *satisfy* (6) *or* (7); $b - 14$, (5),
$h = 1$, $B \equiv 1$, $A \equiv B^2 - 5$, $b - 14$, (6) *or* (7),
$h = 1$, $B \equiv 1$, $A \equiv B^2 - 6$, $b - 14$, (6) *or* (7),
$h = 1$, $B \equiv -1$, $A \equiv B^2 - 6$, $b - 14$, (6) *or* (7),
$h = 1$, $B \equiv 2$, $A \equiv B^2 - 5$, $b - 28$, (6) *or* (7); $b - 14$, $b - 21$, (5),
$h = 1$, $B \equiv -2$, $A \equiv B^2 - 6$, $b - 21$, (6) *or* (7); $b - 14$, (5),
$h = 1$, $B \equiv 3$, $A \equiv B^2 - 3$, $b - 14$, (6) *or* (7),
$h = 1$, $B \equiv -3$, $A \equiv B^2 - 5$, $b - 14$, $b + 14$, (6) *or* (7).

The proofs of Lemmas 3, 4, 5, 6 are omitted since they are simple but

intricate. Next note that if $a_5 = 7$ and $a_6 \not\equiv 0 \pmod 7$ or if $a_5 = a_6 = 7$ and $a_7 \not\equiv 0 \pmod 7$, then the proof applied when $a_5 \not\equiv 0 \pmod 7$ is valid, with a_5 replaced by a_J ($= a_6$ or a_7 respectively). There remain three cases with $a_5 = 7$: case I, $(1,1,2,3,7,14,\cdots)$ with $n \geqq 7$ and $(1,1,2,3,7,7,21,\cdots)$ with $n \geqq 7$ and $(1,1,2,3,7,7,14,\cdots)$ with $n \geqq 7$; case II, $(1,1,2,3,7,7,7)$; case III, $(1,1,2,3,7,7)$. Note that $(1,1,2,3,7)$ and $(1,1,2,3,7,14)$ do not satisfy the necessary conditions (stated later).

Proof for case I. First, if a, b satisfy (5_1) and if a_5 is explicitly in σ: if $h > 1$, use $b' = b + 7$, $\sigma' = \sigma - 7$; but if $h = 1$ and $A \not\equiv -3 \pmod 7$, by Lemmas 3 and 4, use $b' = b + 14$ if also a_6 is explicitly in σ, but use $b' = b - 7$ if a_6 is not explicitly in σ; if $h = 1$ and $A \equiv -3 \pmod 7$, similarly one of $b + 21$, $b \pm 14$, $b - 7$ will be satisfactory. Second, if a, b satisfy (5_1) and if a_5 is not explicitly in σ: by Lemmas 3 and 4, similarly one of $b - 7$, $b + 7$, $b - 14$, $b \pm 21$, will be satisfactory. Next, if a, b satisfy (5_2), by Lemmas 5 and 6 and a lengthy tabulation it was found that it was necessary and sufficient to permit b' to have the values $b \pm 7$, $b \pm 14$, $b \pm 21$, $b \pm 28$; here $g' = g$; also, if $b' = b + \delta$, where δ is one of these multiples of 7, then $a' = 2g' + b'$.

Proof for cases II and III, and for $(1,1,2,3)$. The details for case I were applicable except in certain vital places, where they failed. An independent direct proof, using frequently the device noted in the treatment of $\rho = m - 1$ preceding the statement of Lemma 3, yielded the results tabulated below for b'; always $a' = 2g' + b'$.

$(1,1,2,3,7,7,7)$	b;	$b+7, b-9, b-16, b+16; b-7, b+16, b+9, b-16,$
$(1,1,2,3,7,7)$	b;	$b+7, b-9, b-16;\ b-7, b+9,$
$(1,1,2,3)$	b and $b-9$.	

This completes the proof of Lemma 2. Clearly the values of b cluster around ξ. In fact, if $a_5 = 3, 4, 5, 6$, or 8, the maximum b is $\xi + (t-1) + a_5$ and the minimum b is $\xi + 1 - a_5$. For, the values of b for $\rho = m - 1$ are within this range; also when $\zeta = \xi$ or $\xi + t$ then $b = \zeta \not\equiv 0 \pmod 7$. The consecutive integers $\xi + (t-1) + a_5, \cdots, \xi + 1 - a_5$ are in number $2a_5 + t - 1$. But in Lemma 2 ξ was an arbitrary integer not divisible by t and $\xi \geqq 4t = 28$. Therefore any set of consecutive integers, $2a_5 + t$ in number, whose smallest integer is greater than 20, can serve as values of b in Lemma 2. Define d as the number of consecutive integers in such a set. Therefore, if $a_5 = 3, 4, 5, 6$, or 8, we have $d = 2a_5 + t$. Similarly for case I of $a_5 = 7$ the maximum and minimum values of b are $\xi + (t-1) + 28$ and $\xi + 1 - 28$, so that $d = 9t = (t+2)t$; for case II, they are $\xi + (t-1) + 16$

and $\xi+1-16$, and $d=6t-3$; for $(1,1,2,3,7,7)$ they are $\xi+(t-1)+9$ and $\xi+1-16$, and $d=4t+4$; for $(1,1,2,3)$ they are $\xi+8$ and $\xi-8$, and $d=18$.

LEMMA 7. *In Lemma 2, any set of d consecutive integers, each* $\geqq 21$, *can serve as values of b, if* $d=2a_5+t$ *when* $a_5=3, 4, 5, 6,$ *or* 8; $d=(t+2)t$ *if* $a_5=7$, case I; $d=6t-3$, case II; $d=4t+4$, case III; $d=18$ *if* $(1,1,2,3)$; $d=2a_J+t$, *when* $a_5=7$, $a_J=a_6\not\equiv 0$ $(mod\ 7)$ *and when* $a_5=a_6=7$, $a_J=a_7\not\equiv 0$ (mod 7).

The final step leading to the desired integers a, b, r mentioned preceding (3) is to choose ξ in Lemma 2 so that (3) is satisfied. This determines the integer M mentioned following (4). The method is essentially that given by Dickson[8] in his improved proof of the Fermat theorem on polygonal numbers, and used in my papers I and II. Since the proof is long and intricate, it is omitted here, except the statement that by the equation $N=r+b+m(a-b)/2$ of Lemma 2 the inequalities (3) are transformed to involve N and m instead of a. The result is that if

$$N \geqq (13d^2-65d+87)m+2(13d-83),$$

and if d is defined as in Lemma 7, and if $N=r+b+m(a-b)/2$, then there are d positive integers within the limits on b which involve N and m instead of a. Since these limits on b imply the limits (3) on b, if ξ is chosen appropriately within these limits then Lemma 2 holds and (3) hold.

The value for d of case I, $a_5=7$ can be lowered if the function is such that there is a coefficient $a_J\not\equiv 0$ (mod 7) and $a_J<28$. For then the argument applied to $a_5\not\equiv 0$ (mod 7) in the proof of Lemma 2 holds for a_J. Hence $2a_J+t$ can be used as d. Here $2a_J+t<(t+2)t$ since $a_J<4t=28$.

In section 4 of II necessary and sufficient conditions were found that the functions $(1,1,2,3,\cdots)$ represent the integers $0,1,2,\cdots,34m-16$. They are that $w=m-2$, that $a_k\leqq w_{k-1}+1$ $(5\leqq k\leqq n)$, and that the function be one of the following:

(8)
- $(1,1,2,3)$;
- $(1,1,2,3,\cdots)$, $a_5=3,4,5,6$, $n\geqq 5$;
- $(1,1,2,3,7,\cdots)$, with $a_6=8,\cdots,13$, $n\geqq 6$; also with $a_6=7$, and $n=6$ or $a_7=7,\cdots,14$; also with $a_6=7$, $a_7=15,\cdots,22$ subject to the following conditions (i) or (iii); also with $a_6=14$, $a_7=14,\cdots,28$, $n\geqq 7$;
- $(1,1,2,3,8,\cdots)$, with $n\geqq 5$, subject to the following conditions (i) or (ii);

[8] *Bulletin of the American Mathematical Society*, vol. 33 (1927), p. 715.

(i) $a_k \neq w_{k-1}$ for every $k \geqq 6$;

(ii) $a_k = w_{k-1}$ for at least one $k \geqq 6$, and for every such k there is a coefficient a_K satisfying $a_k < a_K \leqq a_k + 7$;

(iii) $a_k = w_{k-1}$ for at least one $k \geqq 6$, and for every such k there is a coefficient a_K satisfying $a_k < a_K \leqq a_k + 14$ but $a_K \neq a_k + 7$.

THEOREM 1. *Let f satisfy the necessary conditions above. Define $M = (13d^2 - 65d + 87)m + 2(13d - 83)$. Then f is universal except perhaps for integers N such that $34m - 16 < N < M$.*

3. *The functions* $(1, 1, 1, 2, \cdots)$. The fundamental structure of the argument is precisely that of paragraph 2. Hence only new or difficult details are given. Sufficient conditions that integers x, y, z, w, each $\geqq 0$, satisfying $a = x^2 + y^2 + z^2 + 2w^2$ and $b = x + y + z + 2w$, exist are (3) and

$$(9) \qquad 4(5a - b^2) \neq 25^s(5n + e) \qquad (e = 2, 3).$$

These are the conditions of Dickson, since $3 \equiv -2 \pmod 5$. Here $t = 5$.

The conditions which are equivalent to (9) when $a \equiv b \equiv 0 \pmod 5$ are those obtained from (6) and (7) by replacing 7 by 5 and using $e = 2, 3$, and $AB \not\equiv 0 \pmod 5$. A lemma analogous to Lemma 1 holds when $t = 5$, and also one analogous to Lemma 2. If $a_5 = 2, 3, 4, 6$, that is, if $a_5 \not\equiv 0 \pmod 5$, the details are precisely similar to those for $t = 7$ and $a_5 \not\equiv 0 \pmod 7$. The same is true if $a_5 = 5 \not\equiv a_6 \pmod 5$. If $a_5 = 5 \equiv a_6 \pmod 5$, lemmas concerning the pairs involving $b \pm 5$, $b \pm 10$, $b \pm 15$ were proved. These lemmas were distinctly different in statement and proof for the cases $t = 7$ and $t = 5$. The proof was completed as for $t = 7$, and there emerged the following values for d.

LEMMA 8. *Let f satisfy conditions (12) of II (necessary conditions). Let $d = 2a_5 + t$ if $a_5 = 2, 3, 4, 6$; $d = 2a_6 + t$ if $a_5 = 5 \not\equiv a_6 \pmod 5$; $d = (t + 2)t$ if $a_5 = 5 \equiv a_6 \pmod 5$ and $n > 5$ except $d = 27$ if $(1, 1, 1, 2, 5, 5)$; $d = 14$ if $n = 4$. Let $N \geqq \xi \geqq 3t$. Then there are integers a, b, r, each $\geqq 0$, such that $N = r + b + m(a - b)/2$, $a \equiv b \pmod 2$, (9) is satisfied, and $f_4 = r \leqq m - 2 - t$. Any set of d consecutive integers, each $\geqq 3t$, can serve as values of b.*

The value $(t + 2)t$ for d can be lowered if there is a coefficient $a_J \not\equiv 0 \pmod t$ and $a_J < 3t$. For the argument applied to $a_5 \not\equiv 0 \pmod t$ holds for a_J. Hence $2a_J + t$ can be used as d, and $2a_J + t < (t + 2)t$.

In section 4 of II necessary and sufficient conditions were found that the functions $(1,1,1,2,\cdots)$ represent the integers $0,1,\cdots,34m-16$. They are (12) of II.

THEOREM 2. *Let f satisfy* (12) *of* II. *Define*

$$M=(9d^2-45d+61)m+2(9d-49).$$

Then f represents every positive integer N except perhaps $34m-16<N<M$.

The results for $(1,1,1,2,\cdots)$ of Theorem 3 of II are improved upon in this theorem.

4. *The functions* $(1,1,2,4,\cdots)$. The fundamental structure of the argument is that of paragraph 2. Dickson proved that if b is odd or double an odd integer, and if (3) hold, then

$$a=x^2+y^2+2z^2+4w^2,\qquad b=x+y+2z+4w \tag{10}$$

have a solution in integers $\geqq 0$. The new case in which b is divisible by 4 is included in the following theorem proved in section 7:

THEOREM 3. *Let $b=2^iB$, $a=2^{2h-1}A$ or $2^{2h}A$, where i and h are integers $\geqq 1$, while A and B are odd integers. Then* (10) *have solutions in integers $\geqq 0$ if and only if $8a\geqq b^2$ and one of the following conditions hold:*

$$\begin{cases} a=2^{2h}A, & 1\leqq i\leqq h+1, \quad 1\leqq h;\\ a=2^{2h}A, & 3\leqq h+2=i, \quad A\not\equiv 1 \pmod 8;\\ a=2^{2h}A, & 4\leqq h+3\leqq i, \quad A\not\equiv 7 \pmod 8;\end{cases} \tag{11}$$

$$\begin{cases} a=2^{2h-1}A, & 2<i=h+1, \quad B\equiv\pm 1 \pmod 8, \quad A\not\equiv 15 \pmod{16};\\ a=2^{2h-1}A, & 2<i=h+1, \quad B\equiv\pm 3 \pmod 8, \quad A\not\equiv 7 \pmod{16};\\ a=2^{2h-1}A, & 2=i=h+1, \quad A\equiv\pm 3 \pmod 8;\\ a=2^{2h-1}A, & 1\leqq i\neq h+1, \quad 1\leqq h.\end{cases} \tag{12}$$

Lemma 1 holds with $t=8$. In paragraph 4 of II necessary and sufficient conditions were found that the functions $(1,1,2,4,\cdots)$ represent the integers $0,1,\cdots,34m-16$, namely merely that $w=m-2$ and that $n=4$ or that $a_k\leqq w_{k-1}+1$ $(5\leqq k\leqq n)$. A lemma analogous to Lemma 2 holds: if $a_5\not\equiv 0 \pmod 4$, that is, if $a_5=5$, 6, 7, 9 the details are similar to those for $t=5$, 7 and $a_5\not\equiv 0 \pmod t$; but if $a_5=4$, 8 the supplementary lemmas

concerning the pairs involving $b \pm 4$, $b \pm 8$, $b \pm 12$, $b \pm 16$ were extremely intricate. The proof was then completed as for $t = 5, 7$ and the values of d determined.

LEMMA 9. *Let f satisfy $w = m - 2$, and $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$ or $n = 4$. Let $d = 2a_5 + t$ if $a_5 = 5, 7, 9$; $d = 2a_6 + t$ if $a_5 \equiv 0 \not\equiv a_6 \pmod 4$; $d = 5t$ if $a_5 = 8$ and another coefficient is 8 or 16; $d = 4t$ if $a_5 = 8$ and another coefficient is 12; $d = 28$ if $(1, 1, 2, 4, 8)$; $d = 5t/2$ if $a_5 = 4$ and another coefficient is 4 or 8; $d = (t - 1)t/2$ if $a_5 = 4$ and another coefficient is 12; $d = 3t$ if f is $(1, 1, 2, 4, 4)$; $d = 20$ if $(1, 1, 2, 4)$.* Let $N \geqq \xi \geqq 2t$. *Then there are integers a, b, r, each $\geqq 0$, such that $N = r + b + m(a - b)/2$, $a \equiv b \equiv 1$ (mod 2) or (11) or (12) is satisfied, and $f_4 = r \leqq m - 2 - t$. Any set of d consecutive integers, each $\geqq 2t$, can serve as values of b.*

The value for d can be lowered if $a_5 \equiv 0 \pmod 4$ but there exists a coefficient $a_J \not\equiv 0 \pmod 4$ such that $2a_J + t$ is less than the value stated in Lemma 9.

THEOREM 4. *Let f satisfy $w = m - 2$, and $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$ or have $n = 4$. Define $M = (15d^2 - 75d + 100)m + 2(15d - 103)$. Then f represents every positive integer except perhaps $34m - 16 < N < M$.*

5. *The functions* $(1, 1, 1, 1, \cdots)$. Again the fundamental structure of the argument is that of paragraph 2. Sufficient conditions that integers x, y, z, w, each $\geqq 0$, satisfying $a = x^2 + y^2 + z^2 + w^2$, $b = x + y + z + w$, exist are (3) and

$$4a - b^2 \neq 4^s\,(8n + 7). \tag{13}$$

When $a \equiv 0 \equiv b \pmod 2$ the conditions, equivalent to (13), suitable to deriving supplementary lemmas for pairs involving $b \pm 4$, $b \pm 8$, are (14) and (15), using $a = 2^{2h}A$ or $2^{2h-1}A$, $b = 2^iB$, with $AB \not\equiv 0 \pmod 2$:

$$\begin{aligned} &a = 2^{2h}A, \quad 1 \leqq i = h \text{ or } h + 1; \text{ or} \\ &a = 2^{2h}A, \quad 3 \leqq h + 2 = i, \quad A \not\equiv 3 \pmod 8; \text{ or} \\ &a = 2^{2h}A, \quad 4 \leqq h + 3 \leqq i, \quad A \not\equiv 7 \pmod 8; \end{aligned} \tag{14}$$

$$a = 2^{2h-1}A, \quad 1 \leqq h \leqq i. \tag{15}$$

Lemma 1 holds with $t = 4$. In paragraph 4 of II the necessary and sufficient conditions that the functions $(1, 1, 1, 1, \cdots)$ represent the integers

$0, 1, \cdots, 34m - 16$ were merely that $w = m - 2$ and that $n = 4$ or that $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$. A lemma analogous to Lemma 2 holds. The previous treatment of $\rho = m - 1$ is invalid here, because, if a is odd or double an odd, then a, b satisfy (15), but if $a \equiv 0 \pmod 4$, then when

$$b = \zeta = \xi + t + 1, \qquad a = 2g + \zeta$$

fail to satisfy (14) or (15) so also do $b' = \zeta - t - 2$, $a' = 2(g + 1) + b'$. If $a_5 = 1, 2, 3, 5$ then one of ζ or $\zeta' = \zeta \pm a_5$ is satisfactory; similarly if $a_5 = 4 \not\equiv a_6 \pmod 4$; but if $a_5 = 4 \equiv a_6 \pmod 4$ then one of ζ, $\zeta \pm 4$, or $\zeta \pm 8$ is satisfactory (in fact ζ or $\zeta \pm a_J$, if there exists $a_J \not\equiv 0 \pmod 4$ such that $a_J < 8$), if $n > 5$, while one of ζ, $\zeta \pm 4$, $\zeta + 10$ is satisfactory if $n = 5$. The details for $\rho < m - 1$ are similar to those for $t = 5, 7, 8$, if $a_5 = 1, 2, 3, 5$; if $a_5 = 4$ supplementary lemmas involving $b \pm 4$, $b \pm 8$ were used. The proof was completed then, and the values of d determined.

If $n = 4$ the function is known to be universal.[9]

LEMMA 10. *Let f satisfy $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$. Let $d = 2a_5 + 6$ if $a_5 = 1, 2, 3, 5$; $d = 2a_6 + 6$ if $a_5 = 4 \not\equiv a_6 \pmod 4$; $d = 22$ if $a_5 = 4$ and another coefficient is t or $2t$; $d = 5t$ for $(1, 1, 1, 1, 4)$. Let $N \geqq \xi \geqq t$. Then there are integers a, b, r, each $\geqq 0$, such that $N = r + b + m(a - b)/2$, $a \equiv b \pmod 2$, (13) holds, and $f_4 = r \leqq m - 2 - t$. Any set of d consecutive integers, each $\geqq t$, can serve as values of b.*

The value $d = 5t + 2$, when $a_5 = t$, can be lowered if there exists a coefficient $a_J \not\equiv 0 \pmod 4$ such that $a_J < 2t$, for then the argument applied to $a_5 \not\equiv 0 \pmod 4$ is valid, and $d = 2a_J + 6$.

THEOREM 5. *Let f have $w = m - 2$, and $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$. Define $M = (7d^2 - 35d + 48)m + 2(7d - 35)$. Then f represents every positive integer N except perhaps $34m - 16 < N < M$.*

6. *The functions* $(1, 1, 2, 2, \cdots)$. The fundamental structure of the argument is that of paragraph 2. The method is also that which yielded Lemma 7 of II. The results of Lemma 7 of II (in which the upper value $\beta + 5 + a_5$, if $a_5 = 3, 5, 7$ should be $\beta + 7 + a_5$) are not as good as the new results here obtained, because my limits on b in Lemma 6 of II are not as good as (3) which Dickson obtained, and because of improved values for d. If $a_5 = 3, 4, 5, 7$ then for $\rho = m - 1$ satisfactory values of b are ζ or $\zeta + a_5$;

[9] Dickson, *Journal de Mathématiques*, ser. 9, vol. 7 (1928), Theorems 11-15.

if $a_5 = 2, 6$ then ζ, $\zeta + a_5$ or $\zeta - t - 2$ are satisfactory. For $\rho < m - 1$ and $a_5 = 3, 4, 5, 7$ the details are similar to those for $t = 7$ and $a_5 \not\equiv 0 \pmod 7$; for $a_5 = 2, 6$ supplementary lemmas involving $b \pm 2$, $b \pm 4$, $b \pm 6$, $b \pm 8$, $b \pm 10$, $b \pm 12$ were used. The details were different for $a_5 = 2$ and $a_5 = 6$, but a striking uniformity of values for d emerged.

The necessary conditions (13) of II should have been stated so that $(1, 1, 2, 2, 6)$ is excluded.

LEMMA 11. *Let f satisfy* (13) *of* II *excluding* $(1, 1, 2, 2, 6)$. *Let $d = 2a_5 + t + 2$ if $a_5 = 3, 4, 5, 7$; if $a_5 = 2$ or 6 and $a_6 > a_5 + 1$, let $d = 2a_6 + t$; if $a_5 = 2$ or 6 and $a_6 = a_5 + 1$, let $d = 2a_6 + t + 1$;* if $a_5 = 2$ *or 6 and $a_6 = a_5$, let $d = 4a_5 + t$; for $(1, 1, 2, 2, 2)$ and $(1, 1, 2, 2, 6)$ let $d = 2(t + 2) + a_5$; for $(1, 1, 2, 2)$ let $d = 25$. Let $N \geqq \xi \geqq 2t$. Then there are integers a, b, r, each $\geqq 0$, such that $N = r + b + m(a - b)/2$, the equations $a = x^2 + y^2 + 2z^2 + 2w^2$, $b = x + y + 2z + 2w$ have solutions in integers each $\geqq 0$, and $f_4 = r \leqq m - 2 - t$. Any set of d consecutive integers, each $\geqq 2t$, can serve as values of b.*

The values of d can be lowered, if $a_5 = 2$ or 6 and $a_6 = a_5$, to $2a_J + t + 1$ if $a_J = a_5 + 1$ and to $2a_J + t$ if $a_5 + 1 < a_J < 2a_5$.

THEOREM 6. *Let f satisfy* (13) *of* II *excluding* $(1, 1, 2, 2, 6)$. *Define $M = (11d^2 - 55d + 74)m + 2(11d - 65)$. Then f represents every positive integer N except perhaps $34m - 16 < N < M$.*

7. *Solvability of* (10). Dickson showed that necessary conditions are

$$a \equiv b \pmod 2, \tag{16}$$

$$8a - b^2 = F^2 + 2v^2 + 4W^2 \tag{17}$$

$$F = 4w - 2z - y - x, \qquad v = 2z - y - x, \qquad W = y - x. \tag{18}$$

The solution of (18) with (10_2) give

$$8w = b + F, \qquad 8z = b - F + 2v, \qquad 8y = b - F - 2v + 4W, \qquad 8x = b - F - 2v - 4W. \tag{19}$$

Let a and b be even, expressed as in Theorem 3. By examining the conditions for equality, we find the inequality $8a - b^2 \neq 4^m (16n + 14)$ holds only in the following cases:

$$(20)\quad \begin{cases} a = 2^{2h-1}A, & h+1=i, \quad B \equiv \pm 1 \pmod 8, \quad A \not\equiv 15 \pmod{16}; \\ a = 2^{2h-1}A, & h+1=i, \quad B \equiv \pm 3 \pmod 8, \quad A \not\equiv 7 \pmod{16}; \\ a = 2^{2h-1}A, & h+1 \neq i; \end{cases}$$

$$(21)\quad \begin{cases} a = 2^{2h}A, & h+1 \geqq i; \\ a = 2^{2h}A, & h+2=i, \quad A \not\equiv 1 \pmod 8; \\ a = 2^{2h}A, & h+3 \leqq i, \quad A \not\equiv 7 \pmod 8). \end{cases}$$

Use is made of the known

LEMMA 12. *If an even integer is represented by $x^2 + y^2 + 2z^2$, there exists a representation with x and y both even.*

Let a and b satisfy (20) or (21) with $i \geqq 2$. Then $8a - b^2 \equiv 0 \pmod 4$, and in $8a - b^2 = F^2 + 2v^2 + 4W^2$ we have F and v both even. Hence there are integers F_1 and v_1 such that $F = 2F_1$ and $v = 2v_1$ and (22) or (23) holds, according as $a = 2^{2h-1}A$ or 2^{2h}:

$$(22)\qquad 2^{2h}A - 2^{2i-2}B^2 = F_1^2 + 2v_1^2 + W^2,$$

$$(23)\qquad 2^{2h+1}A - 2^{2i-2}B^2 = F_1^2 + 2v_1^2 + W^2.$$

Since $i \geqq 2$, the left member of (22) and that of (23) are divisible by 4. Hence by Lemma 12 we can take F_1 and W both even; that is, there exist integers F_2, W_2 and v_2 such that $F_1 = 2F_2$, $W = 2W_2$ and $v_1 = 2v_2$, and that (22) and (23) become respectively

$$(24)\qquad 2^{2h-2}A - 2^{2i-4}B^2 = F_2^2 + 2v_2^2 + W_2^2,$$

$$(25)\qquad 2^{2h-1}A - 2^{2i-4}B^2 = F_2^2 + 2v_2^2 + W_2^2.$$

By these values $F = 4F_2$, $v = 4v_2$ and $W = 2W_2$, (19) become

$$(26)\quad \begin{array}{ll} 2w = 2^{i-2}B + F_2, & 2z = 2^{i-2}B - F_2 + 2v_2, \\ 2y = 2^{i-2}B - F_2 - 2v_2 + 2W_2, & 2x = 2^{i-2}B - F_2 - 2v_2 - 2W_2. \end{array}$$

If $i \geqq 3$ conditions (26) are equivalent to F_2 even, but if $i = 2$ they are equivalent to F_2 odd, since B is odd. Now, if $i \geqq 3$ by Lemma 12 we can take F_2 even in (25); also in (24), if $i \geqq 3$ and $h > 1$. If $i \geqq 3$ and $h = 1$ in (24), then one of F_2 and W_2 is odd and the other is even, since A is odd, and hence by the symmetry we can take F_2 even. If $i = 2$ in (25), then $B + F_2 + W_2$ is even and hence $F_2 \not\equiv W_2 \pmod 2$, and we can take F_2 odd by the symmetry. If $i = 2$ and $h > 1$ in (24) the same argument holds. There remains therefore the case $i = 2$, $h = 1$ in (24); but in (24), as in

(22), h and i satisfy (20), and hence in fact (20_1) or (20_2). Here (24) becomes $A - B^2 = F_2{}^2 + 2v_2{}^2 + W_2{}^2$.

Lemma 13 states necessary and sufficient conditions on a and b, satisfying (20_1) or (20_2) with $h = 1$, that there exist a representation of

$$A - B^2 = F_2{}^2 + 2v_2{}^2 + W_2{}^2$$

with F_2 odd.

LEMMA 13. *If A and B satisfy $B \equiv \pm 1 \pmod 8$ and $A \not\equiv 15 \pmod{16}$. or $B \equiv \pm 3 \pmod 8$ and $A \not\equiv 7 \pmod{16}$, if A is odd, and if $A - B^2$ is represented by the form $x^2 + y^2 + 2z^2$, then there exists a representation with x odd if and only if $A \equiv \pm 3 \pmod 8$.*

First, let $B \equiv \pm 1 \pmod 8$. If $A \equiv 7 \pmod{16}$ then $A - B^2 \equiv 6 \pmod{16}$. By the proof of Lemma 12, we have $(A - B^2)/2 = s^2 + t^2 + z^2$, and therefore $s \equiv t \equiv z \equiv 1 \pmod 2$, $x \equiv y \equiv 0 \pmod 2$. Similarly, if $A \equiv 1$ or $9 \pmod{16}$, we have $s \equiv t \equiv z \equiv 0 \pmod 2$, $x \equiv y \equiv 0 \pmod 2$. Next let $A \equiv 3, 5, 11$, or $13 \pmod{16}$. Then if in $A - B^2 = x^2 + y^2 + 2z^2$ in fact x and y are even, we have $s \equiv t \pmod 2$. If $z \not\equiv s \pmod 2$, we have

$$(A - B^2)/2 = s^2 + t^2 + z^2, \qquad A - B^2 = (s + z)^2 + (s - z)^2 + 2t^2$$

with $s + z$ odd. But if $z \equiv s \pmod 2$ then in fact each of z, s, t is even or each is odd. If each is even, then $(A - B^2)/2$ is divisible by 4 and therefore $A - B^2$ is divisible by 8; but by the hypotheses $B \equiv \pm 1 \pmod 8$ and $A \equiv 3$, $5, 11, 13 \pmod{16}$, $A - B^2 \not\equiv 0 \pmod 8$. Similarly a contradiction is obtained if $z \equiv s \equiv t \equiv 1 \pmod 2$. This completes the proof when $B \equiv \pm 1 \pmod 8$.

Next, if $B \equiv \pm 3 \pmod 8$, it is shown similarly that $A \equiv 1, 9, 15 \pmod{16}$ imply $x \equiv y \equiv 0 \pmod 2$. But if $A \equiv 3, 5, 11, 13 \pmod{16}$, then $x \equiv y \equiv 0 \pmod 2$ imply $z \equiv s \pmod 2$ and

$$A - B^2 = (s + z)^2 + (s - z)^2 + 2t^2$$

with $s + z$ odd. This completes the proof of Lemma 13.

Therefore if a and b satisfy (20) with $i = 2$ and $h = 1$, there exists a representation (17) such that (19) yield integers if and only if $A \equiv \pm 3 \pmod 8$.

This proves Theorem 3.

Since (11) and (12) are merely (20) and (21) with the case $i = 2$, $h = 1$ of (20) modified to include only $A \equiv \pm 3 \pmod 8$, an alternative statement to Theorem 3 is

THEOREM 7. *If a and b are even integers such that*

$$8a - b^2 \neq 4^m(16n + 14)$$

and that $8a \geqq b^2$ *then there exist integers* x, y, z, w *satisfying* (10) *if and only if* $b \not\equiv 4 \pmod 8$, *or* $a \equiv 0 \pmod 4$, *or* $a \equiv \pm 6 \pmod{16}$.

Dickson proved that if k is an integer $\geqq 0$ and if (17) holds, then the values x, y, z, w from (19) are each $> -k$ if $7a < b^2 + 2bk + 8k^2$. The proof does not depend upon whether b is divisible by four or not.

THEOREM 8. *Let k be an integer $\geqq 0$, and let a and b be integers such that* $a \equiv b \pmod 2$, $8a \geqq b^2$, $b \geqq 8(1 - k)$ *and* $7a < b^2 + 2bk + 8k^2$. *Then there exist integers* x, y, z, w, *each* $> -k$, *satisfying* (10) *if and only if* $b \not\equiv 4 \pmod 8$, *or* $a \equiv 0 \pmod 4$, *or* $a \equiv \pm 6 \pmod{16}$, *and* $8a - b^2 \neq 4m(16n + 14)$.

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A CUBIC ANALOGUE OF THE CAUCHY-FERMAT THEOREM.[1]

By Alvin Sugar.

Introduction. In this paper we shall obtain an ideal universal Waring theorem for the polynomial

$$P(x) = m(x^3 - x)/6 + x, \qquad x \text{ integral and } \geqq 0, \tag{1}$$

where m is an integer $\geqq 16$, i. e. we shall prove $g(P) = m + 3$ for $m \geqq 16$.

In Part II of this paper we evaluate a constant $C_1 = 10^{12}m^{10}$, which maximizes the constants of papers of Dickson,[2] Baker[3] and Webber[4]; this gives us the following theorem.

Theorem 1. *For $m \geqq 7$, every integer $\geqq C_1 = 10^{12}m^{10}$ is a sum of nine or ten values of* (1) *according as the congruence $m \equiv 6 \pmod 9$ does not or does hold.*

In Part I we develop a powerful ascension theorem and ascension methods, and by ascending beyond the constant C_1, prove that every positive integer $\leqq C_1$ is a sum of $m + 3$ values of $P(x)$ for $m \geqq 16$.

Part I. Ascension Methods.

1. *Ascension theorems.* We shall ascend beyond the constant $C_1 = 10^{12}m^{10}$ first for a fixed range and then for an arbitrary range of values of m.

We write

$$F(a) = P(a+1) - P(a)$$

and apply a theorem of Dickson's[5] to our polynomial $P(x)$.

Theorem 2. *Let every integer n, $c < n \leqq g$, be a sum of $k - 1$ values of $P(x)$, and let a be an integer $\geqq 0$ for which $F(a) < g - c$. Then every integer N, $c < N \leqq g + P(a+1)$, is a sum of k values of $P(x)$.*

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[1] Presented to the Society, November 30, 1935.

[2] *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 1-12.

[3] Doctoral dissertation, Chicago, 1934.

[4] *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 493-510.

[5] Theorem 9 in the *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 709.

Before going on to our next theorem, we note that in this series of theorems, g and c need not be integers.

THEOREM 3. *Let every integer n, $c < n \leqq g$, be a sum of $k - 1$ values of $P(x)$, and let y be a real number $\geqq 0$ which satisfies the inequality $F(y+1) < g - c$. Then every integer N,*

$$c \leqq N \leqq g + P(y+1), \tag{2}$$

is a sum of k values of $P(x)$.

By way of proof we observe that $F([y]+1) \leqq F(y+1)$ and $P(y+1) < P([y]+2)$, since $F(x)$ and $P(x)$ are properly monotone increasing functions for $x \geqq 1$.

We are now in a position to introduce an important ascension theorem, which will enable us to breach a huge interval in one step. The inequality

$$F(3t^{(3/2)e}+1) < P(3t^{e}+1) \tag{3}$$

holds for $t \geqq 1$. Let t be a real number $\geqq 1$ which satisfies the inequality $F(3t+1) < pm + q$, and let every integer N_0, $c < N_0 \leqq c + pm + q$, be a sum of k values of $P(x)$. Then from (2) we have that every integer N', $c < N' \leqq c + pm + q + P(3t+1)$, and hence every integer N_1, $c < N_1 \leqq c + P(3t+1)$, is a sum of $k+1$ values. Similarly, since $F(3t^{3/2}+1) < P(3t+1)$, by (3) with $e = 1$, then every integer N_2, $c < N_2 \leqq c + P(3t^{3/2}+1)$, is a sum of $k+2$. And finally, every integer N_2,

$$c < N_2 \leqq c + P(3t^{(3/2)^{s-1}}+1),$$

is a sum of $k+s$. The proof of this last statement is made by an induction on s. Since

$$(9/2)t^{2(3/2)^{s}}m < c + P(3t^{(3/2)^{s-1}}+1),$$

we may state the following theorem.

THEOREM 4. *Let every integer n, $c < n \leqq c + pm + q$, be a sum of k values of $P(x)$, and let t be a real number $\geqq 1$ which satisfies the inequality $F(3t+1) < pm + q$. Then every integer N,*

$$c < N \leqq (9/2)t^{2(3/2)^{s}}m$$

is a sum of $k+s$ values of $P(x)$.

2. *The first ascension*; $16 \leqq m \leqq 1950$. There follows a list of values of $P(x)$.

$$0, 1, a = m + 2,\ b = 4m + 3,\ c = 10m + 4,\ d = 20m + 5,\ e = 35m + 6,$$
$$f = 56m + 7,\ g = 84m + 8,\ h = 120m + 9,\ i = 165m + 10.$$

We are also going to list a set of intervals such that an integer lying in anyone of these intervals will be a sum of $m - 8$ values of $P(x)$. These intervals will overlap for $m \geqq 16$. Therefore $m - 8$ values will suffice over the interval defined by the overlapping intervals.

We shall reconstruct a portion of the following list. We begin with $120m + 9$. By adding $m - 8$ to this we obtain $121m + 1$. It is evident that every integer from $120m + 9$ to and not including $121m + 1$ is a sum of $m - 8$ values. Now consider the integer $120m + 16 = a + e + g$. By adding $m - 10$ to this we obtain $121m + 6$. It is evident that every integer from $120m + 16$ to $121m + 6$ is a sum of $m - 8$ values, and continuing thus we come to the interval $(120m + 24, 121m + 11)$ over which $m - 8$ values will suffice. Since $121m + 11 = h + a$, we can begin all over again as we did with $120m + 9$ by adding $m - 9$ to $121m + 11$ and repeating the above procedure. By inspection it may be verified that the following set of intervals overlap for $m \geqq 16$.

$(h = 120m + 9, 121m + 1)$, $(a + e + g = 120m + 16, 121m + 6)$, $(2b + 2f = 120m + 20, 121m + 9)$, $(2a + b + c + d + g = 120m + 24, 121m + 11)$, $(a + h = 121m + 11, 122m + 2)$, $(2a + e + g = 121m + 18, 122m + 7)$, $(c + d + e + f = 121m + 22, 122m + 11)$, $(3c + e + f = 121m + 25, 122m + 13)$, $(2a + h = 122m + 13, 123m + 3)$, $(c + 2f = 122m + 18, 123m + 8)$, $(a + c + d + e + f = 122m + 24, 123m + 12)$, $(a + 3c + e + f = 122m + 27, 123m + 14)$, $(2a + c + 2d + 2e = 122m + 30, 123m + 16)$, $(1 + 3a + h = 123m + 16, 124m + 4)$, $(a + c + 2f = 123m + 20, 124m + 9)$, $(a + 2b + c + d + g = 123m + 25, 124m + 12)$, $(b + h = 124m + 12, 125m + 3)$, $(a + b + e + g = 124m + 19, 125m + 8)$, $(4c + g = 124m + 24, 125m + 12)$, $(2a + 2b + c + d + g = 124m + 27, 125m + 13)$, $(b + c + 2d + 2e = 124m + 29, 125m + 16)$, $(2 + a + b + h = 125m + 16, 126m + 4)$, $(a + 2d + g = 125m + 20, 126m + 9)$, $(b + c + d + e + f = 125m + 25, 126m + 13)$.

Hence if $m \geqq 16$, then every integer n, $120m + 8 < n \leqq 126m + 12$, is a sum of $m - 8$ values of $P(x)$. Applying Theorem 2 we see that $F(3) = 6m + 1 < 6m + 4$; then $m - 7$ values suffice from $120m + 8$ to $120m + 8 + 16m + 8$. Two more applications of this theorem give the result that $m - 5$ values will suffice from $120m + 8$ to $120m + 8 + 216m$.

The next ascent will be made in one step by employing Theorem 4. Since $t = 19/3$ satisfies the inequality $F(3t+1) < 216m$, then every integer N,

$$120m + 8 < N \leqq c_1 m = (9/2)(19/3)^{2(3/2)^8} m,$$

is a sum of $m+3$ values. It is evident that the inequality $10^{12}m^{10} \leqq c_1 m$ holds for $16 \leqq m \leqq 1950$. Hence we have by employing Theorem 1 the following theorem.

THEOREM 5. *Let m have the range $16 \leqq m \leqq 1950$; then every positive integer $> 120m + 8$ is a sum of $m+3$ values of $P(x)$.*

3. *The second ascension.* Again we construct a set of overlapping intervals. This time we begin with an arbitrary value $P(A) = Rm + A$, and we take $m - r$ as the number of values which will suffice over each interval. By adding $m - r$ (where r is a positive integer) to $P(A)$, we obtain the first interval

$$\big(Rm + A, (R+1)m + A - r\big),$$

We take $r = R - A - 10$. The rest of the intervals can be written at once, as follows:

$$\big((R-1+t)m + 2R - 2 + 2t, (R+t)m + R + t - r\big),$$
$$\big((R+t)m + R + t - r, (R+1+t)m + R - 2r + t - 10)\big).$$
$$(t = 1, \cdots, 10).$$

We observe that $(R-1+t)m + 2R - 2 + 2t = (R-1+t)a$ and that for this range of t the integer[6] $(R+t)m + R + t - r = P(A) + ta + 10 - t$ is a sum of 11 values of $P(x)$. By inspection it is evident that these intervals will overlap for $m \geqq Q(A) = 3R - 2A - 1 = (A^3 - 5A)/2 - 1$. We also see that $r = (A^3 - 7A)/6 - 10$.

LEMMA 1. *For $m \geqq Q(A)$, $A \geqq 5$, every integer n,*

$$Rm + A \leqq n \leqq (R+10)m + A + 20,$$

is a sum of $m - r$ values of $P(x)$.

In the following discussion we shall prove statements (S_1) and (S_2). We begin with $P(A)$ and show (S_1) *that for $Q(A) \leqq m \leqq Q(A+1)$ every*

[6] As a matter of fact the value assigned to r was obtained by requiring that r be the greatest integer for which the inequality $(R+t)m + R + t - r \geqq P(A) + ta$, $(t = 1, \cdots, 10)$ holds.

integer $> P(A)$ *is a sum of* $m+3$ *values, provided* $A \geqq 10$. We also show (S_2) *that for* $m \geqq Q(A)$, $m+3$ *values will suffice from* $P(A)$ *to* $P(A+1)$ *inclusive, when* $A \geqq 10$. Since $Q(A)$ is an increasing function, then, by (S_2) and an induction on A, we conclude (S_3) *that for* $m \geqq Q(A)$, *every integer* n, $P(10) \leqq n \leqq P(A+1)$, *will be a sum of* $m+3$ *values.* Hence from (S_1) and (S_3) we have the following theorem.

THEOREM 6. *Every integer* $\geqq P(10)$ *is a sum of* $m+3$ *values of* $P(x)$ *for* $m \geqq Q(10) = 474$.

This and Theorem 5 give us the next theorem.

THEOREM 7. *Every integer* $\geqq P(10)$ *is a sum of* $m+3$ *values of* $P(x)$ *for* $m \geqq 16$.

There remains yet to be proved, the Statements (S_1) and (S_2). In establishing these statements we make use of a pair of inequalities which are derived from the expansion of k^x, into the power series

$$k^x = 1 + x \log k + \frac{x^2 \log^2 k}{2!} + \frac{x^3 \log^3 k}{3!} + \cdots.$$

Since this series converges for all x, we have the following inequalities holding for a positive x.

$$k^x > \frac{x^2 \log^2 k}{2}, \qquad k^x > \frac{x^3 \log^3 k}{6}. \tag{4}$$

To the results of Lemma 1 we apply Theorems 2 and 4, and we find that every integer N,

$$P(A) \leqq N \leqq c_2 m = (9/2)(10)^{2(3/2)^r} m, \tag{5}$$

is a sum of $m+3$ values of $P(x)$ for $m \geqq Q(A)$. We know that

$$10^{12} m^{10} \leqq c_2 m, \quad \text{when} \quad Q(A) \leqq m \leqq M = (10^{-12} c_2)^{1/9}.$$

Let $y = A^3/10$; it is then evident that [7]

$$M > (10)^{(2/9)(3/2)^y - 12/9};$$

for,

$$r = (A^3 - 7A)/6 - 10 > A^3/10.$$

From (4_1), we have

$$\frac{2}{9}\left(\frac{3}{2}\right)^y - \frac{12}{9} > \frac{y^2 \log^2 1.5}{9} - \frac{12}{9} > 10^{-4} A^6,$$

[7] For the remainder of this discussion we shall take $A \geqq 10$.

Write $z = 10^{-4}A^6$; then $M > 10^z$. Employing (4_2), we obtain

$$10^z > 10^{-12}A^{18} > A^3 > (A^3 + 3A^2 - 2A - 6)/2 = Q(A+1).$$

Therefore $M > Q(A+1)$, and we have proved (S_1).

It is evident that

$$c_2 m > Mm > m\,Q(A) > P(A+1).$$

This result and (5) prove (S_2).

4. *The positive integers* $< P(10)$. We shall prove another lemma.

LEMMA 2. *Every positive integer* $\leqq c = 10m + 4$ *is a sum of* $m + 3$ *values of* $P(x)$ *for* $m \geqq 4$.

It is evident that every integer $< 3m + 6 = 3a$ is a sum of $m + 3$ values. Adding $m + 1$ to $3m + 6$, we see that every integer $< 4m + 7 = 4 + b$ is a sum of $m + 3$ values. Adding $m - 1$ to this, we see that every integer $< 5m + 6 = 1 + a + b$ is a sum of $m + 3$ values. Repetition of this argument gives the following list:

$$6m + 7 = 2a + b, \quad 7m + 8 = 6a + m - 4, \quad 7m + 10 = 1 + 3a + b,$$
$$8m + 9 = 3 + 2b, \quad 9m + 8 = a + 2b, \quad 10m + 4 = c.$$

This completes the proof of the lemma.

The following set of intervals, which overlap for $m \geqq 16$, give rise to the conclusion that every integer n,

$$(6) \qquad 10m + 3 < n \leqq 13m + 9,$$

is a sum of $m - 1$ values of $P(x)$.

$(c = 10m + 4,\ 11m + 3)$, $(6a + b = 10m + 15,\ 11m + 8)$, $(2 + a + c = 11m + 8,\ 12m + 4)$, $(7a + b = 11m + 17,\ 12m + 9)$, $(1 + 2a + c = 12m + 9,\ 13m + 5)$, $(8a + b = 12m + 19,\ 13m + 10)$.

Applying Theorem 2 to (6) four times, we obtain the following result.

LEMMA 3. *For* $m \geqq 16$, *every integer* n, $10m + 3 < n \leqq 217m + 32$, *is a sum of* $m + 3$ *values of* $P(x)$.

This result along with Lemma 2 and Theorem 7 completes the proof of the Principal Theorem.

PRINCIPAL THEOREM. *Every positive integer is a sum of $m+3$ values of $m(x^3-x)/6+x$ for non-negative integers x, where $m \geqq 16$.*

PART II. EVALUATION OF THE CONSTANT.

The proof of Theorem 2 of Dickson's and of similar theorems of Baker's and Webber's [8] depends upon the existence of an integer C lying in each interval of a triple of intervals of the form

$$f(m, b_i) \leqq 3^{i-1}C \leqq F(m, b_i) \qquad (i=1,2,3) \tag{7}$$

where the b_i are suitably chosen positive odd integers. For $m \not\equiv 6 \pmod 9$, (7) takes on the form (8). For $m \equiv 6 \pmod 9$, (7) becomes (9).

$$2m+\gamma+(9/8)mb_i^3 \leqq 3^{i-1}C \leqq (3/2)mb_i^3+m/3 \qquad (i=1,2,3), \tag{8}$$
$$\gamma = \begin{cases} (9m^4+1)/2 & ((m,3)=1) \\ (m^4+81)/6 & ((m,3)\neq 1); \end{cases}$$

$$\frac{125}{24}mb_i^3+\frac{26}{25}m+\gamma \leqq 3^{i-1}C \leqq \frac{125}{18}mb_i^3+\frac{m}{3} \qquad (i=1,\cdots,5), \tag{9}$$

where γ has some value similar in form to those of (8).[9]

An inspection will verify that for $m \geqq 7$,

$$b_i < 7m \qquad (i=1,2,3) \tag{10}$$

for all the [10] b_i of papers A. Replacing b_i by $7m$ in the right hand side of (8), we get

$$3^{i-1}C < 520m^4 \qquad (i=1,2,3). \tag{11}$$

Whence $C < 60m^4$. We also seek a value for 3^{2n} which satisfies [11]

$$\frac{3^{2n}}{b_i(3A_i)^{1/2}} \geqq 8$$

and hence which satisfies

$$3^{4n} \geqq \left[\frac{384b_i}{m}(3^iC-m)-432b_i^4\right]3^{2n}+\frac{384b_i}{m}(m-6-3b_i)+192b_i^2. \tag{12}$$

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[8] *Op. cit.* These papers shall henceforth be referred to as papers A.

[9] f and F originally contained terms of the form β_i divided by a power $p=p(n)$ of 3, but for $n=\nu$ these terms become negligible.

[10] In his paper, Webber did not list the b_i corresponding to the case $m=18e+12$, i.e. $m=3a$, a even and $\equiv 1 \pmod 3$. They are here supplied by the author: $b_1=20e+11$, $b_2=28e+17$, $b_3=40e+23$.

[11] Dickson, *op. cit.*, p. 7, (28).

For convenience we state the following lemma.

LEMMA 4. *For a positive α, the inequality $x^2 > \alpha x + \beta$ is satisfied by $x > M = \max(\alpha, \beta) + 1$.*

For each of three cases, β negative, $\beta > \alpha$, and $\beta < \alpha$, the proof may be made by substituting M for x.

If we write (12) in the form $x^2 > \alpha x + \beta - \delta$, where β is the sum of all the positive terms free of 3^{2n}, we know that this inequality is satisfied for $x > \max(\alpha, \beta) + 1$, by Lemma 4. But by (10) and (11) we have

$$5{,}000{,}000\, m^4 > 384 b_i 3^i C/m > \max(\alpha, \beta) + 1.$$

Hence we have for $m \geqq 7$,

$$C3^{3\nu} < 10^{12} m^{10} = C_1.$$

A similar argument for (9) produces a smaller constant than C_1.

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ON A PROBLEM OF PÓLYA.

By Norman Levinson.[1]

1. Pólya[2] has set the following problem:

If $f(z)$ is an entire function uniformly bounded at $z = 0, \pm 1, \pm 2, \cdots$, and if

$$\overline{\lim_{r\to\infty}} \frac{\log | f(re^{i\theta}) |}{r} \leq 0 \tag{1.0}$$

then $f(z)$ is a constant.

Here we shall prove the following theorem which will yield immediately a generalization of Pólya's results.

Theorem I. *Let $g(z) = g(x + iy)$ be an entire function such that*

$$\overline{\lim_{r\to\infty}} \frac{\log | g(re^{i\theta}) |}{r} \leq \pi \tag{1.1}$$

and

$$g(iy) = O(e^{\pi|y|}) \tag{1.2}$$

as $| y | \to \infty$. If

$$g(z_n) = O(1) \tag{1.3}$$

as $| n | \to \infty$, where $\{z_n\}$ is a sequence of complex quantities such that

$$| z_n - n | \leq a, \qquad -\infty < n < \infty \tag{1.4}$$

for some positive integer a, and

$$| z_n - z_m | \geq \delta \tag{1.5}$$

for $n \neq m$ and for some fixed $\delta > 0$, then

$$g(x) = O(| x |^A) \tag{1.6}$$

where A depends only on a.

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[1] National Research Fellow.

[2] *Jahresbericht der Deutschen Mathematischen Vereinigung,* Bd. 40 (1931), 2te Abteilung, p. 80, problem 105. Solutions have been given by Tschakaloff, by Szegö, and by Pólya. Paley and Wiener in "Fourier transforms in the complex domain," *American Mathematical Society Colloquium Publications,* vol. 19, pp. 81-83, have also given a solution.

As a corollary of this theorem we have the extension of Pólya's result.

THEOREM II. *Let $f(z)$ satisfy condition* (1.0) *and let*

$$f(z_n) = O(1) \tag{1.7}$$

as $|n| \to \infty$, where $\{z_n\}$ is a set of complex quantities such that

$$|z_n - n\alpha| \leq \beta, \qquad -\infty < n < \infty \tag{1.8}$$

for some positive α and β. Then $f(z)$ is a constant.

2. We require several lemmas in proving Theorem I.

LEMMA 1. *Let $\{z_n\}$ satisfy conditions* (1.4) *and* (1.5). *We also assume $|z_n| \geq 1$. If*

$$F(z) = (z - z_0) \prod_1^\infty \left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{z_{-n}}\right) \tag{2.0}$$

then

$$|F(z)| < A_1(|z| + 1)^{20a} e^{\pi|y|}, \tag{2.1}$$

$$\left|\frac{F(z)}{z - z_n}\right| < A_2(|z| + 1)^{20a} e^{\pi|y|}, \tag{2.2}$$

$$|F(z)| > \frac{A_3 e^{\pi|y|}}{(|z| + 1)^{20a}} \quad \text{for } |y| > 2a, \tag{2.3}$$

and

$$|F'(z_n)| > \frac{A_4 \delta^{8a+1}}{(|n| + 1)^{8a+1}}, \qquad -\infty < n < \infty, \tag{2.4}$$

where A_1, A_2, A_3, and A_4 depend only on a.

Since $F(-z)$ satisfies all the requirements we assume with no loss of generality that $x \geq 0$. We begin by proving (2.1) for $|z| \geq 4a$. Clearly if $|n| > 2a$

$$\left|\frac{1 - \dfrac{z}{z_n}}{1 - \dfrac{z}{n}}\right| \leq 1 + \left|\frac{z(z_n - n)}{z_n(n - z)}\right|$$

$$\leq 1 + \frac{a\,|z|}{|n - z|\,(|n| - a)} \leq \exp(2a\,|z|\,/\,|n|\;|n - z|).$$

Or if N is defined by $N - \frac{1}{2} \leq |z| < N + \frac{1}{2}$, then

$$\prod_{2a+1}^\infty \left|1 - \frac{z}{z_n}\right| \left|1 - \frac{z}{z_{-n}}\right|$$

$$\leq \prod_{2a+1}^\infty \left|1 - \frac{z^2}{n^2}\right| \left|\frac{1 - \dfrac{z}{z_N}}{1 - \dfrac{z}{N}}\right| \exp[2a\,|z|\,\Sigma''(1/|n|\;|n - z|)]$$

where Σ'' indicates the sum from $-\infty$ to ∞ with the terms $n=0$ and $n=N$ omitted. Clearly

$$\begin{aligned}\Sigma'' \frac{1}{|n|\,|n-z|} &\leq \sum_{-\infty}^{-N} \frac{1}{n^2} + \frac{1}{|z|} \sum_{-N}^{-1} \frac{1}{|n|} \\ &\quad + \frac{3}{|z|} \sum_{1}^{[\frac{1}{2}N]} \frac{1}{n} + \frac{3}{|z|} \sum_{[\frac{1}{2}N]+1}^{N-1} \frac{1}{N-n-\frac{1}{2}} \\ &\quad + \frac{1}{|z|} \sum_{N+1}^{2N} \frac{1}{(n-N-\frac{1}{2})} + \sum_{2N+1}^{\infty} \frac{1}{(n-N-\frac{1}{2})^2} \\ &\leq \frac{10 \log |z|}{|z|}.\end{aligned}$$

Therefore

$$\prod_{2a+1}^{\infty} \left|1-\frac{z}{z_n}\right| \left|1-\frac{z}{z_{-n}}\right| \leq 2\,|z|^{20a} \left|\frac{z_N - z}{z-N}\right| \left|\frac{\sin \pi z}{\pi z}\right| \prod_{1}^{2a} \left|1-\frac{z^2}{n^2}\right|^{-1}.$$

Or

$$|F(z)| \leq 2\,|z|^{20a} \frac{|z-z_0|\,|z_N - z|\,|\sin \pi z| \prod_{1}^{2a} \left|1-\frac{z}{z_n}\right| \left|1-\frac{z}{z_{-n}}\right|}{\pi\,|z|\,|N-z| \prod_{1}^{2a} \left|1-\frac{z^2}{n^2}\right|}.$$

Recalling that $|z| \geq 4a$ and $|z_n| \geq 1$, we have

$$\begin{aligned}|F(z)| &\leq 10\,|z|^{20a} \frac{(|z|+3a)^{4a+1}(2a)^{4a+1} e^{\pi|y|}}{(|z|-2a)^{4a+1}} \\ &< A\,|z|^{20a} e^{\pi|y|}\end{aligned}$$

where A depends only on a. This holds for $|z| \geq 4a$. If $|z| < 4a$ we can extend this result by observing that $F(z)$ being analytic takes its extreme value on the boundary. Thus we get (2.1).

As regards $F(z)/(z-z_n)$ we observe that for $|z-z_n| \geq 1$, (2.2) is a consequence of (2.1). For $|z-z_n| < 1$ we use the fact that $F(z)/(z-z_n)$ takes its extreme value on $|z-z_n| = 1$. This proves (2.2).

The proof of (2.3) is similar to that of (2.1). Here we consider

$$\left|\frac{1-\dfrac{z}{n}}{1-\dfrac{z}{z_n}}\right| = \left|1-\frac{z(z_n-n)}{n(z_n-z)}\right|.$$

If $-\infty < n < |z| - 2a$ or if $|z| + 2a < n < \infty$, then

$$\left|\frac{1-\dfrac{z}{n}}{1-\dfrac{z}{z_n}}\right| \leq 1 + \frac{a\,|z|}{|n|(|z-n|-a)} \leq \exp(2a\,|z|\,/\,|n|\,\,|z-n|).$$

If we now proceed in a manner similar to that used above and remember that $|y| > 2a$ we get

$$\left|\frac{1}{F(z)}\right| \leq \frac{(|z|+1)^{20a}}{4A_3 |\sin \pi z|}$$

which gives (2. 3) immediately.

In proving (2. 4) we observe that

$$F'(z_n) = -\frac{1}{z_n}(z_n - z_0) \prod_{k=1}^{\infty}{}' \left(1 - \frac{z_n}{z_k}\right)\left(1 - \frac{z_n}{z_{-k}}\right)$$

where the prime on the product indicates that the term zero is omitted. If $n > 3a$,

$$\begin{aligned} |F'(z_n)| &\geq \frac{n-2a}{n+a} \prod_1^{n-2a-1} \left(\frac{n-k-2a}{k+a}\right)\left(\frac{n+k-2a}{k+a}\right) \\ &\quad \times \prod_{n+2a+1}^{\infty} \left(\frac{k-n-2a}{k+a}\right)\left(\frac{k+n-2a}{k+a}\right)\left(\frac{\delta}{n+3a}\right)^{8a+1} \\ &\geq \tfrac{1}{4}\left(\frac{\delta}{n+3a}\right)^{8a+1} \prod_1^{a} \frac{k^2(n+2a+k)^2}{(n-2a-1+k)^2} \\ &\quad \times \prod_1^{n-2a-1} \left|1 - \frac{(n-2a)^2}{k^2}\right| \prod_{n+2a+1}^{\infty} \left(1 - \frac{(n-2a)^2}{k^2}\right). \end{aligned}$$

If we observe that $|\sin \pi(x+k)|/|x|$ approaches π as $x \to 0$ we get, recalling that a is an integer,

$$\prod_1^{n-2a-1} \left|1 - \frac{(n-2a)^2}{k^2}\right| \prod_{n+2a+1}^{\infty} \left(1 - \frac{(n-2a)^2}{k^2}\right) = \tfrac{1}{2} \prod_{n-2a+1}^{n+2a} \left|1 - \frac{(n-2a)^2}{k^2}\right|^{-1}.$$

Thus

$$|F'(z_n)| \geq \tfrac{1}{8}\left(\frac{\delta}{n+3a}\right)^{8a+1} \prod_{n-2a-1}^{n+2a} \left|1 - \frac{(n-2a)^2}{k^2}\right|^{-1} \geq \tfrac{1}{8}\left(\frac{\delta}{n+3a}\right)^{8a+1}.$$

This proves (2. 4) for $n > 3a$. Clearly the same method can be used for $0 \leq n \leq 3a$ and since $F(-z)$ is of the same form as $F(z)$ it holds for all values of n.

LEMMA 2. *If $\psi(z)$ is an entire function satisfying* (1. 0) *and if $\psi(x)$ is uniformly bounded for real x, then $\psi(z)$ is a constant.*

That $\psi(z)$ is bounded in each half-plane (upper and lower) follows from a well known result of Pragmén and Lindelöf.[3] Thus $\psi(z)$ is a constant.

3. In proving Theorem I we make use of the Pragmén-Lindelöf function,

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[3] Pólya-Szegö, *Aufgaben und Lehrsätze*, vol. 1 (1925), p. 147, problem 135.

$h(\theta)$. Since we are dealing exclusively with entire functions of order 1, the following definition will suffice for use here,

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r}. \tag{3.0}$$

The behaviour of this function is characterized by

THEOREM [4] A. *If* $\theta_1 < \theta_2 < \theta_3$ *and* $\theta_3 - \theta_1 < \pi$ *then*

$$h(\theta_1)\sin(\theta_3-\theta_2) + h(\theta_3)\sin(\theta_2-\theta_1) \geq h(\theta_2)\sin(\theta_3-\theta_1). \tag{3.1}$$

Proof of Theorem I. We can assume that $|z_n| \geq 1$ for if this is not the case then it becomes true on discarding a finite number of z_n. The new set which can again be called $\{z_n\}$ clearly satisfies (1.4) for some new a. Thus

$$F(z) = (z - z_0)\prod_1^\infty\left(1-\frac{z}{z_n}\right)\left(1-\frac{z}{z_{-n}}\right)$$

satisfies the requirements of Lemma 1.

We assume that $g(z)$ has an infinite number of zeros (otherwise the theorem is trivial by an application of the Hadamard factorization theorem). If we divide out $8a+3$ of these zeros it is obvious that we obtain a function $g_1(z)$ such that

$$g_1(z_n) = O(|z_n|^{-8a-3}), \qquad |n| \to \infty.$$

Thus by (2.4)

$$\sum_{-\infty}^{\infty}\frac{g_1(z_n)F(z)}{F'(z_n)(z-z_n)}$$

is absolutely convergent. Moreover

$$H(z) = g_1(z) - \sum_{-\infty}^{\infty}\frac{g_1(z_n)F(z)}{F'(z_n)(z-z_n)}$$

vanishes at all points $z = z_n$ and by (1.1) and (2.2) we have

$$\overline{\lim_{r\to\infty}} \frac{\log|H(re^{i\theta})|}{r} \leq \pi. \tag{3.2}$$

Thus by the Hadamard factorization theorem

$$H(z) = F(z)\psi(z) \tag{3.3}$$

where $\psi(z)$ is an entire function of at most order 1.

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[4] Titchmarsh, *Theory of Functions*, Oxford Press (1932), p. 184.

By (1.2) and (2.2) it follows that

$$H(iy) = O(|y|^{20a} e^{\pi|y|}).$$

By (3.3) and (2.3) it follows therefore that

$$\psi(iy) = O(|y|^{40a}). \tag{3.4}$$

We shall now investigate separately the various possible forms of $\psi(z)$.

Case 1. Let us suppose $\psi(z)$ is a polynomial. By (3.4) it is of at most degree $40a$. We recall that

$$g_1(z) = \psi(z)F(z) + \sum_{-\infty}^{\infty} \frac{g_1(z_n)F(z)}{F'(z_n)(z-z_n)}.$$

By (2.1) and (2.2) with $\psi(z)$ a polynomial of degree at most $40a$ it is clear that

$$g_1(x) = O(|x|^{60a}).$$

Therefore

$$g(x) = O(|x|^{68a+3})$$

and the theorem is proved for this case.

Case 2. Let us suppose that $\psi(z)$ is not a polynomial but that

$$\overline{\lim_{|x|\to\infty}} \frac{\log|\psi(x)|}{|x|} \leq 0.$$

Clearly (3.4) gives us

$$\overline{\lim_{|y|\to\infty}} \frac{\log|\psi(iy)|}{|y|} \leq 0. \tag{3.5}$$

Thus applying (3.1) to each quadrant we get

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|\psi(re^{i\theta})|}{r} \leq 0 \tag{3.6}$$

for all values of θ.

We can now conclude that $\psi(z)$ cannot have a finite number of zeros because if it did then the Hadamard factorization theorem would give

$$\psi(z) = P(z)e^{cz}$$

where $P(z)$ is a polynomial. By (3.6), $c = 0$ and therefore $\psi(z)$ is a polynomial contrary to hypothesis.

Thus $\psi(z)$ has an infinite number of zeros and if we divide out $40a$ of these we obtain a function $\psi_1(z)$ such that by (3.4), $\psi_1(iy) = 0(1)$ as

$|y| \to \infty$. Since $\psi_1(z)$ satisfies (3.6), it follows from Lemma 2 that $\psi_1(iz)$ is a constant. But this means $\psi(z)$ is a polynomial which is contrary to our assumption. Thus Case 2 is impossible.

Case 3. Finally we suppose that

$$\overline{\lim_{|x|\to\infty}} \frac{\log|\psi(x)|}{|x|} = \alpha > 0. \tag{3.7}$$

With no loss of generality we can assume that (3.7) holds for $x > 0$ (for otherwise we could deal with $\psi(-x)$). Thus $h(0) = \alpha > 0$. Using (3.5) and applying (3.1) to $0 < \theta < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < \theta < 0$, we have

$$h(\theta) \leq \alpha \cos\theta, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi. \tag{3.8}$$

If we set $\theta_1 = -\theta$, $\theta_3 = \theta$, and $\theta_2 = 0$ in (3.1) we have

$$h(-\theta)\sin\theta + h(\theta)\sin\theta \geq \alpha \sin 2\theta, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

Or

$$h(-\theta) + h(\theta) \geq 2\alpha\cos\theta.$$

Combining this with (3.8) we have

$$h(\theta) = \alpha\cos\theta, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

That is

$$\overline{\lim_{r\to\infty}} \frac{\log|\psi(re^{i\theta})|}{r} = \alpha\cos\theta, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi. \tag{3.9}$$

By (2.3) and (3.2) it follows that α cannot be infinite. Let us take $\theta_0 = \tan^{-1}\pi/\alpha$. Then using (2.1), (2.3), and (3.3) we have

$$\begin{aligned}\overline{\lim_{r\to\infty}} \frac{\log|H(re^{i\theta_0})|}{r} &= \lim_{r\to\infty} \frac{\log|F(re^{i\theta_0})|}{r} + \overline{\lim_{r\to\infty}} \frac{\log|\psi(re^{i\theta_0})|}{r} \\ &= \pi\sin\theta_0 + \alpha\cos\theta_0 = (\pi^2+\alpha^2)^{\frac{1}{2}}.\end{aligned}$$

But by (3.2), α must be zero, contrary to hypothesis. Thus Case 3 is impossible.

These three cases exhaust all possibilities. Only Case 1 is possible and this leads immediately to the completion of the theorem.[5]

We can now readily prove Theorem II. We observe that there is no restriction in assuming that $\alpha > 3\beta$ in (1.8), for if this is not the case then let N be an integer so large that $N\alpha > 3\beta$. We can then use the sequence

[5] Clearly, condition (1.1) need only hold for θ close to $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and $g(z)$ be known to be of order 1, in order that the above proof of Theorem I go through.

$\{z_{nN}\}$ in place of $\{z_n\}$ and $\alpha_1 = \alpha N$ in place of α in (1.8) and now we have $\alpha_1 > 3\beta$.

Thus assuming $\alpha > 3\beta$, we have, if $n \neq m$,

$$|z_n - z_m| \geq |n\alpha - m\alpha| - |z_n - n\alpha| - |z_m - m\alpha| \geq \alpha - 2\beta > \beta. \tag{3.10}$$

We observe that $f(z/\alpha)$ is uniformly bounded at the points $\{z_n/\alpha\}$. By (1.8)

$$\left|\frac{\alpha}{z_n} - n\right| \leq \frac{\beta}{\alpha}. \tag{3.11}$$

And by (3.10)

$$\left|\frac{z_n}{\alpha} - \frac{z_m}{\alpha}\right| \geq \frac{\beta}{\alpha}.$$

Thus $f(z/\alpha)$ satisfies the requirements of Theorem I and therefore

$$f\left(\frac{x}{\alpha}\right) = O(|x|^A), \qquad |x| \to \infty,$$

for some A. Or

$$f(x) = O(|x|^A).$$

By the Hadamrd factorization theorem, Theorem II is trivial if $f(z)$ has only a finite number of zeros. If we assume it has an infinite number we divide out A of these zeros and apply Lemma 2, which shows at once that $f(z)$ is an algebraic polynomial and therefore cannot have an infinite number of zeros. This completes the proof of Theorem II.

PRINCETON UNIVERSITY AND THE
INSTITUTE FOR ADVANCED STUDY.

GÉOMÉTRIE DES SYSTÈMES DE CHOSES NORMÉES.

Par V. Glivenko.

I. Introduction.

1. On sait que, dans plusieurs questions d'Analyse, il est permis de ne prendre pas en considération les ensembles de mesure nulle, de sorte qu'il y est permis d'identifier les deux ensembles mesurables A et B chaque fois où les points de A n'appartenant pas à B et ceux de B n'appartenant pas à A forment un ensemble de mesure nulle. Autrement dit, on peut y remplacer les ensembles mesurables par leurs *types métriques*.

Les types métriques eux-mêmes n'étant pas des ensembles, il est toutefois naturel d'établir entre eux des relations propres aux ensembles. Ainsi, on peut dire qu'un type a fait partie d'un type b, en signes $a \subset b$, si tout ensemble A du type a est contenu, à un sous-ensemble de mesure nulle près, dans tout ensemble B du type b. Pareillement, on peut introduire la notion de la partie commune ab de deux types a et b, celle de la somme $a + b$ de ces types etc.

Nous exprimons tout cela en disant que les types métriques, de même que les ensembles eux-mêmes, forment un système de choses, dont nous allons préciser la définition.

2. Nous appelons *système de choses* un ensemble S d'éléments $a, b, c, \cdots$ lorsque les conditions suivantes sont remplies:

1°. L'ensemble S contient des couples d'éléments a, b liés entre eux par une relation $a \subset b$ telle que:

$$a \subset b \text{ et } b \subset a \text{ entraîne } a = b \text{ et inversement;}$$
$$a \subset b \text{ et } b \subset c \text{ entraîne } a \subset c.$$

2°. A tout couple d'éléments a, b de l'ensemble S correspond un élément ab de S tel que

$$ab \subset a,$$
$$ab \subset b,$$

et que, x étant un élément de S,

$$x \subset a \text{ et } x \subset b \text{ entraîne } x \subset ab.$$

3°. A tout couple d'éléments a, b de l'ensemble S correspond un élément $a+b$ de S tel que

$$a \subset a+b,$$
$$b \subset a+b,$$

et que, y étant un élément de S,

$$a \subset y \text{ et } b \subset y \text{ entraîne } a+b \subset y.$$

4°. L'ensemble S contient un élément 0 tel que, quel que soit l'élément z de S,

$$0 \subset z.$$

Il est aisé de voir que les propriétés ci-dessus définissent les éléments ab, $a+b$ et 0 d'une manière univoque.

3. A tout ensemble mesurable correspond un nombre bien défini, sa mesure. De même, à tout type métrique a on peut attacher un nombre bien défini $|a|$, mesure d'un quelconque ensemble de ce type.

Il y a cependant une différence essentielle entre ces deux cas. Il existe plusieurs ensembles ayant la mesure nulle, mais il n'y a qu'un seul type $a=0$ tel que $|a|=0$; c'est porquoi tout ensemble peut être augmenté sans changer sa mesure, ce qui est impossible pour les types.

Nous exprimons tout cela en disant que les types métriques forment le système de choses normées.

4. Nous disons que S est le système de choses *normées* lorsqu'à toute chose a de S correspond un nombre non négatif $|a|$, *norme* de cette chose, possédant les propriétés suivantes:

$$a \subset b \text{ et } a \neq b \text{ entraîne } |a| < |b|;$$
$$|a+b| + |ab| = |a| + |b|;$$
$$|0| = 0.$$

5. Outre le système des types métriques, on connaît plusieurs systèmes de choses normées. En voici quelques exemples.

1°. Dans la théorie des probabilités, il est quelquefois favorable d'identifier les deux événements chaque fois où la probabilité que l'un d'eux se produit tandis que l'autre ne se produit pas, est nulle. Autrement dit, il y est favorable de remplacer les événements par leurs *types stocastiques*.

Les types stocastiques forment un système de choses normées si l'on con-

vient d'écrire $a \subset b$ chaque fois où, A étant un événement arbitraite du type a et B un événement arbitraire du type b, la probabilité que A se produit tandis que B ne se produit pas, est nulle. Ici, $|a|$ est la probabilité d'un quelconque événement du type a.

2°. Dans la logique mathématique, il est quelquefois favorable d'identifier les propositions équivalentes. Autrement dit, il y est favorable de remplacer les propositions par leurs *types logiques.*

Les types logiques forment un système de choses normées si l'on convient d'écrire $a \subset b$ chaque fois où, A étant une proposition arbitraire du type a et B une proposition arbitraire du type b, la proposition A implique la proposition B. Ici, $|a|$ est la valeur logique d'une quelconque proposition du type a, égale à *un* pour les proposition vraies et égale à *zéro* pour les propositions fausses.

3°. Prenons les domaines bornés aux frontières quarrables, et considérons ces domaines conjointement avec leurs frontières. Les ensembles fermés ainsi obtenus et l'ensemble vide forment un système de choses normées où $a \subset b$, ab, $a + b$ et 0 ont le sens usuel sauf le cas où la partie commune de a et de b ne contient aucun domaine: alors, on prend pour ab l'ensemble vide. Ici, $|a|$ est l'étendue de a.

4°. Prenons un anneau d'ensembles finis. Ceux-ci et l'ensemble vide forment un système de choses normées où $a \subset b$, ab, $a + b$ et 0 ont le sens usuel. Ici, $|a|$ est le nombre d'éléments de a.

5°. Les nombres entiers positifs forment un système de choses normées où $a \subset b$ signifie que a est un diviseur de b, de sorte que ab est le plus grand diviseur commun de a et de b, $a + b$ est le plus petit multiple commun de ces nombres, 0 est le nombre 1. Ici, on a $|a| = \log a$.

6°. Les nombres non négatifs forment un système de choses normées où $a \subset b$ signifie que a ne dépasse pas b, de sorte que ab est le plus petit des nombres a et b, $a + b$ est le plus grand de ces nombres, 0 est le nombre 0. Ici, on a simplement $|a| = a$.

6. Abordons maintenant la question principale dont nous nous occupons. On verra que tout système de choses normées est un espace métrique. Cela résulte du fait qu'on peut y former une expression, à savoir $|a + b| - |ab|$, qui possède tous les propriétés de la distance de a et de b.

C'est précisement l'étude de la structure de cet espace qui est le but du présent article. Nous définirons un espèce d'espaces métriques que nous appelerons espaces *presque ordonnés,* et nous démontrerons les deux théorèmes suivants:

THÉORÈME DIRECT. *Tout système de choses normées est un espace métrique presque ordonné où la distance de a et de b est égale à*

$$| a + b | - | ab |.$$

THÉORÈME INVERSE. *Tout espace métrique presque ordonné est un système de choses normées où l'expression*

$$| a + b | - | ab |$$

est égale à la distance de point a et de point b.

7. Il nous paraît que telles considérations, où les types métriques, par exemple, se présentent comme des points d'un espace, sont dignes d'intérêt puisque les types métriques peuvent être définis effectivement comme des éléments indépendants, en laissant de côté toute la théorie des ensembles mesurables.

Convenons de désigner, en effet, par $A, B, \cdots$ les sommes finies et bornées d'intervalles, et par $| A |, | B |, \cdots$ les sommes de ses longueurs; ces sommes d'intervalles forment un espace métrique où la distance de A et de B est égale à

$$| A + B | - | AB |.$$

L'espace en question n'est pas *complet,* mais nous pouvons le faire complet en y ajoutant des nouveaux éléments à l'aide des suites convergentes, en se servant d'un procédé bien connu, de la même manière qu'on obtient tous les nombres réels à partir des nombres rationnels.[1] Les éléments de l'espace ainsi complété seront précisement les types métriques.

8. Il est à remarquer, entre autres, que tous les exemples cités plus haut, des systèmes de choses normées, possèdent une propriété importante.

On sait que les propriétés 1°-4° du n°2, caractéristiques pour les systèmes de choses, ont pour conséquence la relation suivante:

$$ac + bc \subset (a + b)c. \tag{1}$$

Quant à la relation inverse,

$$(a + b)c \subset ac + bc, \tag{2}$$

elle est, au contraire, indépendante des propriétés en question. Il existe, en effet, des systèmes de choses dans lesquels la relation (2) n'est pas nécessaire-

[1] Cf. F. Hausdorff, *Mengenlehre,* 1927, p. 106.

ment remplie. L'exemple bien connu d'un tel système est celui du système des corps convexes. Si, a et b étant deux corps convexes quelconques, on attribue à $a \subset b$, à ab et à 0 le sens usuel et si l'on prend pour $a + b$ le plus petit corps convexe contenant a et b, on voit sans peine que les conditions 1°-4° du $n^o 2$ y seront remplies tandis qu'il n'en sera pas, en général, pour la relation (2).

En d'autres termes, convenons de dire qu'un système de choses est *distributif* si, quels que soient a, b, c, on a nécessairement

$$ac + bc = (a + b)c.$$

Ce-ci est équivalent à la couple des relations (1) et (2).

La propriété importante, mentionnée ci-dessus, des exemples cités plus haut, consiste en ce que c'étaient toujours des exemples des systèmes distributifs. On pourrait croire que c'est inévitable pour les systèmes de choses normées. Mais on verra dans la suite qu'il existe aussi des systèmes des choses normées qui ne sont pas distributifs. Cependant, les systèmes distributifs de choses normées méritent, sans doute, une attention particulière et nous leur consacrerons, dans ce qui suit, une étude détaillée.

II. Espaces métriques presque ordonnés.

9. Rappolons qu'un ensemble D est dit *espace métrique*,[2] et que ses éléments $a, b, c, \cdots$ se nomment *points*, lorsqu'à tout couple d'éléments a, b de l'ensemble D correspond un nombre non négatif (a, b), *distance* de point a et de point b, possédant les propriétés suivantes:

$$(a, b) = 0 \text{ entraîne } a = b \text{ et inversement;}$$
$$(a, b) = (b, a);$$
$$(a, b) \leq (a, c) + (c, b).$$

Quand on a $(a, b) = (a, c) + (c, b)$, on dit que le point c se trouve *entre* les points a et b. Il est aisé de voir qu'entre les points a et b se trouvent, en particulier, a et b eux-mêmes.

Nous disons qu'un espace métrique D est *presque ordonné* s'il contient un point que nous appelerons *origine* et qui possède les propriétés suivantes. Convenons de dire qu'un point a est plus *prochain* qu'un point b, ou que b est plus *lointain* que a, si a se trouve entre l'origine et b. Alors:

1°. Quels que soient les deux points a et b, de D, et quel que soit le point c se trouvant entre a et b, chaque point plus prochain que a et b est aussi plus

[2] Il s'agit des espaces découverts par M. M. Fréchet et qu'il a nommé *espaces* (D).

prochain que c; et, de même, chaque point plus lointain que a et b est aussi plus lointain que c.

2°. Quels que soient les deux points a et b, de D, parmi les points se trouvant entre a et b il existe un qui est le plus prochain et il existe un autre qui est le plus lointain.

Pour avoir un exemple d'espace métrique presque ordonné, prenons un ensemble arbitraire de nombres réels, où $(a, b) = | a - b |$. Il est aisé de voir que le rôle de l'origine peut être joué par un nombre quelconque appartenant à cet ensemble.

Pour avoir un exemple d'espace métrique qui n'est pas presque ordonné, prenons un ensemble de nombres complexes, contenant au moins les trois nombres qui ne se trouvent pas sur une droite du plan complexe, d'ailleurs arbitraire, et où, comme auparavant, $(a, b) = | a - b |$. Essayons de prendre un certain nombre c de cet ensemble pour l'origine, et considérons deux nombres a et b tels que c, a et b ne se trouvent pas tous les trois sur une droite. D'après la propriété 2° de l'origine, il y doit exister un nombre d se trouvant entre a et b et tel que, pour tout autre nombre x se trouvant entre a et b, on ait

$$| c - d | + | d - x | = | c - x |.$$

En particulier, comme entre les nombres a et b se trouvent a et b eux-mêmes, on doit y avoir

$$| c - d | + | d - a | = | c - a |$$

et

$$| c - d | + | d - b | = | c - b |.$$

Or, il est manifeste que la première de ces égalités n'est possible que si c, d et a se trouvent sur une droite, de même, la seconde n'est possible que si c, d et b se trouvent sur une droite. Ce-ci contredit à notre supposition que c, a et b ne se trouvent pas sur une droite. On voit ainsi qu'il n'y a aucun nombre possédant les propriétés de l'origine. L'espace n'est pas presque ordonné.

III. Systèmes de choses.

10. Nous commençons par établir une série de propositions auxilières concernant les systèmes de choses et qui nous seront utiles dans notre théorie. La plupart de ces propositions est bien connue dans la Logique mathématique. Les autres, même quand elles ne se rencontrent jamais dans les travaux antérieures, peuvent être reproduites sans peine. Ainsi, ce chapitre peut être omis par le lecteur familier avec les méthodes de la Logique mathématique.

Soit donné donc un système de choses $a, b, c, \cdots$. Alors, on a

$$ab = ba, \tag{3}$$

$$a + b = b + a. \tag{4}$$

Ceci est immédiat.

$$aa = a, \tag{5}$$

$$a + a = a. \tag{6}$$

En effet, d'une part, on a $aa \subset a$. D'autre part, $a \subset aa$, ce qui est une conséquence de $a \subset a$. En comparant, on obtient (5). Pareillement, on établit (6).

Lemme I. *Si l'on a*

$$a \subset c \quad et \quad c \subset d$$

on a aussi

$$ab \subset cd \quad et \quad a + b \subset c + d.$$

Démonstration. On a $ab \subset a$ et $ab \subset b$, donc, lorsque les conditions du lemme, $a \subset c$ et $b \subset d$, sont remplies, on a aussi $ab \subset c$ et $ab \subset d$ d'où $ab \subset cd$. Pareillement, on établit que $a + b \subset c + d$.

Lemme II. *Si l'on a*

$$a \subset b,$$

on a aussi, quel que soit c,

$$ac \subset bc \quad et \quad a + c \subset b + c.$$

Démonstration. Lorsque la condition du lemme, $a \subset b$, est remplie, $ac \subset bc$ est, en vertu du Lemme I, une conséquence de $c \subset c$. Il en est de même pour $a + c \subset b + c$.

On a

$$(ac)(bc) = (ab)c, \tag{7}$$

$$(a + c) + (b + c) = (a + b) + c. \tag{8}$$

En effet, on a, d'une part, $(ac)(bc) \subset ab$, ce qui, en vertu du Lemme I, est une conséquence de $ac \subset a$ et $bc \subset b$. De la même manière, on démontre que $(ac)(bc) \subset c$. Par suite,

$$(ac)(bc) \subset (ab)c.$$

D'autre part, on a $(ab)c \subset ac$, ce qui est, en vertu du Lemme II, und con-

séquence de $ab \subset a$. De la même manière, on démontre que $(ab)c \subset bc$. Par suite

$$(ab)c \subset (ac)(bc).$$

En comparant, on obtient (7). Pareillement, on établit (8).

$$(9) \qquad ac + bc \subset c,$$

$$(10) \qquad c \subset (a+c)(b+c).$$

En effet, on a $ac \subset c$ et $bc \subset c$, d'où (9). Pareillement, on établit (10).

$$(11) \qquad ac + bc \subset a + b,$$

$$(12) \qquad ab \subset (a+c)(b+c).$$

En effet, on a $ac \subset a$ et $bc \subset b$, d'où, en vertu du Lemme I, on obtient (11). Pareillement on établit (12).

$$(13) \qquad ac + bc \subset (a+b)c,$$

$$(14) \qquad ab + c \subset (a+c)(b+c).$$

En effet, (13) est une conséquence de (9) et (11). Pareillement, (14) est une conséquence de (10) et (12).

$$(15) \qquad ac + bc \subset ab + c,$$

$$(16) \qquad (a+b)c \subset (a+c)(b+c).$$

En effet, (15) est une conséquence de (9) et de $c \subset ab + c$. Pareillement, (16) est une conséquence de $(a+b)c \subset c$ et de (10).

Lemme III. *La relation*

$$a \subset b$$

est équivalente à chacune des relations

$$ab = a \quad et \quad a + b = b.$$

Démonstration. On a, d'une part,

$$ab \subset a.$$

D'autre part, lorsque $a \subset b$, on a, en vertu du Lemme II,

$$a \subset ab.$$

En comparant, on voit que, lorsque $a \subset b$, on a

$$ab = a.$$

Inversement, lorsque $ab = a$, on obtient $a \subset b$ en vertu de $ab \subset b$. Pareillement, on établit l'équivalence de $a \subset b$ et de $a + b = b$.

On a, quel que soit a,

$$0a = 0, \tag{17}$$

$$0 + a = a. \tag{18}$$

En effet, (17) est, en vertu du Lemme III, une conséquence de $0 \subset a$. Il en est de même pour (18).

$$a(ab) = ab \quad \text{et} \quad a + ab = a, \tag{19}$$

$$b(ab) = ab \quad \text{et} \quad b + ab = b, \tag{20}$$

$$a(a + b) = a \quad \text{et} \quad a + (a + b) = a + b, \tag{21}$$

$$b(a + b) = b \quad \text{et} \quad b + (a + b) = a + b. \tag{22}$$

Pour s'en convaincre, remarquons que la relation $ab \subset a$ est une identité; par suite, les relations (19) qui sont, en vertu du Lemme III, équivalentes à $ab \subset a$, sont, eux-aussi, des identités. Pereillement, en partant de l'identité $ab \subset b$ on obtient (20); en partant de l'identité $a \subset a + b$, on obtient (21); en portant de l'identité $b \subset a + b$, on obtient (22).

Lemme IV. *Si l'on a*

$$a \subset c, \; c \subset b \quad et \quad a = b,$$

on a aussi

$$a = c = b.$$

Démonstration. Si les conditions du lemme, $c \subset b$ et $a = b$, sont remplies, on a

$$c \subset a;$$

si, en outre, la condition du lemme

$$a \subset c$$

est aussi remplie, on a, en comparant,

$$a = c.$$

Pareillement, on établit que $c = b$.

LEMME V. *Si l'on a*

$$ab = a + b,$$

on a aussi

$$a = b.$$

DÉMONSTRATION. On a $ab \subset a \subset a + b$ et $ab \subset b \subset a + b$, donc lorsque la condition du lemme, $ab = a + b$, est remplie, on a, en vertu du Lemme IV, $ab = a = a + b$ et $ab = b = a + b$. Il en résulte immédiatement que $a = b$.

LEMME VI. *Si l'on a*

$$x \subset c \subset y,$$
$$a \subset x \subset b,$$
$$a \subset y \subset b,$$

on a aussi

$$a \subset c \subset b.$$

DÉMONSTRATION. Si les conditions du lemme, $x \subset c$ et $a \subset x$, sont remplies, on a $a \subset c$. Pareillement, on établit que $c \subset b$.

On a

$$(ab)c = a(bc), \tag{23}$$

$$(a + b) + c = a + (b + c). \tag{24}$$

En effet, on a, d'une part,

$$(ab)c \subset a$$

ce qui est une conséquence de $(ab)c \subset ab$ et $ab \subset a$. D'autre part, on a

$$(ab)c \subset bc$$

ce qui est, en vertu du Lemme II, une conséquence de $ab \subset a$. Par suite,

$$(ab)c \subset a(bc).$$

De la même manière, on démontre la réciproque,

$$a(bc) \subset (ab)c.$$

En comparant, on obtient (23). Pareillement, on établit (24).

$$ab \subset a + b. \tag{25}$$

En effet, (25) est une conséquence soit de $ab \subset a$ et $a \subset a + b$, soit de $ab \subset b$ et $b \subset a + b$.

LEMME VII. *Si le système est distributif, on a*

$$ab + c = (a + c)(b + c).$$

DÉMONSTRATION. La condition de distributivité étant

$$(a + b)c = ac + bc,$$

on en déduit successivement:

$$(a + c)(b + c) = a(b + c) + c(b + c) = (ab + ac) + (cb + c).$$

Il en résulte, en tenant compte de ce que, d'après (19), $cb + c = c$:

$$(a + c)(b + c) = (ab + ac) + c.$$

De là, en vertu de (24),

$$(a + c)(b + c) = ab + (ac + c).$$

Il en résulte, en tenant compte de ce que, d'après (19), $ac + c = c$:

$$(a + c)(b + c) = ab + c.$$

C'est ce qu'il fallait démontrer.

On appelle parfois *première loi distributive* la relation

$$ac + bc = (a + b)c$$

et *seconde loi distributive* la relation

$$ab + c = (a + c)(b + c).$$

Nous venons de démontrer que la première loi distributive a pour conséquence la seconde; de la même manière, on pourrait aussi démontrer la réciproque.

IV. Systèmes de choses normées.

11. Soit donné maintenant un système de choses normées $a, b, c, \cdots$. Etablissons un principe général concernant ces systèmes, à savoir:

PRINCIPE GÉNÉRAL. *Si l'on sait que* $|a| = |b|$ *et que* $a \subset b$, *on peut affirmer que* $a = b$.

En effet, de $a \subset b$ et $a \neq b$ il résulterait $|a| < |b|$.

12. Ce-ci posé, nous pouvons aborder la démonstration des théorèmes fondamentaux mentionnés dans l'Introduction. Dans la suite, ce seront les Théorèmes I et IV.

THÉORÈME I. *Tout système de choses normées est un espace métrique presque ordonné où la distance de a et de b est égale à*

$$| a + b | - | ab |.$$

PREMIÈRE PARTIE DE DÉMONSTRATION. Commençons par démontrer que tout système de choses normées est un espace métrique où la distance de a et de b est égale à $| a + b | - | ab |$, ça veut dire que la distance définie par l'égalité

$$(a, b) = | a + b | - | ab |$$

possède effectivement toutes les propriétés de la distance de deux points d'un espace métrique.

Etablissons d'abord que $(a, b) = 0$ entraîne $a = b$ et inversement.

Soit $(a, b) = 0$. Ce-ci n'est autre chose que $| ab | = | a + b |$. Or, d'après (25), on a $ab \subset a + b$. En vertu du principe général, on en conclut que $ab = a + b$. Donc, d'après le Lemme V, on a $a = b$. Inversement, soit $a = b$. Autrement dit, considérons la distance (a, a). Elle est égale à $| a + a | - | aa |$. Or, d'après (5) et (6), on a $aa = a$ et $a + a = a$. Donc, on a $(a, a) = 0$.

Etablissons maintenant que

$$(a, b) = (b, a).$$

(a, b) est égale à $| a + b | - | ab |$ et (b, a) est égale à $| b + a | - | ba |$. Or, d'après (3) et (4), on a $ab = ba$ et $a + b = b + a$. Donc, on a $(a, b) = (b, a)$.

Il nous reste à établir que

$$(a, b) \leq (a, c) + (c, b).$$

D'après (8), on a $(a + b) + c = (a + c) + (b + c)$ et, d'après (16), $(a + b)c \subset (a + c)(b + c)$. Donc, en tenant compte des propriétés de la norme,

$$(26) \qquad | a + b | + | c | = |(a + b) + c | + |(a + b)c |$$
$$\leq |(a + c) + (b + c)| + |(a + c)(b + c)| = | a + c | + | b + c |.$$

Puis, d'après (7), on a $(ab)c = (ac)(bc)$ et, d'après (15), $ac + bc \subset ab + c$. Donc, en tenant compte des mêmes propriétés de la norme,

$$
\begin{aligned}
(27) \qquad & |ab| + |c| = |ab + c| + |(ab)c| \\
& \geq |ac + bc| + |(ac)(bc)| = |ac| + |bc|.
\end{aligned}
$$

En comparant les inégalités (26) et (27), on obtient

$$|a+b| - |ab| \leq |a+c| - |ac| + |b+c| - |bc|.$$

Or, ce n'est autre chose que $(a, b) \leq (a, c) + (c, b)$.

13. Avant de terminer la démonstration du Théorème I, nous démontrerons les deux autres théorèmes.

THÉORÈME II. *Dans l'espace métrique formé par un système de choses normées, un point c se trouve entre deux points a et b si et seulement si l'on a*

$$ac + bc = c = (a + c)(b + c).$$

DÉMONSTRATION. Supposons d'abord que c se trouve entre a et b et proposons-nous d'établir la relation

$$ac + bc = c.$$

A cet effet, remarquons qu'on a, d'après notre supposition,

$$(a, b) = (a, c) + (c, b),$$

ou bien que

$$(28) \qquad |a+b| - |ab| = |a+c| - |ac| + |b+c| - |bc|.$$

En vertu de la propriété fondamentale de la norme, on peut remplacer, dans (28), $|a+b|$ par $|a| + |b| - |ab|$, puis $|a+c|$ par $|a| + |c| - |ac|$ et enfin $|b+c|$ par $|b| + |c| - |bc|$, de sorte que (28) s'ecrira comme il suit:

$$(29) \qquad -|ab| = |c| - |ac| - |bc|.$$

Or, d'après (7), on a $(ac)(bc) = (ab)c$. Donc, en vertu de la même propriété de la norme,

$$(30) \qquad -|(ab)c| = |ac + bc| - |ac| - |bc|.$$

Puis, on a $(ab)c \subset ab$, d'où $|(ab)c| \leq |ab|$. Il en résulte d'après les égalités (29) et (30), que

$$(31) \qquad |c| \leq |ac + bc|.$$

D'autre part, d'après (9), on a $ac + bc \subset c$, d'où

$$(32) \qquad |ac + bc| \leq |c|.$$

En comparant les inégalités (31) et (32), on obtient l'égalité

$$| ac + bc | = | c |.$$

Cette dernière égalité et la relation $ac + bc \subset c$ fournissent, en vertu du principe général, la relation à démontrer $ac + bc = c$.

Pareillement, on établit la relation $c = (a + c)(b + c)$.

Inversement, supposons qu'on a

$$ac + bc = c = (a + c)(b + c)$$

et proposons-nous d'établir que c se trouve entre a et b, c'est-à dire que $(a, b) = (a, c) + (c, b)$. A cet effet, démontrons d'abord que notre supposition entraîne les deux relations suivantes:

$$ab = (ac)(bc), \tag{33}$$

$$a + b = (a + c) + (b + c). \tag{34}$$

Quant à la première, on a, d'après (7),

$$(ac)(bc) = (ab)c.$$

Puis, d'après (12), on a $ab \subset (a + c)(b + c)$, d'où, en tenant compte de notre supposition, $ab \subset c$. En vertu du Lemme III, ce-ci équivant à

$$(ab)c = ab.$$

En comparant, on obtient (33). Pareillement, on établit (34).

Cela posé, on a

$$| a + b | - | ab | = |(a + c) + (b + c)| - |(ac)(bc)|.$$

On en déduit, en se servant encore une fois de notre supposition:

$$\begin{aligned} | a + b | - | ab | = {} & |(a + c) + (b + c)| \\ & + |(a + c)(b + c)| - | ac + bc | - |(ac)(bc)|. \end{aligned}$$

Ce-ci fournit, en vertu de la propriété fondamentale de la norme:

$$| a + b | - | ab | = | a + c | + | b + c | - | ac | - | bc |,$$

ou, ce qui revient au même,

$$| a + b | - | ab | = | a + c | - | ac | + | b + c | - | bc |.$$

Or, c'est précisement l'égalité $(a, b) = (a, c) + (c, b)$, ce qui termine la démonstration de notre théorème.

THÉORÈME III. *Lorsque, dans l'espace métrique formé par un système de choses normées, on prend* 0 *pour l'origine, un point a est plus prochain qu'un point b si et seulement si l'on a*

$$a \subset b.$$

DÉMONSTRATION. "a est plus prochain que b" n'est autre chose que "a se trouve entre l'origine et b." En vertu du Théorème II, ce-ci équivant à la couple des relation que voici:

$$0a + ba = a, \tag{35}$$

$$a = (0 + a)(b + a). \tag{36}$$

Or, d'après (17), on a $0a = 0$ et, d'après (18), $0 + ba = ba$, de sorte que (35) peut s'écrire $ba = a$. En vertu du Lemme III, ceci équivaut à $a \subset b$. Quant à (36), elle ne fournit aucune restriction, car ce n'est qu'une identité. Pour s'en convaincre, remarquons que, d'après (18), on a $0 + a = a$ de sorte que (36) peut s'écrire $a = a(b + a)$. Or, en vertu du Lemme III, ce-ci équivant à $a \subset b + a$, ce qui est une identité. Ainsi, la couple des relations (35) et (36) équivaut, elle-aussi, à $a \subset b$.

14. Nous pouvons maintenant terminer la démonstration du Théorème I en établissant que tout système de choses normées est un espace métrique *presque ordonné*.

SECONDE PARTIE DE LA DÉMONSTRATION DU THÉORÈME I. Dans l'espace métrique formé par un système de choses normées, prenons 0 pour l'origine et démontrons que les deux axiomes caractérisant l'espace métrique presque ordonné seront vrais pour cet espace.

1°. Quels que soient les deux points a et b et quel que soit le point c se trouvant entre a et b, chaque point plus prochain que a et b est aussi plus prochain que c et chaque point plus lointain que a et b est aussi plus lointain que c.

En effet, soit x un point plus prochain que a et b, c'est-à dire, d'après le Théorème III, que $x \subset a$ et $x \subset b$. Alors, $x \subset a + c$ et $x \subset b + c$, d'où

$$x \subset (a + c)(b + c).$$

Lorsque c se trouve entre a et b et, par suite,

$$(a + c)(b + c) = c,$$

on en conclut que $x \subset c$, c'est-à-dire que x est plus prochain que c. De même, soit y un point plus lointain que a et b, c'est-à-dire, d'après le Théorème III, que $a \subset y$ et $b \subset y$. Alors, $ac \subset y$ et $bc \subset y$, d'où

$$ac + bc \subset y.$$

Lorsque c se trouve entre a et b et, par suite,

$$c = ac + bc,$$

on en conclut que $c \subset y$, c'est-à-dire que y est plus lointain que c.

2°. Quels que soient les deux points a et b, il existe parmi les points que se trouvent entre a et b un point qui est le plus prochain et il en existe un autre qui est le plus lointain.

En effet, ces points sont ab (le plus prochain) et $a + b$ (le plus lointain). Premièrement, ils se trouvent effectivement entre a et b. Pour s'en convaincre, remarquons que, d'une part, d'après (19) et (20), on a

$$\begin{aligned} a(ab) &= ab, \\ b(ab) &= ab, \\ a &= a + ab, \\ b &= b + ab, \end{aligned}$$

d'où l'on déduit tout de suite, en se servant de la formule (6) $a + a = a$, que

$$a(ab) + b(ab) = ab = (a + ab)(b + ab).$$

Ceci nous montre que ab se trouve entre a et b. D'autre part, d'après (21) et (22), on a

$$\begin{aligned} a(a + b) &= a, \\ b(a + b) &= b, \\ a + b &= a + (a + b), \\ a + b &= b + (a + b), \end{aligned}$$

d'où l'on déduit tout de suite, en se servant de la formule (5) $a = aa$, que

$$a(a + b) + b(a + b) = a + b = (a + (a + b))(b + (a + b)).$$

Ceci nous montre que $a + b$ se trouve, lui-aussi, entre a et b.

Deuxièmement, si un point c se trouve entre a et b, ab est plus prochain que c et $a + b$ est plus lointain que c. Autrement dit, en tenant compte du Théorème III,

$$ab \subset c \subset a + b.$$

Pour s'en convaincre, remarquons que, d'après (13) et (14), on a

$$ac + bc \subset (a + b)c \subset c \subset ab + c \subset (a + c)(b + c). \tag{37}$$

Or, lorsque c se trouve entre a et b, on a

$$ac + bc = c = (a + c)(b + c).$$

Il en résulte, en vertu du Lemme IV, que les relations (37) prennent la forme:

$$ac + bc = (a + b)c = c = ab + c = (a + c)(b + c). \tag{38}$$

Mais la trosième et à la deuxième des relations (38) :

$$c = ab + c,$$
$$(a + b)c = c$$

sont en vertu du Lemme III équivalentes à ce que nous avons à démontrer:

$$ab \subset c \subset a + b.$$

Le Théorème I est ainsi complétement établi.

15. Abordons la démonstration du théorème inverse.

THÉORÈME IV. *Tout espace métrique presque ordonné est un système de choses normées où l'expression*

$$|a + b| - |ab|$$

est égale à la distance du point a et du point b.

PREMIÈRE PARTIE DE LA DÉMONSTRATION. Ecrivons, par définition, $a \subset b$ si le point a est plus prochain que le point b, prenons pour ab le plus prochain des points se trouvant entre a et b, pour $a + b$ le plus lointain de ces points et pour 0 l'origine de l'espace, et posons, enfin, $|a| = (0, a)$.

Démontrons que tous les axiomes caractérisant les systèmes de choses y seront remplis.

1°. La relation $a \subset b$ possède les propriétés suivantes:

$a \subset b$ et $b \subset a$ entraîne $a = b$ et inversement:

$a \subset b$ et $b \subset c$ entraîne $a \subset c$.

Démontrons la première de ces propriétés. Les relations $a \subset b$ et $b \subset a$ ne sont autres choses que

$$(0, a) + (a, b) = (0, b),$$
$$(0, b) + (b, a) = (0, a).$$

En additionnant ces égalités et en remarquant qu'on a $(a, b) = (b, a)$, on obtient $(a, b) = 0$, c'est-à-dire que $a = b$. Inversement, si $a = b$, on a $(a, b) = (b, a)$ et $(0, a) = (0, b)$, de sorte que

$$(0, a) + (a, b) = (0, b),$$
$$(0, b) + (b, a) = (0, a),$$

c'est-à dire que $a \subset b$ et $b \subset a$.

Démontrons la deuxième propriété. Les relations $a \subset b$, $b \subset c$ et $a \subset c$ sont équivalentes respectivement aux inégalités:

$$(0, a) + (a, b) \leq (0, b),$$
$$(0, b) + (b, c) \leq (0, c),$$
$$(0, a) + (a, c) \leq (0, c).$$

Admettant les deux premières inégalités, on en déduit la dernière, car

$$(0, a) + (a, c) \leq (0, a) + (a, b) + (b, c) \leq (0, b) + (b, c) \leq (0, c),$$

2°. Le point ab possède les propriétés suivantes:

$$ab \subset a,$$
$$ab \subset b,$$
$$x \subset a \text{ et } x \subset b \text{ entraîne } x \subset ab.$$

La dernière de ces propriétés est évidente, puisque le point ab est, par définition, un des points se trouvant entre a et b. Or, dans l'espace métrique presque ordonné, chaque point x qui est plus prochain que a et b est aussi plus prochain que tout point se trouvant entre a et b. En particulier, x est plus prochain que ab.

Pour démontrer les deux premières propriétés, rappelons que a se trouve toujours entre a et b et que b se trouve aussi entre a et b. Or, ab est, par définition, le point qui est plus prochain que tout autre point se trouvant entre a et b. En particulier, ab est plus prochain que les points a et b eux-mêmes.

3°. Le point $a + b$ possède les propriétés suivantes:

$$a \subset a + b,$$
$$b \subset a + b,$$
$$a \subset y \text{ et } b \subset y \text{ entraîne } a + b \subset y.$$

La démonstration est tout-à-fait analogue à la précédente.

4°. Le point 0 possède la propriété que, quel que soit le point z, on a

$$0 \subset z.$$

En effet, cette propriété n'est autre chose que l'égalité évidente:

$$(0, 0) + (0, z) = (0, z).$$

16. Il nous reste à démontrer qu'avec nos conventions tous les axiomes caractérisant la norme sont remplis et qu'on a, de plus,

$$(a, b) = |a + b| - |ab|.$$

SECONDE PARTIE DE LA DÉMONSTRATION DU THÉORÈME IV. Le fait que $a \subset b$ et $a \neq b$ entraîne $|a| < |b|$, est la conséquence immédiate du fait que

$$(0, a) + (a, b) = (0, b)$$

et $(a, b) > 0$ entraîne

$$(0, a) < (0, b).$$

Démontrons maintenant que

$$|a + b| + |ab| = |a| + |b|.$$

A cet effet, exprimons $|ab|$ à l'aide de $|a|$, de $|b|$ et de (a, b). On a $ab \subset a$ et $ab \subset b$, c'est-à-dire que

$$(0, ab) + (ab, a) = (0, a),$$
$$(0, ab) + (ab, b) = (0, b).$$

En additionnant, on obtient

$$2(0, ab) + (ab, a) + (ab, b) = (0, a) + (0, b).$$

Or, ab se trouve entre a et b, donc

$$(ab, a) + (ab, b) = (a, b).$$

Par suite,

$$2(0, ab) + (a, b) = (0, a) + (0, b),$$

ou, ce qui revient au même,

$$(39) \qquad |ab| = \frac{|a| + |b| - (a, b)}{2}.$$

Pareillement, on établit que

$$(40) \qquad |a + b| = \frac{|a| + |b| + (a, b)}{2}.$$

Remarquons en passant que ce sont les généralisations des expressions connues pour *le plus petit des nombres a et b*,

$$\frac{|a| + |b| - |a - b|}{2},$$

et pour *le plus grand de ces nombres,*

$$\frac{|\,a\,| + |\,b\,| + |\,a - b\,|}{2}.$$

En additionnant les égalités (39) et (40), on obtient ce que nous avions à démontrer, à savoir $|\,a + b\,| + |\,ab\,| = |\,a\,| + |\,b\,|$.

Enfin, le fait que $|\,0\,| = 0$ est la conséquence immédiate de ce que $(0, 0) = 0$.

Quant à l'égalité

$$(a, b) = |\,a + b\,| - |\,ab\,|,$$

elle est, elle-aussi, une conséquence de (39) et de (40).

Le Théorème IV est ainsi complétement établi.

V. Espaces transitifs.

17. Ayant en vue l'étude particulière des systèmes distributifs de choses normées, nous allons introduire une définition nouvelle. Il existe un espèce d'espaces métriques presque ordonnés qui jouent dans la théorie des systèmes distributifs le méme rôle que les espaces métriques presque ordonnés quelconques jouent dans la théorie des systèmes arbitraires de choses normées. Ce sont les espaces métriques presque ordonnés D que nous appelerons *transitifs* et qui possèdent, par définition, la propriété suivante:

T. *Si un point c, de D, se trouve entre x et y et si tous les deux points x et y se trouvent entre a et b, le point c se trouve, lui-aussi, entre a et b.*

Pour avoir un exemple d'espace métrique presque ordonné transitif, il suffit de rappeler l'exemple déjà cité d'un ensemble arbitraire de nombres réels, où $(x, y) = |\,x - y\,|$.

Pour avoir un exemple d'espace métrique presque ordonné qui n'est pas transitif, prenons l'espace formé de cinq points abstraits 0, a, b, c et d, où la distance (x, y) est définie par le Tableau I.

I. (x, y)

$x \backslash y$	0	a	b	c	d
0	0	1	1	1	2
a	1	0	2	2	1
b	1	2	0	2	1
c	1	2	2	0	1
d	2	1	1	1	0

Ici, on a

$$(0, c) + (c, d) = (0, d),$$
$$(a, 0) + (0, b) = (a, b),$$
$$(a, d) + (d, b) = (a, b),$$

de sorte que c se trouve entre 0 et d, que 0 se trouve entre a et b et que d se trouve entre a et b. Cependant, on a

$$(a, c) + (c, b) \neq (a, b),$$

de sorte que c ne se trouve pas entre a et b.

Le fait que cet espace est métrique peut être prouvé par une simple comparaison des distances du Tableau I. Le fait que cet espace est, de plus, presque ordonné s'éclaircira de lui-même dans le Chapitre VII.

VI. Systèmes distributifs.

18. Dans ce qui suit, chaque fois que l'on considérera un espace métrique presque ordonné, les notations du Chapitre IV seront employées, c'est à dire que nous écrirons $a \subset b$ pour exprimer que le point a est plus prochain que le point b, et nous désignerons par ab le plus prochain des points se trouvant entre a et b par $a + b$ le plus lointain de ces points, par 0 l'origine de l'espace et par $| a |$ la distance $(0, a)$. Cette convention faite, nous aurons un théorème qui nous sera utile dans tout ce qui suit.

Théorème V. *La condition nécessaire et suffisante pour qu'un espace métrique presque ordonné D soit transitif est qu'il possède la propriété suivante:*

T bis. *Un point c, de D, se trouve entre deux points a et b si et seulement si l'on a*

$$ab \subset c \subset a + b.$$

Démonstration. Remarquons d'abord que la condition T bis peut s'exprimer plus simplement en y omettant les mots “et seulement si.” En effet, nous savons déjà que, lorsque c se trouve entre a et b, on a nécessairement $ab \subset c \subset a + b$.

Supposons maintenant que l'espace en question est transitif, c'est-à-dire que la condition T est remplie, et soit $ab \subset c \subset a + b$. Nous allons voir que c se trouve alors entre ab et $a + b$. En effet, on a

$$(ab, a + b) = |(a + b) + ab| - |(a + b)(ab)|,$$
$$(ab, c) = | c + ab | - | c(ab)|,$$
$$(c, a + b) = |(a + b) + c| - |(a + b)c|.$$

Mais on a, d'après (25), $ab \subset a + b$, et, d'après notre supposition, $ab \subset c$ et $c \subset a + b$. En vertu du Lemme III, on en conclut que

$$\begin{aligned} (a+b) + ab &= a + b \quad \text{et} \quad (a+b)(ab) = ab, \\ c + ab &= c \quad \text{et} \quad c(ab) = ab, \\ (a+b) + c &= a + b \quad \text{et} \quad (a+b)c = c. \end{aligned}$$

Par suite, on a

$$\begin{aligned} (ab, a+b) &= |a+b| - |ab|, \\ (ab, c) &= |c| - |ab|, \\ (c, a+b) &= |a+b| - |c|, \end{aligned}$$

d'où l'on tire ce qu'il fallait déduire:

$$(ab, a+b) = (ab, c) + (c, a+b).$$

Mais ab et $a + b$ se trouvent entre a et b. Par suite, d'après T, le point c se trouve, lui-aussi, entre a et b. On voit ainsi que, si l'on a $ab \subset c \subset a + b$, le point c se trouve entre a et b, ça veut dire que, d'après la remarque ci-dessus, la condition T bis est remplie.

Inversement, supposons que la condition T bis soit remplie, que c se trouve entre x et y et que x et y se trouvent entre a et b. Alors, nous savons qu'on a

$$\begin{aligned} xy &\subset c \subset x + y, \\ ab &\subset x \subset a + b, \\ ab &\subset y \subset a + b. \end{aligned}$$

Ceci peut s'écrire aussi, en vertu du Lemme I et d'après les formules (5) $aa = a$ et (6) $a + a = a$,

$$\begin{aligned} xy &\subset \quad c \quad \subset x + y, \\ ab &\subset \quad xy \quad \subset a + b, \\ ab &\subset x + y \subset a + b. \end{aligned}$$

On en déduit, en vertu du Lemme VI,

$$ab \subset c \subset a + b.$$

Par suite, d'après T bis, c se trouve entre a et b. On voit ainsi que, si le point c se trouve entre x et y, et que x et y se trouvent entre a et b, le point c se trouve, lui-aussi, entre a et b. C'est la condition T, ça veut dire que l'espace est transitif.

Cela posé, on peut donner une propriété caractéristique des systèmes distributifs.

THÉORÈME VI. *La condition nécessaire et suffisante pour qu'un système de choses normées soit distributif est que l'espace métrique presque ordonné formé par ce système soit transitif.*

DÉMONSTRATION. Supposons d'abord que le système en question soit distributif. Soit, dans l'espace formé par ce système,

$$ab \subset c \subset a + b.$$

Alors, en vertu du Lemme III, on a

$$(a + b)c = c = ab + c.$$

Or, le système étant supposé distributif et en tenant compte du Lemme VII, on a

$$(a + b)c = ac + bc,$$
$$ab + c = (a + c)(b + c).$$

Donc, on a

$$ac + bc = c = (a + c)(b + c).$$

Ceci nous montre que c se trouve entre a et b.

Ainsi, on voit que, si l'on a $ab \subset c \subset a + b$, le point c se trouve entre a et b. La réciproque étant toujours vraie, le théorème V nous apprend que l'espace en question est transitif.

Inversement, supposons que, dans un espace métrique presque ordonné, la condition de distributivité,

$$ac + bc = (a + b)c,$$

n'est pas remplie pour au moins trois points a, b, c et démontrons qu'il existe alors un point x tel que $ab \subset x \subset a + b$ et qui cependant ne se trouve pas entre a et b. En vertu du Théorème V, cela suffira pour affirmer que l'espace n'est pas transitif.

Ainsi, soit

$$ac + bc \neq (a + b)c.$$

On sait toutefois, d'après (13), que

$$ac + bc \subset (a + b)c.$$

Tout espace métrique presque ordonné étant un système de choses normées, il en résulte que

$$|ac + bc| < |(a + b)c|.$$

Or, en vertu de la propriété fondamentale de la norme et d'après la formule (7) $(ac)(bc) = (ab)c$, l'expression $|ac + bc|$ peut être remplacée par $|ac| + |bc| - |(ab)c|$, d'où

$$|ac| + |bc| < |(a+b)c| + |(ab)c|.$$

Comme on a $ac \subset a$ et $a \subset a + b$, puis $bc \subset b$ et $b \subset a + b$, et, enfin, $(ab)c \subset ab$ et $ab \subset a + b$, (la dernière des ces relations étant la formule (25)), on a aussi $ac \subset a + b$, $bc \subset a + b$ et $(ab)c \subset a + b$. Autrement dit, en tenant compte du Lemme III, on a $ac = (a + b)(ac)$, $bc = (a + b)(bc)$ et $(ab)c = (a + b)(ab)c$. Donc, on a

$$|(a+b)(ac)| + |(a+b)(bc)| < |(a+b)c| + |(a+b)(ab)c|.$$

D'après les formules (3) $ab = ba$ et (23) $(ab)c = a(bc)$, ceci peut s'écrire encore comme il suit:

$$|((a+b)c)a| + |((a+b)c)b| < |(a+b)c| + |((a+b)c)(ab)|.$$

En vertu de la propriété fondamentale de la norme, les expressions $|((a+b)c)a|$, $|((a+b)c)b|$ et $|((a+b)c)(ab)|$ peuvent être remplacées respectivement par

$$\begin{gathered}|(a+b)c| + |a| - |(a+b)c + a|,\\ |(a+b)c| + |b| - |(a+b)c + b|,\\ |(a+b)c| + |ab| - |(a+b)c + ab|.\end{gathered}$$

Donc, on a

$$|a| - |(a+b)c + a| + |b| - |(a+b)c + b| < |ab| - |(a+b)c + ab|.$$

Comme on a, d'une part, $ab \subset a$ et $a \subset (a+b)c + a$ et, d'autre part, $ab \subset b$ et $b \subset (a+b)c + b$, on a aussi

$$\begin{gathered}ab \subset (a+b)c + a,\\ ab \subset (a+b)c + b.\end{gathered}$$

Autrement dit, en tenant compte du Lemme III,

$$\begin{gathered}(a+b)c + a = ((a+b)c + a) + ab,\\ (a+b)c + b = ((a+b)c + b) + ab.\end{gathered}$$

Donc, on a

$$\begin{aligned}|a| - |((a+b)c + a) + ab| + |b|&\\ - |((a+b)c + b) + ab| &< |ab| - |(a+b)c + ab|.\end{aligned}$$

D'après les formules (4) $a+b=b+a$ et (24) $(a+b)+c=a+(b+c)$, ceci peut s'écrire encore comme il suit :

$$|a|-|((a+b)c+ab)+a|+|b| \\ -|((a+b)c+ab)+b|<|ab|-|(a+b)c+ab|,$$

ou, ce qui revient au même,

$$|a|+|b|-2|ab|<2|((a+b)c+ab)+a|-|(a+b)c+ab|-|a| \\ +2|((a+b)c+ab)+b|-|(a+b)c+ab|-|b|.$$

En s'appuyant encore une fois sur la propriété fondamentale de la norme, on en obtient :

$$|a+b|-|ab|<|((a+b)c+ab)+a|-|((a+b)c+ab)a| \\ +|((a+b)c+ab)+b|-|((a+b)c+ab)b|.$$

Ceci n'est autre chose que

$$(a,b)<(a,(a+b)c+ab)+((a+b)c+ab,b).$$

On voit donc que, si l'on pose

$$x=(a+b)c+ab,$$

on aura

$$(a,b)<(a,x)+(x,b),$$

c'est-à-dire que x ne se trouve pas entre a et b.

Cependant, en s'appuyant sur la relation (25) $ab\subset a+b$, on verra tout de suite que

$$ab\subset x\subset a+b.$$

VII. Exemple de système non distributif de choses normées.

19. Pour avoir un exemple de système non distributif de choses normées, prenons le système des choses abstraites 0, a, b, c, d, où, par définition, toutes ces cinq choses sont différentes et où l'on a

$$0\subset a,\quad 0\subset b,\quad 0\subset c,\quad 0\subset d, \\ a\subset d,\quad b\subset d,\quad c\subset d.$$

Définissons ensuite xy et $x+y$ par les Tableaux II et III.

II xy

$x \diagdown y$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
d	0	a	b	c	d

III $x+y$

$x \diagdown y$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	d	d	d
b	b	d	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

Posons aussi

$$|0| = 0, \quad |a| = 1, \quad |b| = 1, \quad |c| = 1, \quad |d| = 2.$$

Il est aisé de prouver que ce système-ci est effectivement un système de choses. Si $x \subset y$ et $x \neq y$, on a nécessairement $|x| < |y|$, car on a

$$|0| < |a|, \quad |0| < |b|, \quad |0| < |c|, \quad |0| < |d|,$$
$$|a| < |d|, \quad |b| < |d|, \quad |c| < |d|.$$

La propriété fondamentale de la norme

$$|x+y| + |xy| = |x| + |y|$$

se démontre par un simple calcul dont les résultats sont exposés dans le Tableau IV.

IV $|x+y| + |xy| = |x| + |y|$

$x \diagdown y$	0	a	b	c	d
0	0	1	1	1	2
a	1	2	2	2	3
b	1	2	2	2	3
c	1	2	2	2	3
d	2	3	3	3	4

Enfin, nous avons déjà vu que, par la définition même, $|0| = 0$. Le système en question est donc un système de choses normées.

Ce système n'est pas distributif, car on a

$$(a+b)c = dc = c,$$

tandis que

$$ac + bc = 0 + 0 = 0.$$

En considérant ce système comme espace métrique, on obtient l'espace que nous avons défini, par le Tableau I, dans le Chapitre V. (En vertu du Théorème I, celui-ci est presque ordonné.)

On peut indiquer une interprétation élégante du système en question. A cet effet, prenons pour 0 l'ensemble vide, pour a, b et c trois points d'une droite, et pour d la droite même. Attribuons à $x \subset y$ et à xy le sens usuel, et prenons pour $x + y$ la plus petite des cinq choses nommées embrassant à la fois x et y. Pour $| x |$, nous prenons le nombre de dimensions de x augmenté de 1.

Un exemple analogue nous fournirait le système de tous les sous-espaces linéaires d'un espace projectif donné.

VIII. Espaces métriques presque ordonnés qui sont simplement ordonnés.

20. En introduisant le nom de l'espace métrique *presque ordonné* nous avions en vue, bien entendu, la notion d'ordre $a \prec b$ dont l'expression précise se compose de deux affirmations, savoir $a \subset b$ et $a \neq b$. Il n'est pas dépourvu d'intérêt de se demander quand l'espace métrique presque ordonné est un espace simplement ordonné avec la même notion d'ordre. Autrement dit, quand l'espace en question est tel que, quels que soient ses points a et b, il existe nécessairement entre eux l'une des trois relations $a \prec b$, $b \prec a$ ou $a = b$, ou bien, ce qui revient au même, l'une des deux relations $a \subset b$ ou $b \subset a$. La réponse nous est donné dans le théorème:

THÉORÈME VII. *La condition nécessaire et suffisante pour qu'un espace métrique presque ordonné D soit simplement ordonné est qu'il possède la propriété suivante:*

O. *Un point c, de D, se trouve entre deux points a et b si et seulement si l'on a, ou bien*

$$a \subset c \subset b,$$

ou bien

$$b \subset c \subset a.$$

DÉMONSTRATION. Rappelons que si, dans un espace métrique presque ordonné, on a $a \subset b$, on a aussi, en vertu du Lemme III, $ab = a$ et $a + b = b$.

Supposons maintenant que l'espace en question est ordonné et soit, pour fixer les idées, $a \subset b$ (ce qui correspond à la première des éventualités de la condition O). Soit c un point se trouvant entre a et b. Alors

$$ab \subset c \subset a + b,$$

et par suite, d'après la remarque précédente

$$a \subset c \subset b.$$

Pareillement, on démontre que si l'on a $b \subset a$, on a aussi $b \subset c \subset a$.

Inversement, soit, pour tout point c se trouvant entre a et b,

$$a \subset c \subset b.$$

Alors, les points a et b se trouvant eux-mêmes entre a et b, on a, en particulier,

$$a \subset b.$$

Pareillement, on démontre que si l'on a $b \subset c \subset a$, on a aussi $b \subset a$. On voit donc que la condition O y est remplie, c'est-à-dire que l'espace est ordonné.

Terminons par une remarque concernant les espaces transitifs.

Théorème VIII. *Tout espace métrique presque ordonné qui est simplement ordonné, est nécessairement transitif.*

Démonstration. Supposons que, dans l'espace en question, un point c se trouve entre x et y et que x et y se trouvent entre a et b. Alors, en vertu du Théorème VII, on aura, par exemple,

$$x \subset c \subset y,$$
$$a \subset x \subset b,$$
$$a \subset y \subset b.$$

On en conclut, en vertu du Lemme VI, qu'on aura aussi

$$a \subset c \subset b.$$

Par suite, en vertu du Théorème VII, c se trouvera, lui-aussi, entre a et b. Ainsi, on voit que si le point c se trouve entre x et y et si x et y se trouvent entre a et b, le point c se trouve, lui-aussi, entre a et b. C'est la condition T, c'est-à-dire que l'espace est transitif.

APPENDICE.

Sur les diverses définitions des systèmes de choses.

Plusieurs travaux ont déjà été consacrés à l'étude des systèmes de choses. Outre un Mémoire de R. Dedekind,[3] on peut citer les recherches récentes de MM. K. Menger,[4] Fritz Klein,[5] Garrett Birkhoff [6] et O. Ore.[7] C'est M. Menger qui a proposé le nom System von Dingen. M. Klein l'appelle Verband, M. Birkhoff préfère le nom Lattice, M. Ore le nom Structure.

Au fond, il s'agit toujours de la même chose à quelques détails près. Il est à noter seulement que la forme de la définition du système de choses que j'ai adoptée dans le présent article diffère de celle qui a été adoptée par tous les auteurs cités sauf M. Ore.[8] Tandis que, pour moi et pour M. Ore, le point de départ est la relation $a \subset b$, les autres commencent par les opérations ab et $a + b$. Dans ce dernier cas, on construit le système d'axiomes pour le système de choses comme il suit:

1*. A tout couple de choses a, b correspond une chose déterminée ab telle que

$$(ab)c = a(bc), \quad ab = ba, \quad aa = a.$$

2*. A tout couple de choses a, b correspond une chose déterminée $a + b$ telle que

$$(a + b) + c = a + (b + c), \quad a + b = b + a, \quad a + a = a.$$

3*. $ab = a$ entraîne $a + b = b$, et inversement.

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[3] "Über die von drei Moduln erzeugte Dualgruppe," *Mathematische Annalen*, vol. 53 (1900), pp. 371-403.

[4] "Axiomatik der endlichen Mengen und der elementargeometrischen Verknüpfungsbeziehungen," *Jahr. d. Deutsch. Math.-Ver.*, vol. 37 (1928), pp. 309-325.

[5] "Zur Theorie der abstrakten Verknüpfungen," *Mathematische Annalen*, vol. 105 (1931), pp. 308-323.

[6] "On the combination of subalgebras" et "Applications of lattice algebra," *Proceedings of the Cambridge Philosophical Society*, vol. 23 (1933), pp. 441-464, resp. vol. 30 (1934), pp. 115-122.

[7] "On the foundation of abstract algebra," I, *Annals of Mathematics*, vol. 36 (1935), pp. 406-437.

[8] Dans un article récent, "New foundations of projective and affine geometry," *Annals of Mathematics*, vol. 37 (1936), pp. 456-482, qui a parû après la rédaction de mon mémoire, K. Menger est parti du même point de vue que moi. Ses méthodes diffèrent et certains de ses résultats se dirigent dans une direction différente des miens.

Puis, on introduit la relation $a \subset b$ en la définissant comme équivalente à celle qui figure dans le dernier de ces axiomes, c'est-à-dire à $ab = a$ ou à $a + b = b$.

On y ajoute parfois l'axiome:

4*. Il existe une chose 0 telle qu'on a toujours

$$0 \subset z;$$

et parfois aussi l'axiome:

5*. Il existe une chose 1 telle qu'on a toujours

$$z \subset 1.$$

Il n'y a aucune difficulté à établir l'équivalence des axiomes 1*, 2*, 3*, et des axiomes $1^\circ, 2^\circ, 3^\circ$ du $n^\circ 2$ du présent article.

Les systèmes satisfaisant aux axiomes 1*, 2*, 3* méritent le nom des systèmes de choses dans le sens le plus large. En y ajoutant l'axiome 4*, on obtient précisemment les systèmes de choses dont nous nous avons occupé. Enfin, en y ajoutant l'axiome 5*, on obtient les systèmes de choses dans le sens le plus étroit.

Je veux noter encore que certains raisonnements que nous avons employés, ont été utilisés déjà par les auteurs cités. Ainsi, on trouve chez M. Menger l'ensemble des sous-espaces linéaires d'un espace projectif comme un système de choses normées; et l'on trouve chez M. Birkhoff l'emploi de l'expression $(a + b)c + ab$ dans l'étude des propriétés caractéristiques des systèmes distributifs.

Kliazma, près de
Moscou, U. S. S. R.

A SYMMETRIC REDUCTION OF THE PLANAR THREE-BODY PROBLEM.

By F. D. Murnaghan.

Since the center of mass of a dynamical system, consisting of a collection of particles, in which the only forces are the mutual attractions of the various particles, remains at rest in an inertial reference frame, we shall suppose the center of mass of the three particles, of masses m_1, m_2, m_3, respectively, to be the origin of our inertial frame. The coördinates of the point P_k occupied by m_k being denoted by (x_k, y_k), $k = 1, 2, 3$, this implies the equations

$$(1) \qquad \Sigma\, m_k x_k = 0; \qquad \Sigma\, m_k y_k = 0$$

and the system is accordingly one with four degrees of freedom—there being six coördinates (x_k, y_k) connected by the two relations (1). The kinetic energy of the system is $T = \Sigma\, \frac{1}{2} m_k v_k^2$ where $v_k^2 = (\dot{x}_k^2 + \dot{y}_k^2)$ is the squared velocity of the k-th particle and the relations (1) enable us to readily express T in terms of the relative velocities of the three particles.[1] In fact, on squaring the relation $\Sigma\, m_k \dot{x}_k = 0$ and replacing each product term $\dot{x}_p \dot{x}_q$ by its equivalent $\frac{1}{2}[\dot{x}_p^2 + \dot{x}_q^2 - (\dot{x}_p - \dot{x}_q)^2]$, we find

$$(\Sigma\, m_k)(\Sigma\, m_k \dot{x}_k^2) = \Sigma\, m_p m_q (\dot{x}_p - \dot{x}_q)^2.$$

This implies

$$(2) \qquad 2T = (\Sigma\, m_p m_q v_{pq}^2)/\Sigma\, m_k$$

where $v_{pq}^2 = (\dot{x}_p - \dot{x}_q)^2 + (\dot{y}_p - \dot{y}_q)^2$ is the squared relative velocity of the particles m_p and m_q. We now denote by (a_1, a_2, a_3) and (A_1, A_2, A_3) the sides and angles of the triangle formed by the three particles and by $(\theta_1, \theta_2, \theta_3)$ the inclinations of the sides, whose lengths are (a_1, a_2, a_3) respectively, to the x-axis of our inertial frame (the sides having the sense found by traversing the triangle in the order $1 \to 2 \to 3 \to 1$). It is clear that $\theta_2 - \theta_3 \equiv A_1 \pmod{\pi}$ and hence $\dot{\theta}_2 - \dot{\theta}_3 = \dot{A}_1$; similarly $\dot{\theta}_3 - \dot{\theta}_1 = \dot{A}_2$, $\dot{\theta}_1 - \dot{\theta}_2 = \dot{A}_3$. These equations enable us to express each of the angular velocities $\dot{\theta}_k$ in terms of $\dot{\phi}$ and the $\dot{A}_k$ where $3\phi = \theta_1 + \theta_2 + \theta_3$. For instance

$$3\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 = 3\dot{\theta}_3 + \dot{A}_1 - \dot{A}_2$$

[1] E. J. Routh, *Stability of Motion* (1877), p. 67.

so that $\dot{\theta}_3 = \dot{\phi} + (\dot{A}_2 - \dot{A}_1)/3$. Now the polar coördinates of P_2 relative to P_1 being (a_3, θ_3) we have

$$v_{21}^2 = \dot{a}_3^2 + a_3^2\dot{\theta}_3^2 = \dot{a}_3^2 + a_3^2\{\dot{\phi} + (\dot{A}_2 - \dot{A}_1)/3\}^2.$$

Similarly

$$v_{32}^2 = \dot{a}_1^2 + a_1^2\{\dot{\phi} + (\dot{A}_3 - \dot{A}_2)/3\}^2$$
$$v_{13}^2 = \dot{a}_2^2 + a_2^2\{\dot{\phi} + (\dot{A}_1 - \dot{A}_3)/3\}^2.$$

On substituting these values in the expression (2) for $2T$ we readily find

$$2T = R_2\dot{\phi}^2 + 2R_1\dot{\phi} + R_0$$

where the coefficients (R_2, R_1, R_0) are furnished by the formulae

$$\begin{aligned} \mu R_2 &= \Sigma\, m_2 m_3 a_1^2 \\ \mu R_1 &= \tfrac{1}{3}\Sigma\, m_1(m_3 a_2^2 - m_2 a_3^2)\dot{A}_1 \\ \mu R_2 &= \Sigma\, m_2 m_3 \dot{a}_1^2 + \tfrac{1}{9}\Sigma\, m_2 m_3 a_1^2(\dot{A}_3 - \dot{A}_2)^2 \end{aligned} \tag{3}$$

where $\mu = m_1 + m_2 + m_3$ and the summation sign implies the sum of the term written and the two others obtained from it by cyclic permutation of the labels 1, 2, 3. The coefficients (R_2, R_1, R_0) are easily expressible in terms of the sides (a_1, a_2, a_3) and their time derivatives. In fact the relation $2a_2a_3 \cos A_1 = a_2^2 + a_3^2 - a_1^2$ furnishes (on differentiation with respect to t and making use of the relations $a_2 = a_3 \cos A_1 + a_1 \cos A_3$, etc.) the relations $2\Delta\dot{A}_1 = a_1(\dot{a}_1 - \dot{a}_2 \cos A_3 - \dot{a}_3 \cos A_2)$ etc., where Δ is the area of the triangle $P_1P_2P_3$.

Regarding, now, the three sides (a_1, a_2, a_3) and the angle ϕ as the four coördinates of the dynamical system it is clear that ϕ is an ignorable coördinate with the momentum integral

$$c = \partial T/\partial\dot{\phi} = R_2\dot{\phi} + R_1 = (1/\mu)\Sigma\, m_2 m_3 a_1^2 \dot{\theta}_1. \tag{4}$$

It follows, by a reasoning similar to that already given for the kinetic energy T, that c is the angular momentum of the system about its center of mass. Thus if we multiply $\Sigma\, m_k x_k = 0$ by $\Sigma\, m_k \dot{y}_k = 0$ we find

$$\Sigma\, m_k^2 x_k \dot{y}_k + \Sigma\, m_p m_q(x_p\dot{y}_q + x_q\dot{y}_p) = 0$$

and on replacing $x_p\dot{y}_q + x_q\dot{y}_p$ by the equivalent expression

$$x_p\dot{y}_p + x_q\dot{y}_q - (x_p - x_q)(\dot{y}_p - \dot{y}_q)$$

we find

$$\mu\Sigma\, m_k x_k \dot{y}_k = \Sigma\, m_p m_q(x_p - x_q)(\dot{y}_p - \dot{y}_q).$$

On interchanging the variables x and y and subtracting we see that the angular momentum of the system about the center of mass is the quotient, by the total mass μ, of the expression $\Sigma\, m_p m_q k_{pq}$ where

$$k_{pq} = (x_p - x_q)(\dot{y}_p - \dot{y}_q) - (y_p - y_q)(\dot{x}_p - \dot{x}_q)$$

is the angular momentum (per unit mass) of the particle m_p relative to the particle m_q. In terms of the relative polar coördinates (a, θ), k_{23}, for example, is $a_1{}^2\dot{\theta}_1$ and hence the angular momentum of the system about its center of mass is the constant c of formula (4).

We now make use of the angular momentum integral to reduce the problem to one of three degrees of freedom in which the coördinates are the lengths (a_1, a_2, a_3) of the sides of the triangle formed by the three particles. The potential energy being

$$V = -\gamma\left(\frac{m_2 m_3}{a_1} + \frac{m_3 m_1}{a_2} + \frac{m_1 m_2}{a_3}\right)$$

(where γ is the gravitational constant) the Lagrangian function of the original problem is $L = T - V$ and the modified Lagrangian (in the sense of Routh) is $L^* = L - c\dot{\phi}$. The modified Hamiltonian function is

$$H^* = \left(\Sigma\, \dot{a}_k \frac{\partial L^*}{\partial \dot{a}_k}\right) - L^*$$

and since

$$\frac{\partial L^*}{\partial \dot{a}_k} = \frac{\partial L}{\partial \dot{a}_k} + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \dot{\phi}}{\partial \dot{a}_k} - c\frac{\partial \dot{\phi}}{\partial \dot{a}_k} = \frac{\partial L}{\partial \dot{a}_k} \quad \left(\text{since } \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = c\right)$$

we have

$$H^* = \left(\Sigma\, \dot{a}_k \frac{\partial L}{\partial \dot{a}_k}\right) + c\dot{\phi} - L = \left(\Sigma\, \dot{a}_k \frac{\partial L}{\partial \dot{a}_k}\right) + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L = H.$$

Hence, in order to find the modified Hamiltonian function, we have merely to express the original energy function $H = T + V$ in terms of the modified momenta $(\omega_1, \omega_2, \omega_3) = (\partial L^*/\partial \dot{a}_1, \partial L^*/\partial \dot{a}_2, \partial L^*/\partial \dot{a}_3)$. This is readily done if we observe that $2T$ is the sum of the squared magnitudes of the linear momentum vectors of the three masses each divided by the corresponding mass. The x-component of the linear momentum of the particle m_1 is

$$m_1\dot{x}_1 = \frac{\partial L}{\partial \dot{x}_1} = \left(\Sigma \frac{\partial L}{\partial \dot{a}_k}\,\frac{\partial \dot{a}_k}{\partial \dot{x}_1}\right) + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \dot{\phi}}{\partial \dot{x}_1} = \left(\Sigma\, \omega_k \frac{\partial \dot{a}_k}{\partial \dot{x}_1}\right) + c\frac{\partial \dot{\phi}}{\partial \dot{x}_1}.$$

But $\partial \dot{a}_k/\partial \dot{x}_1 = \partial a_k/\partial x_1$ and so is zero if $k = 1$, whilst if $k = 2$ or 3 it is the

x-direction cosine of the corresponding side of the triangle (with the sense *towards* the vertex P_1). Furthermore the relation

$$a_2{}^2\dot{\theta}_2 = (x_3 - x_1)(\dot{y}_3 - \dot{y}_1) - (y_3 - y_1)(\dot{x}_3 - \dot{x}_1)$$

yields

$$a_2 \frac{\partial \dot{\theta}_2}{\partial \dot{x}_1} = (y_3 - y_1)/a_2; \qquad a_2 \frac{\partial \dot{\theta}_2}{\partial \dot{y}_1} = -(x_3 - x_1)/a_2$$

so that the vector whose components are $(\partial\dot{\theta}_2/\partial\dot{x}_1, \partial\dot{\theta}_2/\partial\dot{y}_1)$ has magnitude $1/a_2$ and is 90° ahead of the side P_3P_1. Similarly for $(\partial\dot{\theta}_3/\partial\dot{x}_1, \partial\dot{\theta}_3/\partial\dot{y}_1)$ whilst $(\partial\dot{\theta}_1/\partial\dot{x}_1, \partial\dot{\theta}_1/\partial\dot{y}_1)$ is the zero vector. Hence the vector $(c\,\partial\dot{\phi}/\partial\dot{x}_1, c\,\partial\dot{\phi}/\partial\dot{y}_1)$ is the sum of two vectors of magnitudes $c/3a_2$ and $c/3a_3$ and 90° ahead of the sides P_3P_1 and P_2P_1 respectively. Hence the linear momentum of the particle m_1 may be analysed into the sum of four vectors: (1) a vector, of magnitude ω_2, along P_3P_1; (2) a vector, of magnitude ω_3, along P_2P_1; (3) a vector, of magnitude $c/3a_2$, 90° ahead of P_3P_1; and (4) a vector, of magnitude $c/3a_3$, 90° ahead of P_2P_1. Hence the squared magnitude of the linear momentum of the particle m_1 is

$$\omega_2{}^2 + \omega_3{}^2 + 2\omega_2\omega_3 \cos A_1 + \frac{c^2}{9}\left(\frac{1}{a_2{}^2} + \frac{1}{a_3{}^2}\right) + \tfrac{2}{9}\frac{c^2}{a_2a_3}\cos A_1 + \tfrac{2}{3}c \sin A_1\left(\frac{\omega_2}{a_3} - \frac{\omega_3}{a_2}\right)$$

and it follows that the Hamiltonian function of the reduced problem is

$$H^* = H = \Sigma \frac{1}{2m_i}\Big[\omega_j{}^2 + \omega_k{}^2 + 2\left(\omega_j\omega_k + \frac{c^2}{9a_ja_k}\right)\cos A_i + \frac{c^2}{9}\left(\frac{1}{a_j{}^2} + \frac{1}{a_k{}^2}\right) + \tfrac{2}{3}\,c \sin A_i\left(\frac{\omega_j}{a_k} - \frac{\omega_k}{a_j}\right)\Big] + V$$

where (i, j, k) is a cyclic arrangement of the labels $(1, 2, 3)$ (i. e. $= (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$).[2]

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[2] References to previous reductions of the plane three body problem may be found in Whittaker, *Enzykl. d. Math. Wiss.*, Bd. 6, 2^1 (Art. 12). In addition the reader is referred to the article by Grammel, *Handbuch der Physik*, Bd. 5, pp. 346-349. The present reduction is a result of a discussion with our colleague, Professor Wintner, who has applied it effectively to the "regularisation" of Levi-Civita.

ON A GENERALIZED TANGENT VECTOR. II.[1]

By H. V. Craig.

1. *Introduction.* In this paper we proceed with certain of the ideas and results given in a previous paper.[2] In particular, we define by means of the tangent vector T_r two metric tensors and develop from each of them a connection. These connections give rise to a scheme of parallel displacement enjoying the following properties: (a) the auto-parallel curves are the extremals associated with F; (b) the scalar product of any two vectors undergoing parallel displacement is constant.

2. *Notation.* In addition to the notation employed in I, we shall introduce the operator ${}^vO_r$ defined by[3]

$$(2.1)\qquad {}^vO_r F = \sum_{u=v}^{m} (-1)^u \, {}_uC_v \, F^{\cdot\cdot(u-v)}_{(u)r}$$

F being any sufficiently differentiable function of the coördinates and their derivatives with respect to a parameter t up to and including order m. The quantities T_r, defined by

$$(2.2)\qquad T_r = -{}^1O_r F,$$

were adopted in I as the components of the tangent vector. Furthermore, wherever it is necessary to stress the fact that the highest order of derivative occurring in F is m, we shall write $F(m)$, and for the associated tangent vector $T_r(m)$.

3. *Some properties of T_r and related vectors.* It is well known that the invariance of $\int F(1)\,dt$ under a parameter change implies the identity

$$(3.1)\qquad x'^r T_r(1) = \mathrm{F}(1),$$

[1] Presented to the American Mathematical Society, October 27, 1934.

[2] *American Journal of Mathematics*, vol. 57 (1935), p. 457. This paper will be referred to as I.

[3] The quantities ${}^vO_r$ were first found by J. L. Synge who established their vector character. See J. L. Synge, "Some intrinsic and derived vectors in a Kawaguchi space," *American Journal of Mathematics*, vol. 57 (1935), p. 679. Soon after the writer encountered them he received from Professor Synge a copy of the abstract of his paper.

since $T_r(1) = F_{(1)r}(1)$. Also, it was shown in I that if $\int F(m)dt$ is invariant under a transformation of parameter, then the following is an identity:

$$(3.2) \qquad x'^r\, T_r(m) = F(m), \qquad (m = 1, 2, \cdots).$$

In addition, by performing the indicated operations, one may verify that

$$(3.3) \qquad x'^r\, {}^1O_r\, F^2(1) = -2\, F^2(1),$$

and that

$$(3.4) \qquad {}^1O_r\, [F(1)'] = -{}^0O_r\, F(1).$$

And thus we are led to investigate the more general expressions

$$x'^r\, {}^1O_r\, F^2(m) \quad \text{and} \quad {}^vO_r[F(m)'].$$

As a first step, we shall demonstrate the formula

$$(3.5) \qquad {}^1O_r\, F^2 = 2 \sum_{w=0}^{m-1} (w+1) F^{(w)} \cdot {}^{w+1}O_r\, F.$$

Now, by virtue of the rule for differentiating a product,

$$(3.6) \qquad (F\, F_{(u)r})^{(u-1)} = \sum_{w=0}^{u-1} {}_{u-1}C_w\, F^{(w)}\, F_{(u)r}^{\cdot\cdot(u-1-w)};$$

also, from (2.1),

$$(3.7) \qquad {}^1O_r\, F^2 = \sum_{u=1}^{m} (-1)^u\, u[(F^2)_{(u)r}]^{(u-1)}.$$

Hence, we may write

$$(3.8) \qquad {}^1O_r\, F^2 = 2 \sum_{u=1}^{m} (-1)^u \cdot u \cdot \sum_{w=0}^{u-1} {}_{u-1}C_w\, F^{(w)}\, F_{(u)r}^{\cdot\cdot(u-1-w)}.$$

If, in this, we first change the order of summation and then replace $u \cdot {}_{u-1}C_w$ with $(w+1) \cdot {}_uC_{w+1}$, we obtain the equalities

$$\begin{aligned}(3.9) \qquad {}^1O_r\, F^2 &= 2 \sum_{w=0}^{m-1} F^{(w)} \sum_{u=w+1}^{m} (-1)^u\, u \cdot {}_{u-1}C_w\, F_{(u)r}^{\cdot\cdot(u-w-1)} \\ &= 2 \sum_{w=0}^{m-1} (w+1) F^{(w)} \sum_{u=w+1}^{m} (-1)^u\, {}_uC_{w+1}\, F_{(u)r}^{\cdot\cdot(u-w-1)} \\ &= 2 \sum_{w=0}^{m-1} (w+1) F^{(w)}\, {}^{w+1}O_r\, F,\end{aligned}$$

and (3.5) is established.

Remark. Examining (3.5), we observe that, if $x'^r \cdot {}^cO_r\, F(m) = -\delta_1{}^c F(m)$, the first of the expressions to be investigated $x'^r\, {}^1O_r\, F^2(m)$ reduces to $-2\, F^2(m)$. As a matter of fact, if $\int F(m)dt$, $(m > 1)$ is invariant x'^r is

normal [4] to each of ${}^{m}O_r F$ and ${}^{0}O_r F$ while $x'^{r}\,{}^{1}O_r$ F, as we know, reduces to $-F$. Thus we are led to consider the following theorem:

THEOREM 3.1. *If $\int F(m)dt$ is invariant under an admissible parameter transformation, then* $x'^{r}\,{}^{c}O_r F = -\delta_1{}^{c} F$, $(c = 0, 1, \cdots, m)$.

Proof. Since

$$L_a = \sum_{u=a}^{m} {}_{u}C_a\, x^{(u-a+1)r} F_{(u)r} = \delta_a{}^{1} F,^{5,6} \tag{3.10}$$

it follows at once that for $c > 1$, L_{a+c} and $\sum_{a=0}^{m-c} (-1)^{a+c}\, {}_{a+c}C_c.\ L_{a+c}{}^{(a)}$ vanish.

Therefore, since the special cases of the theorem $c = 0$, $c = 1$ have been demonstrated, it will suffice to establish that for c larger than one

$$\sum_{a=0}^{m-c} (-1)^{a+c}\, {}_{a+c}C_c\, L_{a+c}{}^{(a)} = x'^{r}\,{}^{c}O_r F \tag{3.11}$$

is an identity.

Dropping the index r from x and F, and employing first the definition of L_a and then the rule for differentiating a product, we obtain the set of equalities

$$\begin{aligned} L_{a+c}{}^{(a)} &= \sum_{u=a+c}^{m} {}_{u}C_{a+c}[x^{(u-a-c+1)} F_{(u)}]^{(a)} \\ &= \sum_{u=a+c}^{m} {}_{u}C_{a+c} \sum_{v=0}^{a} {}_{a}C_v x^{(u-a-c+v+1)} F_{(u)}^{\cdot\cdot(a-v)} \end{aligned} \tag{3.12}$$

and, replacing the umbral index u with $u + a + c$, the relationship

$$L_{a+c}{}^{(a)} = \sum_{u=0}^{m-(a+c)} {}_{u+a+c}C_{a+c} \sum_{v=0}^{a} {}_{a}C_v x^{(u+v+1)} F_{(u+a+c)}^{\cdot\cdot(a-v)}. \tag{3.13}$$

Substituting (3.13) in (3.11), we find that the coefficient of the free x' in the left member of (3.11) is

$$\sum_{a=0}^{m-c} (-1)^{a+c}\, {}_{a+c}C_c F_{(a+c)}{}^{(a)},$$

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[4] See H. V. Craig, "On the solution of the Euler equations for their highest derivatives," *Bulletin of the American Mathematical Society*, vol. 14 (1930), p. 559.

[5] See I, p. 461.

[6] For a more general discussion of the invariance of integrals under a parameter transformation, reference may be made to Théophile de Donder, *Théorie invariantive du calcul des variations*, Paris (1935), pp. 53-57. De Donder's exposition is based on the work of R. Deladrière and J. Géhéniau and treats of multiple integrals of functions containing higher order derivatives. My attention was called to it by a referee.

which, upon replacing a with $a - c$, becomes

$$\sum_{a=c}^{m} (-1)^a \, {}_aC_c F_{(a)}{}^{(a-c)},$$

the coefficient of x' in the right member. Likewise, if we put $v = 0$, $u = 1$ and then $u = 0$, $v = 1$, we obtain the coefficient of x'',

$$\sum_{a=0}^{m-c-1} (-1)^{a+c} \, {}_{a+c}C_c \cdot {}_{1+a+c}C_{a+c} \, {}_aC_0 F_{(a+c+1)}{}^{(a)} + \sum_{a=1}^{m-c} (-1)^{a+c} \, {}_{a+c}C_c \, {}_{a+c}C_{a+c} \, {}_aC_1 F_{(a+c)}{}^{(a-1)}$$

or

$$\sum_{w=0}^{1} \sum_{a=w}^{m-c-1+w} (-1)^{a+c} \, {}_{a+c}C_c \cdot {}_{a+c-w+1}C_{a+c} \cdot {}_aC_w F_{(a+c+1-w)}{}^{(a-w)}.$$

In general, the coefficient of $x^{(e+1)}$ is

$$\sum_{w=0}^{e} \sum_{a=w}^{m-c-e+w} (-1)^{a+c} \, {}_{a+c}C_c \cdot {}_{a+c+e-w}C_{a+c} \cdot {}_aC_w F_{(a+c+e-w)}{}^{(a-w)}$$

or, replacing a with $a + w$,

$$\sum_{w=0}^{e} \sum_{a=0}^{m-c-e} (-1)^{a+w+c} \, {}_{a+w+c}C_c \cdot {}_{a+c+e}C_{a+w+c} \cdot {}_{a+w}C_w F_{(a+c+e)}{}^{(a)}.$$

But, for $e > 0$ the foregoing expression vanishes, since

$$\sum_{w=0}^{e} (-1)^{a+w+c} \frac{(a+w+c)!\,(a+c+e)!\,(a+w)!}{(a+w)!\,c!\,(e-w)!\,(a+w+c)!\,a!\,w!} = \frac{(a+c+e)!}{a!\,c!\,e!} (-1)^{a+c} \sum_{w=0}^{e} (-1)^w \, {}_eC_w,$$

and the theorem is established. We now turn to the generalization of (3. 3).

Theorem 3. 2. *If $\int F(m)\,dt$ is invariant under an admissible parameter transformation, then*

$$x'^r \, {}^1O_r \, F^2(m) = -2F^2(m).$$

This theorem is an immediate consequence of (2. 2), (3. 2), (3. 5), and Theorem 3. 1.

A generalization of Theorem 3. 2 is as follows:

Theorem 3. 3. *If $\int F(m)\,dt$ is invariant under an admissible parameter transformation, then*

$$x'^r \cdot {}^cO_r \, \phi(F) = -[\delta_0{}^c(d\phi/dF)' + \delta_1{}^c d\phi/dF]F \qquad (c = 0, 1, \cdots, m).$$

Proof. By employing the definition of ${}^cO_r$ and the rule for differentiating a product and then changing the order of summation, we derive the set of equalities

$$(3.14) \qquad \begin{aligned} x'^r\, {}^cO_r\, \phi(F) &= x'^r \sum_{u=c}^{m} (-1)^u\, {}_uC_c\, [d\phi/dF\, F_{(u)r}]^{(u-c)} \\ &= x'^r \sum_{u=c}^{m} (-1)^u\, {}_uC_c \sum_{w=0}^{u-c} {}_{u-c}C_w\, (d\phi/dF)^{(w)} F_{(u)r}^{\cdot\cdot(u-c-w)} \\ &= x'^r \sum_{w=0}^{m-c} (d\phi/dF)^{(w)} \sum_{u=w+c}^{m} (-1)^u\, {}_uC_c\, {}_{u-c}C_w\, F_{(u)r}^{\cdot\cdot(u-c-w)}. \end{aligned}$$

But ${}_uC_c \cdot {}_{u-c}C_w = {}_{c+w}C_c \cdot {}_uC_{w+c}$, hence the last member of (3.14) may be expressed in the form

$$x'^r \sum_{w=0}^{m-c} {}_{c+w}C_c (d\phi/dF)^{(w)} \cdot {}^{w+c}O_r\, F,$$

and our proposition follows by way of Theorem 3.1. We now turn to the generalization of the identity (3.4).

THEOREM 3.4. *If $F(m)$ is any function such that ${}^vO_r\, F'$ exists, and if ${}^{-1}O_r\, F$ represents zero, then ${}^vO_r\, F' = -\,{}^{v-1}O_r\, F$, $(v = 0, 1, \cdots, m)$.*

Proof. If we define the symbol $F_{(-1)r}$ to be zero then we may write

$$(3.15) \qquad F' = \sum_{u=0}^{m} x^{(u+1)r} F_{(u)r}; \qquad F'_{(u)r} = F_{(u)r}{}' + F_{(u-1)r}$$

and, dropping the index r,

$$(3.16) \quad {}^vO\, F' = \sum_{u=v}^{m+1} (-1)^u\, {}_uC_v\, F'^{(u-v)}_{(u)} \qquad = \sum_{u=v}^{m+1} (-1)^u {}_uC_v [F_{(u)}{}' + F_{(u-1)}]^{(u-v)}.$$

Evidently, we may replace $m + 1$ with m in the first term of the last member of (3.16), and the index u with $u + 1$ in the second. The result is

$$(3.17) \quad {}^vO\, F' = -\sum_{u=v}^{m} (-1)^u [-\,{}_uC_v + {}_{u+1}C_v] F_{(u)}{}^{(u-v+1)} - (-1)^{v-1} F_{(v-1)};$$

and this, since ${}_{u+1}C_v - {}_uC_v = {}_uC_{v-1}$, reduces to $-\,{}^{v-1}O\, F$.

Another property of the Finsler tangent vector $-\,{}^1O_r\, F(1)$ or $F_{(1)r}$ due to Cartan,[7] may be stated as follows: if $x = x(s, t)$ is a parametric representation of a tube, such that, for each fixed s, $x = x(s, t)$ is an extremal while

[7] See E. Cartan, *Leçons sur les invariants intégraux*, Paris, Hermann (1922), chapters 1 and 18.

for each fixed t, $x = x(s, t)$ is a closed curve and if (varying our notation) $F_{(1)r}(e)$ is the covariant tangent vector to the extremal through the point in question, then the integral $\int F_{(1)r}(e)\,(\partial x^r/\partial s)\,ds$, taken around an s-curve, i. e. a curve on which t is constant, is independent of t. Unfortunately, this property does not carry over in full strength to $F(m)$.

As a first step in the investigation of this matter we shall demonstrate that, if V^r is any vector and $V^{(u)r}$ denotes $d^u V^r/dt^u$, then

$$(3.18)\quad \sum_{w=1}^{m} (-1)^w (V^r \cdot {}^wO_r F)^{(w-1)} = \sum_{v=1}^{m} (-1)^{v-1} \sum_{u=0}^{m-v} V^{(u)r} F_{(u+v)r}^{\,..(v-1)}$$

is an identity. This may be accomplished by comparing the corresponding coefficients of the derivatives of V^r. Thus, suppressing the index r, the coefficient of $V^{(a)}$ in the right member of (3.18), which we shall represent by CR, is given by

$$(3.19)\qquad CR = \sum_{v=1}^{m-a} (-1)^{v-1} F_{(a+v)}^{\,..(v-1)}$$

Whereas, since the left member of (3.18) may be written in the form

$$\sum_{w=1}^{m} (-1)^w \sum_{u=0}^{w-1} {}_{w-1}C_u\, V^{(u)} ({}^wOF)^{(w-1-u)},$$

the corresponding coefficient, CL, is

$$\sum_{w=a+1}^{m} (-1)^w\, {}_{w-1}C_a ({}^wOF)^{(w-1-a)}.$$

Upon expanding ${}^wOF$ in CL and changing the order of summation, we obtain successively

$$(3.20)\qquad CL = \sum_{w=a+1}^{m} (-1)^w\, {}_{w-1}C_a \sum_{v=w}^{m} (-1)^v\, {}_vC_w\, F_{(v)}^{\,.(v-1-a)}$$

$$= \sum_{v=a+1}^{m} \sum_{w=a+1}^{v} (-1)^{w+v}\, {}_{w-1}C_a \cdot {}_vC_w\, F_{(v)}^{\,.(v-1-a)}$$

and, replacing v with $v + a$ and w with $w + a + 1$,

$$(3.21)\qquad CL = \sum_{v=1}^{m-a} \sum_{w=0}^{v-1} (-1)^{w+v+1}\, {}_{w+a}C_a\, {}_{v+a}C_{w+a+1}\, F_{(a+v)}^{\,..(v-1)}.$$

Evidently, $CL = CR$ if

$$\sum_{w=0}^{v-1}(-1)^w\ {}_{w+a}C_a\cdot{}_{v+a}C_{w+a+1}=1 \tag{3.22}$$

or, expressed otherwise, if

$$\sum_{w=0}^{v-1}(-1)^w\ {}_{v-1}C_w/(w+a+1)=a!\,(v-1)!/(v+a)! \tag{3.23}$$

If in the left member of (3.22) we set $a=0$, replace w with $w-1$ and subtract unity there results the expression $-\sum_{w=0}^{v}(-1)^w\ {}_vC_w$ which, as we know, vanishes. Therefore, (3.23) is valid for each integer v if $a=0$, and we proceed by induction. Specifically, we show that if for some a, (3.23) is an identity in V, then (3.23) is an identity in V for $a+1$. Thus, substituting $a+1$ for a in the left member of (3.23), we obtain

$$\sum_{w=0}^{v-1}(-1)^w\ {}_{v-1}C_w/(w+a+2).$$

But, upon replacing w with $w-1$ and performing certain simple operations, this expression may be transformed as follows:

$$\begin{aligned}\sum_{w=0}^{v-1}(-1)^w\ {}_{v-1}C_w/(w+a+2)&=\sum_{w=1}^{v}(-1)^{w-1}\ {}_{v-1}C_{w-1}/(w+a+1)\\ &=-1/v\sum_{w=1}^{v}(-1)^w\ {}_vC_w\,w/(w+a+1)\\ &=-1/v\sum_{w=0}^{v}(-1)^w\ {}_vC_w\left(1-\frac{a+1}{w+a+1}\right)\\ &=\frac{(a+1)}{v}\ \frac{a!\,v!}{(v+a+1)!},\end{aligned} \tag{3.24}$$

and the theorem (3.23) is established.

We shall now apply (3.18) to transform the first variation of $\int Fdt$. Thus, let $x=x(s,t)$ be a surface such that for each fixed s, $x=x(s,t)$ is an extremal and let $J(s)$ represent $\int_{t_1}^{t_2}Fdt$; then the first variation is given by (3.25)

$$dJ/ds=\int_{t_1}^{t_2}\sum_{v=0}^{m}(\partial x^{(v)r}/\partial s)F_{(v)r}dt,\quad\text{where}\quad x^{(v)r}=\partial^v x^r/\partial t^v. \tag{3.25}$$

Now, by the usual integration by parts, (3.25) can be expressed, since ${}^0O_r\,F(e)$ vanishes, in the form

$$(3.26) \qquad dJ/ds = \sum_{v=1}^{m} [(-1)^{v-1} \sum_{u=v}^{m} F_{(u)r}(e)^{(v-1)} \partial x^{(u-v)r}/\partial s]_{t_2}^{t_1}.^{8}$$

And finally, by virtue of (3.18) with $V^r = \partial x^r/\partial s$ and $u - v$ substituted for u, (3.25) takes the form

$$(3.27) \qquad dJ/ds = [T_r \partial x^r/\partial s]_{t_1}^{t_2} + [\sum_{w=2}^{m} (-1)^w (\partial x^r/\partial s \cdot {}^w O_r F(e))^{(w-1)}]_{t_1}^{t_2}.$$

This accomplished, we are ready to consider the generalization of Cartan's theorem.

THEOREM 3.5. *If $x = x(s, t)$ is an extremal tube then*

$$\sum_{w=1}^{m} (-1)^w [\int \{{}^w O_r F(e)\} (\partial x^r/\partial s) ds]^{(w-1)}$$

taken around an s-curve is independent of t.

Proof. The integral of dJ/ds around an s-curve vanishes and the theorem follows.

COROLLARY. *If* $\sum_{w=2}^{m} (-1)^w (\partial x^r/\partial s) \cdot {}^w O_r F)^{(w-1)}$ *vanishes over an extremal tube then* $\int T_r(\partial x^r/\partial s) ds$ *taken around an s-curve is independent of t.*

Now let us consider an extremal surface $x = x(s, t)$, not necessarily a tube, and let u represent s on an s-curve and t on an extremal or t-curve. If, in addition, Fdt is invariant under the transformation $t = t(T)$, the integral around a parameter mesh: $s_1, t_1; s_2, t_2$ of the quantity Sdu,

$$(3.28) \qquad S = \sum_{w=1}^{m} [(\partial x^r/\partial u)(-1)^w \, {}^w O_r F(e)]^{(w-1)},$$

vanishes. For, if $J(s)$ denotes the integral $\int_{t_1}^{t_2} F(s, t) dt$, then, since S reduces to F when u becomes t, the difference $\int_{t_1}^{t_2} S(s_1, t) dt - \int_{t_1}^{t_2} S(s_2, t) dt$ is $J(s_1) - J(s_2)$, which may be written $-\int_{s_1}^{s_2} (dJ/ds) ds$. But dJ/ds is $S(s, t_2) - S(s, t_1)$ with u replaced by s.

This result may be extended to other closed curves: $s = s(u)$, $t = t(u)$ if we replace the notation for the tangent vector $\partial x^r/\partial u$ with dx^r/du,

[8] See J. L. Synge, *loc. cit.*, p. 682, equation 2.7.

$dx^r/du = (\partial x^r/\partial s)ds/du + x'^r dt/du$ and interpret $(dx^r/du)^{(w)}$ to be $[\partial x^{(w)r}/\partial s]ds/du + x^{(w+1)r}dt/du$.

To summarize, we have seen that the vector $T_r(m)$ is in some respects analogous to the covariant tangent vector $T_r(1)$. We shall now turn to the development of a scheme of parallel displacement.

4. *Parallel displacement.* We assume throughout this section that Fdt is invariant under the parameter transformation: $t = t(T)$, $T = T(t)$, and observe that this change of variable induces the following transformation:

$$(4.1)\quad \begin{cases} x(t(T)) = X(T); \qquad dx/dt = (dX/dT)dT/dt; \\ d^m x/dt^m = (d^m X/dT^m)(dT/dt)^m + \cdots; \\ F(x, \cdots, x^{(m)}) = F(X, \cdots, d^m X/dT^m)dT/dt \text{ or } F(x) = F(X)dT/dt; \\ dT/dt\, F_{(m)r}(X) = F_{(m)r}(x) \cdot (dT/dt)^m; \end{cases}$$

and, therefore, $F^{2m-1} F_{(m)r(m)s}$ is invariant. Furthermore, if $t = aT + b$, then

$$(4.2)\qquad F_{(w)r}(X) = F_{(w)r}(x) \cdot (1/a)^{w-1}$$

and consequently $F^{\cdot\cdot(w-1)}_{(w)r}$ transforms by invariance and likewise T_r. Hence, we shall suppose in what follows that the parameter is linearly related to the arc length and adopt the quantities f_{rs},

$$(4.3)\qquad f_{rs} = F^{2m-1} F_{(m)r(m)s} + T_r T_s,$$

as the components of our metric tensor.

Following the procedure outlined in I, we construct an auxiliary connection and then modify it by means of the vectors T_r and S_r. As a first step we differentiate the equation of transformation of the fundamental tensor, thus

$$(4.4)\qquad \bar{f}_{ij}' = f_{rs}' X_i^r X_j^s + f_{rs} X_i^{r\prime} X_j^s + f_{rs} X_i^r X_j^{s\prime}.$$

Then recalling the relationships [9]

$$(4.5)\qquad X^{(m-1)s}_{(m-1)j} = X_j^s; \qquad X^{(m)s}_{(m-1)j} = mX'^s_j = mX_j^{s\prime},$$

and the tensor character of $F_{(m)r}$ and T_r, we derive the transformations:

$$(4.6)\qquad \bar{F}_{(m)i(m-1)j} = F_{(m)r(m-1)s} X_i^r X_j^s + F_{(m)r(m)s} X_i^r m X_j^{s\prime};$$

[9] See I, p. 457.

$$(4.7)\qquad \bar{T}_j' = T_s'X_j^s + T_sX_j^{s\prime};$$

$$(4.8)\qquad \begin{aligned}\bar{F}^{2m-1}\bar{F}_{(m)i(m-1)j} &+ m\bar{T}_i\bar{T}_j' \\ &= (F^{2m-1}F_{(m)r(m-1)s} + mT_rT_s')X_i^rX_j^s + mf_{rs}X_i^rX_j^{s\prime}.\end{aligned}$$

If we permute the indices i and j in (4.8), subtract the result from (4.8) and indicate this operation of permutation and subtraction in the well known way by means of brackets, we may write

$$(4.9)\qquad \begin{aligned}\bar{F}^{2m-1}\bar{F}_{(m)[i(m-1)j]} &+ m\bar{T}_{[i}\bar{T}_{j]}' \\ &= (F^{2m-1}F_{(m)[r(m-1)s]} + mT_{[r}T_{s]}')X_i^rX_j^s + mf_{rs}X^r{}_{[i}X^s{}_{j]}'.\end{aligned}$$

Multiplying this last equation by $1/m$ and subtracting the result from (4.4), we obtain our auxiliary connection.

$$(4.10)\qquad \begin{aligned}\overline{\{i,j\}} &= \{r,s\}X_i^rX_j^s + f_{rs}X_i^{r\prime}X_j^s,\\ 2\{r,s\} &= f_{rs}' - [1/mF^{2m-1}F_{(m)[r(m-1)s]} + T_{[r}T_{s]}'].\end{aligned}$$

Assuming that F is positive, the determinant $|F_{(m)r(m)s}|$ of rank $n-1$ and that the "F one" function of the calculus of variations is non-vanishing, the determinant [10] $|f_{rs}|$ is likewise non-vanishing and the normalized cofactors f^{rs} of $|f_{rs}|$ exist. Consequently, we may raise the second index j in (4.10) by multiplying by $Y_t{}^jf^{tu} = \bar{f}^{jm}X_m{}^u$ thus,

$$(4.11)\qquad X_m{}^u\overline{\left\{{m \atop i}\right\}} = \left\{{u \atop r}\right\}X_i^r + X_i^{u\prime};\qquad \left\{{u \atop r}\right\} = f^{tu}\{r,t\}.$$

This accomplished, the formula for $\theta T^{r\cdots}_{s\cdots}$ and the theorems concerning the derivatives of sums, products, the Kronecker delta, and scalars expressed by contraction of tensors may be obtained as in Finsler geometry.

Furthermore, from the definitions of $\{r,s\}$ and $\left\{{u \atop r}\right\}$, we observe at once that $\{r,s\} + \{s,r\} = f_{rs}'$, $f_{us}\left\{{u \atop r}\right\} = \{r,s\}$ and consequently, by expanding θf_{rs}, we have the relationship

$$(4.12)\qquad \theta f_{rs} = f_{rs}' - f_{us}\left\{{u \atop r}\right\} - f_{ru}\left\{{u \atop s}\right\} = 0.$$

Obviously, if we alter $\{r,s\}$ by adding a skew symmetric tensor, the above equality will hold as before. With this in mind, we introduce our second connection $\{r,s\}^*$, $\left\{{t \atop r}\right\}^*$ defined as follows:

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[10] See H. V. Craig, "On the solution of the Euler equations for their highest derivatives," *Bulletin of the American Mathematical Society*, vol. 36 (1930), p. 562.

$$\begin{aligned} &\{r, s\}^* = \{r, s\} + S_r T_s - T_r S_s, \\ (4.13) \qquad &\{^{\,t}_{r}\}^* = f^{st}\{r, s\}^* = \{^{\,t}_{r}\} + f^{st} T_s S_r - T_r S^t, \\ &(S_r = F_{(0)r} + \sum_{u=2}^{m} (u-1)(-1)^{u-1} F_{(u)r}{}^{(u)} - T_t\{^{\,t}_{r}\}). \end{aligned}$$

Let us next examine the vector $\theta^* T_r$, which is to be computed, as the presence of the asterisk suggests, by means of the new connection. Thus,

$$(4.14) \qquad \theta^* T_r = T_r' - T_t\{^{\,t}_{r}\} - f^{st} T_s T_t S_r + T_r T_t S^t.$$

By virtue of the invariance of Fdt we have the equalities: $x'^r T_r = F$, $x'^r F_{(m)r(m)s} = 0$ and, therefore, $x'^r f_{rs} = FT_s$. Consequently, we may write $x'^t = x'^r f_{rs} f^{st} = FT_s f^{st}$ and, multiplying by T_t, $F = F f^{st} T_s T_t$. That is, anticipating a definition, T_s is a unit vector regardless of the parameter. Hence, since E_r is $\theta T_r - S_r$, $\theta^* T_r$ reduces to $E_r + T_r S_t x'^t / F$. But, again by virtue of the invariance of Fdt, $x'^t E_t$ vanishes [11] and $x'^t S_t$ becomes $x'^t \theta T_t$. Furthermore, due to the vanishing of θf_{rs} and $\theta \delta_s{}^r$, we conclude that θf^{rs} is likewise zero and the following holds:

$$(4.15) \qquad 2(\theta T_r) x'^r / F = 2(\theta T_r) f^{rs} T_s = \theta(f^{rs} T_r T_s) = 0;$$

hence

$$(4.16) \qquad \theta^* T_r = E_r.$$

Also, if the parameter is the generalized arc, then $x'^r f_{rs}$ is T_s and, therefore, if we write E^t for $f^{st} E_s$, it follows that $\theta^* x'^t$ is E^t.

Finally, if we define the generalized magnitude of a vector, angle, and parallel displacement in the usual way, which may be indicated by the following identities and conditional equality

$$(4.17) \qquad |V|^2 = f_{rs} V^r V^s, \quad |U|\,|V| \cos(U, V) = f_{rs} U^r V^s, \quad \theta^* V^r = 0,$$

we have a theory of displacement possessing the cardinal properties of the Synge-Taylor parallelism.[12] Briefly these characteristics are: (a) the magnitudes of and angles between vectors undergoing parallel displacement are constant, and (b) the autoparallel curves are the extremals associated with F.

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[11] See H. V. Craig, "On the solution of the Euler equations for their highest derivatives," *loc. cit.*, p. 560.

[12] Developed independently by J. L. Synge and J. H. Taylor. See J. L. Synge, "A generalization of the Riemannian line element," *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 61-67, and J. H. Taylor, "A generalization of Levi-Civita's parallelism and the Frenet formulas," *ibid.*, pp. 246-264.

Furthermore, relative to the foregoing connections, we may define derivatives [13] (covariant in character) of tensors whose components are functions of $x, x', \cdots, x^{(m)}$ by means of the identities

$$(4.18) \quad S_{,r} = S_{(m-1)r} - mS_{(m)s}\{{}^{s}_{r}\}; \quad T^{r}{}_{s,t} = T^{r}{}_{s(m-1)t} - mT^{r}{}_{s(m)u}\{{}^{u}_{t}\}.$$

A second metric tensor (from which we will develop a theory of parallelism) may be obtained quite naturally from considerations based on the interpretation of metric space given in I. Briefly, this view is as follows: We suppose that there is given a Euclidean $n+1$ space (x^r, z) together with the set of all arcs, $x^r = x^r(t)$ of class C^m lying in the subspace (x^r) and associate with each of these base arcs the warped arc,

$$x^r = x^r(t), \qquad z = \int_0^t \sqrt{F^2 - E_{rs}x'^r x'^s}\, dt + k.$$

The quantities E_{rs} are merely the components of the Euclidean metric tensor and k is a constant. The coördinate systems to be admitted are either rectangular Cartesian or those obtainable from such by transformation of the x's only. Hence, the element of arc is given by the equality

$$(ds/dt)^2 = E_{rs}x'^r x'^s + z'^2 = F^2.$$

By way of illustration let us suppose that we have given the surface $z = z(x^1, x^2)$. Then

$$\begin{aligned}(ds/dt)^2 &= E_{rs}x'^r x'^s + Z_1{}^2(x'^1)^2 + 2Z_1Z_2x'^1x'^2 + Z_2{}^2(x'^2)^2 \\ &= (E_{rs} + Z_rZ_s)x'^r x'^s = F^2, \qquad (Z_1 = \partial z/\partial x^1,\ Z_2 = \partial z/\partial x^2).\end{aligned}$$

It is of course well known that we can study the warped or surface arcs through their corresponding base curves $x^1(t)$, $x^2(t)$ if we replace the Euclidean metric E_{rs} with the Riemannian metric $E_{rs} + Z_rZ_s$. The functions F dealt with in this paper are not necessarily Riemannian and, hence, the warped curves need not lie on a surface.

Returning to the general case, we shall select as parameter the arc length $\bar{s}$ of the warped curve in question, consequently x'^r, z' is the unit tangent vector and F maintains the value unity along this curve. Evidently, if the arc is such that the quantities

[13] See H. V. Craig, "On a covariant differentiation process," *Bulletin of the American Mathematical Society*, vol. 37 (1931), pp. 731-734; also Paper II, *ibid.*, vol. 39 (1933), pp. 919-922.

$$Z_r = (FT_r - E_{ra}x'^a)(F^2 - E_{ab}x'^a x'^b)^{-\frac{1}{2}} \tag{4.19}$$

exist, we may write

$$x'^r x'^s(E_{rs} + Z_r Z_s) = E_{rs}x'^r x'^s + z'^2 = 1, \tag{4.20}$$

and are thus led to define f_{rs}, the metric tensor, to be $E_{rs} + Z_r Z_s$. We note in passing, distinguishing the warped curves from the base curves by means of a wave, that

$$(F^2 - E_{ab}x'^a x'^b)^{\frac{1}{2}} = [1 - E_{ab}dx^a/ds\, dx^b/ds\, (ds/d\tilde{s})^2]^{\frac{1}{2}} = \sin(c, \tilde{c}). \tag{4.21}$$

Furthermore, $x'^s(E_{rs} + Z_r Z_s) = FT_r = T_r$; that is, our generalized tangent vector is the covariant description of the vector x'^r. Similarly, we may raise indices by means of the normalized cofactors f^{rs} whose existence depends upon the non-vanishing of the determinant $|f_{rs}|$, a fact which may be established readily. For, by employing a rectangular Cartesian coördinate system and certain well known rules for evaluating determinants, we have at once

$$|E_{rs} + Z_r Z_s| = 1 + \Sigma(Z_r)^2 \neq 0. \tag{4.22}$$

Now, if C_1 and C_2 are any two intersecting base arcs, we can select the k's so that the associated warped arcs will intersect. And if, further, these are restricted as above the angles of intersection will be given by

$$\cos\theta = [E_{rs} + Z_r(1)Z_s(2)]x'^r(1)x'^s(2), \tag{4.23}$$

the symbols (1), (2) indicating the arcs from which the quantities involved are computed. Thus, in computing angles, the value of the metric tensor is determined by two curves.

To form a connection, we start, as before, by differentiating the equation of transformation of the fundamental tensor, thus:

$$\bar{f}_{ij}' = f_{rs}'X_i^r X_j^s + f_{rs}X_i^{r\prime} X_j^s + f_{rs}X_i^r X_j^{s\prime}. \tag{4.24}$$

Designating $x'^t E_{rt}$ by E_r and differentiating the relationship $\bar{E}_i = E_r X_i^r$ with respect to y^j, we obtain the equation,

$$\bar{E}_{i(0)j} = E_{r(0)s}X_i^r X_j^s + E_{rs}X_i^r X_j^{s\prime} + E_r X^r_{ij}. \tag{4.25}$$

To this we add the equality obtained by multiplying

$$\bar{Z}_j' = Z_s' X_j^s + Z_s X_j^{s\prime} \quad \text{and} \quad \bar{Z}_i = Z_r X_i^r. \tag{4.26}$$

The result is

$$(4.27) \quad \bar{E}_{i(0)j} + \bar{Z}_i \bar{Z}_j{}' = (E_{r(0)s} + Z_r Z_s{}') X_i{}^r X_j{}^s + f_{rs} X_i{}^r X_j{}^{s\prime} + E_r X^r{}_{ij},$$

from which we derive

$$(4.28) \quad \bar{E}_{[i(0)j]} + \bar{Z}_{[i} \bar{Z}_{j]}{}' = (E_{[r(0)s]} + Z_{[r} Z_{s]}{}') X_i{}^r X_j{}^s + f_{rs} X^r{}_{[i} X^s{}_{j]}{}'.$$

Finally, by subtracting (4. 28) from (4. 24) and dividing by two, we obtain a connection

$$(4.29) \quad \overline{\{i, j\}} = \{r, s\} X_i{}^r X_j{}^s + f_{rs} X_i{}^{r\prime} X_j{}^s, \quad 2\{r, s\} = f_{rs}{}' - E_{[r(0)s]} - Z_{[r} Z_{s]}{}',$$

which is such that, in the case of the Riemannian geometry discussed, $\{r, s\}$ is $\{rt, s\} x'^t$.

By means of the procedure applied to equation (4. 10), we could develop a second theory of parallelism having the properties asserted of the first.

TEXAS UNIVERSITY.

THE FUNDAMENTAL THEOREM FOR RIEMANN INTEGRALS.

By E. R. VAN KAMPEN.

1. The fundamental theorem for Riemann integrals states that a function defined on a closed bounded interval has an integral if and only if the function is bounded and its discontinuity points form a 0-set. The proofs of this theorem, as given in the literature,[1] still follow almost exactly the historical development. The upper and lower Darboux sums are defined and the condition that the upper and lower integrals of the function, defined by means of these sums, are equal is transformed into the condition that the discontinuity points of the function form a 0-set. In this note a shorter treatment will be given. This treatment is closely related to the usual treatment in case of a continuous function and is independent of the Darboux sums. The desirability of an investigation of the Riemann integral along these lines was pointed out to me by Wintner. The same method may be applied to n-dimensional Riemann integrals and to Riemann-Stieltjes integrals.

2. A *partition* Δ of the interval $a \leqq x \leqq b$ of the real variable x is a finite number of values $x_0 = a, x_1, \cdots, x_n = b$ such that $x_{i-1} < x_i$, $(i = 1, \cdots, n)$. The segments of the partition Δ are the intervals $x_{i-1} \leqq x \leqq x_i$ of the variable x. The *degree of fineness of* Δ is the maximum of the lengths of the segments of Δ and will be denoted by $d(\Delta)$. An intermediate partition Γ of Δ is a set of n values y_i of the variable x such that $x_{i-1} \leqq y_i \leqq x_i$.

3. Let $\{I_n\}$ be a sequence of open intervals I_n of the variable x. The length of I_n being denoted by $| I_n |$, the length $l\{I_n\}$ (in which the I_n need not be disjoint) will be considered to be the sum of the series $\Sigma | I_n |$, provided this series is convergent. The sequence $\{I_n\}$ is said to cover a set D of real numbers if any number in D is contained in at least one of the intervals I_n. A set D is said to be a 0-*set* if there exists, for any given $\epsilon > 0$, a sequence $\{I_n\}$ covering D such that $l\{I_n\} < \epsilon$. For the easier part of the fundamental theorem use is made of the following well known

[1] Cf. H. Lebesgue, *Leçons sur l'intégration*, (2nd ed., 1928), chap. II; E. Kamke, *Das Lebesgueesche Integral* (1925), § 5; E. W. Hobson, *The Theory of Functions of a Real Variable*, (2nd ed., 1921), chap. VI.

LEMMA. *If $D_1, D_2, \cdots$ is a sequence of 0-sets and D is the set of those numbers contained in at least one D_n, then D is a 0-set.*

In fact, if $\epsilon > 0$ is given, $\{I_n{}^m\}$ is a sequence of open intervals covering D_m, and $l\{I_n{}^m\} < \epsilon/2^m$, then the enumerable collection of open intervals in all $\{I_n{}^m\}$ covers D and its length is less than ϵ.

4. Let $f(x)$ be a real-valued function defined on the interval $a \leqq x \leqq b$, which will be denoted by J. If $f(x)$ is bounded on J, the *oscillation* of $f(x)$ on an interval K, which has at least one point in common with J, is defined as the difference between the least upper bound and the greatest lower bound of $f(x)$ on K, while the *fluctuation* of $f(x)$ at a given point x of J is the greatest lower bound of the oscillations of $f(x)$ on all intervals having x as interior point. Thus $f(x)$ is continuous at x if and only if its fluctuation at x is zero.

5. Let Δ (defined by x_i, $i = 0, \cdots, n$) be a partition of J and let Γ (defined by y_i, $i = 1, \cdots, n$) be an intermediate partition of Δ. The Riemann sum $S(\Delta, \Gamma)$ of $f(x)$ is defined as the sum

$$S(\Delta, \Gamma) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(y_i),$$

and the function $f(x)$ is said to have a Riemann integral if there exists a number S such that one can choose for every $\epsilon > 0$ a $\delta > 0$ which has the property that

$$| S - S(\Delta, \Gamma) | < \delta, \quad \text{whenever} \quad d(\Delta) < \epsilon.$$

FUNDAMENTAL THEOREM. *A function $f(x)$ defined on J has there a Riemann integral if and only if $f(x)$ is bounded on J and the set D of all discontinuity points of $f(x)$ is a 0-set.*

The necessity of these conditions will be proved in 6, their sufficiency in 7, 8, 9.

6. If $f(x)$ is not bounded, it is not bounded on at least one segment of any partition of J. Hence $S(\Delta, \Gamma)$ is, for any fixed Δ, an unbounded function of Γ, and so $f(x)$ cannot have a Riemann integral.

Now let $f(x)$ be bounded. Let the set of points at which the fluctuation of $f(x)$ is more than $1/n$ be denoted by D_n. Then the set D of all discontinuity points of $f(x)$ is the set of points contained in at least one D_n. If D is not a

0-set, it follows from the lemma in 3 that there exists an integer k such that D_k is not a 0-set. Hence there exists a $\gamma > 0$ such that any sequence of open intervals covering D_k has a length larger than γ. But then, if Δ is any partition of J, the total length of those segments of Δ which contain points of D_k is more than γ. Since the oscillation of $f(x)$ in these intervals is more than $1/k$, the oscillation of $S(\Delta, \Gamma)$ as a function of Γ is more than γ/k for any fixed Δ. Hence $f(x)$ cannot have a Riemann integral.

7. Let $f(x)$ be bounded on J, say $|f(x)| < c$, and let the set D of its discontinuity points be a 0-set. Let $\epsilon > 0$ be given, let $\{I_n\}$ be a sequence of open intervals covering D and such that $l\{I_n\} < \epsilon$, and let $\{K_n\}$ be the collection of those open intervals with rational end points on which the oscillation of $f(x)$ is less than ϵ. Since $\{I_n\} + \{K_n\}$ obviously covers J, a finite set covering J can be selected from $\{I_n\} + \{K_n\}$ in view of the Borel covering theorem. This finite set $I_1, \cdots, I_p; K_1, \cdots, K_q$ has then the following properties: The set covers J, the oscillation of $f(x)$ on each of the intervals $K_1, \cdots, K_q$ is less than ϵ, finally $|I_1| + \cdots + |I_p| < \epsilon$. Let δ be a positive number less than half the distance of any two distinct endpoints of the intervals $I_1, \cdots, I_p; K_1, \cdots, K_q$.

8. Let Δ, Δ' be two partitions of J such that $d(\Delta) < \delta$, $d(\Delta') < \delta$ and let Γ, Γ' be arbitrary intermediate partitions of Δ, Δ'. Let the distinct points occurring in Δ and Δ' together be arranged in increasing order and denoted by $x_0 = a, x_1, \cdots, x_n = b$. Then

$$(*) \qquad S(\Delta, \Gamma) - S(\Delta', \Gamma') = \sum_{i=1}^{n} (x_i - x_{i-1})\{f(y_i) - f(y'_i)\},$$

where y_i, y'_i are those points of Γ, Γ' contained in the segments s, s' of Δ, Δ' respectively, and s and s' both contain the set $x_{i-1} \leqq x \leqq x_i$. Since the total length of s and s' together is less than 2δ, it follows from the definition of δ that s and s' both are contained either in *one* of the intervals $I_1, \cdots, I_p$ or in *one* of the intervals $K_1, \cdots, K_q$. Let Σ_1 and Σ_2 denote the sums of those terms on the right side of $(*)$ for which the first and the second of these possibilities occur. The total length of the segments in Σ_1 is less than ϵ, since these segments do not overlap and are all contained in one of the intervals $I_1, \cdots, I_p$. Since $|f(x)| < c$, hence $|f(y_i) - f(y'_i)| < 2c$ it follows that $|\Sigma_1| < 2c\epsilon$. The total length of the segments in Σ_2 is not larger than the length $b - a$ of J. Since for each of the latter segments $|f(y_i) - f(y'_i)| < \epsilon$ in view of the fact that the oscillation of $f(x)$ on any interval J_m is less than ϵ, it follows that $|\Sigma_2| < (b - a)\epsilon$. Thus

$$(**)\quad | S(\Delta, \Gamma) - S(\Delta', \Gamma') | \leqq | \Sigma_1 | + | \Sigma_2 | < (b - a + 2c)\epsilon, \text{ if } d(\Delta) < \delta,\ d(\Delta') < \delta.$$

9. Since $| S(\Delta, \Gamma) | < c(b - a)$, it is possible to select a sequence of partitions Δ_n, Γ_n in such a way that $d(\Delta_n) \to 0$ and that the sequence $S(\Delta_n, \Gamma_n)$ has a limit. On denoting this limit by S and substituting into (**) for Δ', Γ' successively the partitions Δ_n, Γ_n, it follows by passing to the limit that

$$| S - S(\Delta, \Gamma) | \leqq (b - a + 2c)\epsilon, \text{ whenever } d(\Delta) < \delta.$$

This completes the proof of the fundamental theorem.

10. The method in 8 can be used to prove in a simple manner the following fact: If $f(x)$ is defined on J, and $\{I_n\}$ is a sequence of mutually disjoint intervals covering the set D of discontinuity points of $f(x)$, finally O_n is the oscillation of $f(x)$ on I_n if $f(x)$ is bounded on I_n, then O_n is defined for sufficiently large n and $\lim_{n \to \infty} O_n = 0$.

Addendum: After correcting the proofs I noticed that a paper of Professor A. B. Brown, appearing in the last issue of the *American Mathematical Monthly* (vol. 43, 1936, pp. 396-398), contains a treatment of the same subject along lines very similar to those of the present note.

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ON THE CANONICAL TRANSFORMATIONS OF HAMILTONIAN SYSTEMS.

By E. R. van Kampen and Aurel Wintner.

1. *Comparison of the method of the present paper with the usual treatment.* The object of this paper is to present an approach to the theory of canonical transformations. This approach is, in contrast to the usual treatment, one based directly on point transformations of a phase-space into another phase-space and is, therefore, symmetric in the coördinates and impulses. By this approach one can proceed in a manner which is quite straightforward and, in the main, algebraical in nature. In fact, the method in question is the extension to the case of a general canonical transformation of the algebraic treatment previously given by one of the authors [1] for the case of linear conservative canonical transformations. The difference between this particular case and the general case is the same as that between tensor algebra and tensor analysis. Correspondingly, it would be desirable to adapt the local theory of displacements in differential geometry to the group of canonical transformations, conservative or not. There also arises the question whether or not it is possible to imbed Birkhoff's theory of successive normalizations [2] into the tensor-analytical treatment. This question is connected with the problem whether or not it is possible to obtain Birkhoff's two invariants [3] of a non-degenerate fixed-point of an area-preserving surface transformation by an adaptation to canonical transformations of the treatment by means of which the partly topological invariant of the total curvature is obtained in the theory of surfaces by tensor-analytical methods, thus eliminating the use of formal series.

It turns out that the Jacobian matrices of canonical transformations define linear substitutions under which the invariant bilinear form of the complex group is *relative* invariant,[4] and that the multiplier, which will be denoted by s, is independent both of the position in the $2n$-dimensional phase-space and of the time t. In the usual theory, based on Pfaffians, the Jacobian

[1] A. Wintner, "On the linear conservative dynamical systems," *Annali di Matematica*, ser. 4, vol. 13 (1934), pp. 105-112.

[2] G. D. Birkhoff, *Dynamical Systems*, New York (1927), Chap. III.

[3] G. D. Birkhoff, *loc. cit.*, Chap. VIII.

[4] A. Wintner, *loc. cit.*, p. 107.

matrices turn out to define linear substitutions under which the bilinear form of the complex group is an *absolute* invariant. Actually, the invariance is not absolute but relative. In fact, an analysis of the usual treatment [5] reveals a mistake in the latter, since from the fact that two Pfaffians vanish simultaneously one is not justified to conclude that the two Pfaffians are identical, but merely that they are proportional. Correspondingly, the condition $s = \text{const.}$, which in the theory to be presented is *proved* as the integrability condition of a system of partial differential equations, is in the traditional treatment implied by the tacit *hypothesis* $s \equiv 1$. As a matter of fact, s may be any non-vanishing real constant. However, since s turns out to be a non-vanishing *constant,* one can reduce the case of an arbitrary s to the case $s = 1$ by means of a linear conservative transformation which alters the Hamiltonian functions.

The situation with regard to the relative invariance is illustrated by the following point: It is shown [6] in the usual theory by means of Lagrangian brackets that if a canonical transformation is "completely canonical," then the transformation is volume-preserving, the *square* of the Jacobian being 1. Actually, it will be seen without any calculation that if s is assumed to be 1, then, whether the canonical transformation is completely canonical or not, the transformation is always volume-preserving. Furthermore, it follows from a theorem of Frobenius [7] on relative invariants that if $s = 1$, then the Jacobian also is 1, so that the transformation not only is volume-preserving but *orientation-preserving* as well. This generalizes the fact, well-known from the theory of surface transformations in the particular case $n = 1$, that the area-preserving transformations considered there are all orientation-preserving.[8]

2. *Hamiltonian matrices.* For a fixed n, let G denote the $2n$-rowed square matrix

$$(1) \qquad G = \begin{pmatrix} \omega & -\epsilon \\ \epsilon & \omega \end{pmatrix},$$

where ω denotes the n-rowed zero matrix and ϵ the n-rowed unit matrix. Thus

[5] Cf., e. g., G. Prange, "Die allgemeinen Integrationsmethoden der analytischen Mechanik," *Encyklopädie der Mathematischen Wissenschaften,* vol. IV, 1_{II} (1935), p. 757.

[6] Cf. G. Prange, *loc. cit.*, pp. 768-769.

[7] G. Frobenius, "Ueber die schiefe Invariante einer bilinearen oder quadratischen Form," *Journal für reine und angewandte Mathematik,* vol. 86 (1876), pp. 44-71, more particularly p. 48.

[8] G. D. Birkhoff, *loc. cit.*, Chap. VIII. Cf. also T. Levi-Civita and U. Amaldi, *Lezioni di Meccanica Razionale,* vol. II_2, Bologna (1927), p. 318.

(2) $$G = -G^{-1} = -G',$$

where the prime denotes the operation of transposition. Obviously, G is the matrix of what is called in the theory of Pfaffians the bilinear covariant and in algebra the invariant of the usual representation of the complex group, i. e., G is the normal form of an arbitrary non-singular skew-symmetric matrix.

A $2n$-rowed square matrix C will be said to be a Hamiltonian matrix if it is real and satisfies the condition

(3) $$C'GC = sG$$

for some non-vanishing number s, which will be called the multiplier of C. Since $\det G \neq 0$, it is clear from (3) that

(4) $$|\det C| = |s|^n,$$

so that, since $s \neq 0$, the Hamiltonian matrices are non-singular. It is easily verified from (3) that the Hamiltonian matrices form a group and that the multiplier of the Hamiltonian matrix $C^{(2)}C^{(1)}$ is $s^{(2)}s^{(1)}$, if $s^{(i)}$ is the multiplier of the Hamiltonian matrix $C^{(i)}$, where $i = 1, 2$. Since G is a Hamiltonian matrix in view of (2), and since (3) may be written in the form $C' = sGC^{-1}G^{-1}$, it follows that if C is a Hamiltonian matrix, then so is C', and that C and C' have the same multiplier. It is also seen that if $s = 1$, then C and C^{-1} have the same characteristic numbers and the characteristic equation of C is a reciprocal equation.

For an arbitrary non-vanishing real number c, put

(5) $$C_1 = \begin{pmatrix} c\epsilon & \omega \\ \omega & c\epsilon \end{pmatrix}, \qquad C_2 = \begin{pmatrix} c\epsilon & \omega \\ \omega & -c\epsilon \end{pmatrix}.$$

It is easily verified that these two matrices are Hamiltonian matrices and that if s_1, s_2 denote their multipliers, then

(6_1) $$s_1 = c^2, \qquad \det C_1 = s_1{}^n;$$

(6_2) $$s_2 = -c^2, \qquad \det C_2 = s_2{}^n.$$

These examples show that there exist for every n Hamiltonian matrices which have an arbitrary non-vanishing real number as multiplier.

It follows that (4) may be replaced by the more precise relation

(7) $$\det C = s^n.$$

In fact, since the multiplier of $C^{(2)}C^{(1)}$ is the product of the multipliers of

$C^{(2)}$ and $C^{(1)}$, and since there exists for every n and for every $s \neq 0$ a Hamiltonian matrix, it is clear that (7) holds for an arbitrary s if it holds in the case $s = 1$. In other words, it is sufficient to know that if C is a matrix for which $C'GC = G$, then $\det C = 1$. Now that $C'GC = G$ implies $\det C = 1$ is easily seen from (1) in view of a theorem of Frobenius on relative invariants.[9]

3. *A lemma on gradients.* Let an m-rowed square matrix $A = A(Y)$ and a vector $B = B(Y)$ with m components be defined in an m-dimensional domain of the space of the vector $Y = (y_1, \cdots, y_m)$. Suppose that the pair $A(Y)$, $B(Y)$ has the property that one can find for *every* scalar function $f = f(Y)$ another scalar function $g = g(Y)$ such that

$$(8) \qquad B + A \operatorname{grad} f = \operatorname{grad} g$$

holds in the whole Y-domain under consideration. All functions and variables are supposed to be real. Let it be assumed for simplicity that A, B and f, g have continuous partial derivatives of the first and second order respectively. It is understood that $A \operatorname{grad} f$ in (8) denotes a matrix product, while a vector is a matrix with m rows and 1 column.

Now there exists[10] for a given pair A, B a suitably chosen g for *every* f if and only if B is a gradient and A of the form sE, where s is a scalar independent of Y and E denotes the m-rowed unit matrix. The sufficiency of these conditions is obvious from (8). In order to prove their necessity, suppose that there exists a g for every f. On choosing $f = f(Y)$ independent of Y, it is clear from (8) that B is a gradient. Thus (8) implies that $A \operatorname{grad} f$ is a gradient for every f. Hence, on choosing $f = f(y_1, \cdots, y_m)$ as an arbitrary polynomial $p(y_i)$ of the i-th component of $Y = (y_1, \cdots, y_m)$ and denoting by $A_i = A_i(Y)$ the vector the components of which are the elements of the i-th column of $A = A(Y)$, it is seen that the vector $q(y_i)A_i(Y)$ is a gradient for every polynomial $q(y_i)$. It follows, therefore, from the integrability condition satisfied by a gradient that $A(Y)$ is a diagonal matrix and that the i-th diagonal element of $A(Y)$ is a function of y_i alone. Since $A \operatorname{grad} f$ is a gradient for every f, on choosing i and j arbitrarily and placing $f(Y) = y_i y_j$, it follows that all diagonal elements of $A(Y)$ are equal. This proves that $A = sE$ for some constant scalar s.

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[9] As to § 2, cf. A. Wintner, *loc. cit.*, p. 107.

[10] This fact is the analytical analogue to the algebraical fact used by A. Wintner, *loc. cit.*, p. 106.

4. *Canonical transformations.* The prime being the symbol for the transposition, let total differentiation with respect to the time t be denoted by a dot. Partial differentiations with respect to t or a coördinate or an impulse will be denoted by subscripts, as illustrated by the relation $\dot{a}(q, t) = a_q \dot{q} + a_t$. Thus a canonical system with n degrees of freedom may be written in the form

$$(9) \qquad \dot{q}_i = h_{p_i}, \qquad \dot{p}_i = -h_{q_i}, \qquad (i = 1, \cdots, n),$$

where the Hamiltonian function $h = h(q_1, \cdots, q_n, p_1, \cdots, p_n; t)$ may contain t and will be supposed to have continuous partial derivatives of the second order in the $(2n+1)$-dimensional domain under consideration. Put

$$(10) \qquad x_i = q_i, \quad x_{i+n} = p_i, \qquad (i = 1, \cdots, n),$$

and let X denote the vector with the $2n$ components $x_1, \cdots, x_{2n}$. Then (9) may be written in view of (1) in the form

$$(11) \qquad G\dot{X} = \mathrm{grad}_X h, \quad \text{where} \quad h = h(X; t).$$

The subscript of grad refers to the space in which one carries out the partial differentiations.

Now transform the phase-space X for every fixed t under consideration into a phase-space Y by means of a transformation

$$(12) \qquad Y = Y(X; t), \quad \text{where} \quad Y = (y_1, \cdots, y_{2n}), \quad X = (x_1, \cdots, x_{2n}).$$

Suppose that the $2n$ functions (of $2n+1$ variables) which occur in these transformation formulae have continuous partial derivatives of the second order and that the transformation determines for every fixed t a locally one-to-one correspondence between the two phase-spaces, the $2n$-rowed Jacobian being everywhere distinct from zero. A transformation (12) which satisfies these requirements is said to be a canonical transformation if it transforms *every* canonical system (11) into a system of differential equations which is again canonical, i. e., of the form

$$(13) \qquad G\dot{Y} = \mathrm{grad}_Y h^*, \quad \text{where} \quad h^* = h^*(Y; t),$$

it being understood that the new Hamiltonian function h^* need not be the same function as the Hamiltonian function h of (11).

If a canonical transformation (12) is such that h^* always is the same function as h, i. e., if $h^*(Y(X; t); t) = h(X; t)$ holds for every h, and if in

addition t does not occur in (12), then (12) is said to be a completely canonical transformation.[11]

5. *Partial differential equations characterizing canonical transformations.* Let

$$X = X(Y; t) \tag{14}$$

be the inverse of the transformation (12). Let C denote the Jacobian matrix of the transformation (12) for a fixed t, so that the j-th element in the k-th column of the non-singular $2n$-rowed square matrix C is the partial derivative of y_j with respect to x_k. The matrix C may be considered in view of (14) as a function $C(Y; t)$ of Y and t instead of as a function of X and t. It is easily verified by straightforward differentiations that

$$\dot{Y} = C\dot{X} + Y_t \tag{15}$$

and that, for an arbitrary f,

$$\operatorname{grad}_X f = C' \operatorname{grad}_Y f, \tag{16}$$

where C' is the transposed matrix, finally that

$$\operatorname{grad}_Y Y_t = C_t C^{-1}, \tag{17}$$

where A_t denotes the matrix or vector obtained by partial differentiation of every element of A, and the gradient of a vector denotes, of course, a matrix. Needless to say, Y_t is understood in the sense that one first differentiates (12) partially with respect to t and expresses then X by means of (14) as a function of Y and t. It is not assumed that (12) is a canonical transformation.

It is clear from (15) and (16) that (12) transforms (11) into

$$GC^{-1}(\dot{Y} - Y_t) = C \operatorname{grad}_Y h,$$

a relation which may be written in view of (1) in the form

$$G\dot{Y} = GY_t - GCGC' \operatorname{grad}_Y h.$$

On comparing this with (13), it is seen that (12) is a canonical transformation if and only if there exists for every h an $h^* = h^*(Y; t)$ such that

$$GY_t - GCGC' \operatorname{grad}_Y h = \operatorname{grad}_Y h^*, \tag{18}$$

where the Hamiltonian function $h = h(X; t)$ of (11) is thought of as expressed by means of (14) as a function of Y and t. On keeping t fixed and

[11] Cf. T. Levi-Civita and U. Amaldi, *loc. cit.*, p. 314.

comparing then (18) with (8), it follows from the lemma of § 3 that (12) is a canonical transformation if and only if the vector $B = GY_t$ is, for every fixed t, the gradient of a scalar function with respect to Y and the matrix $A = -GCGC'$ is, for every fixed t, of the form sE, where s is a scalar independent of Y and E denotes the $2n$-rowed unit matrix. This means in view of (1) that the transformation (12) is canonical if and only if there exist two scalars

(19₁) $\quad r = r(Y; t);$ $\qquad$ (19₂) $\quad s = s(t)$

such that

(20₁) $\quad GY_t = \mathrm{grad}_Y r;$ $\qquad$ (20₂) $\quad C'GC = sG.$

If this condition is satisfied, the Hamiltonian function of the system (13) into which (11) is transformed by (12) is

(21) $$h^* = sh + r$$

in view of (18).

6. *The integrability conditions of* (20_1). Condition (20_1) cannot be applied directly to a given transformation (12), since the function r is unknown. It is, however, easy to see that there exists, for a given transformation (12), a function (19_1) satisfying (20_1) if and only if the matrix $C'GC$, considered as a function of Y and t, is independent of t.

In fact, there exists, for a given transformation (12), an r satisfying (20_1) if and only if $\mathrm{grad}_Y GY_t$ is a symmetric matrix at every point of the $(2n+1)$-dimensional region under consideration. Now it is easily verified that $\mathrm{grad}_Y GY_t = G\,\mathrm{grad}_Y Y_t$, so that $\mathrm{grad}_Y GY_t = GC_tC^{-1}$ in view of the identity (17). Hence there exists an r if and only if GC_tC^{-1} is a symmetric matrix, i. e., equal to the matrix $(GC_tC^{-1})'$. Now this condition means in view of (2) that

$$C'_tGC + C'GC_t \equiv (C'GC)_t$$

is the zero matrix for every Y and t, i. e., that $C'GC$ is a function of Y alone.

7. *Characterization of the canonical transformation by means of Hamiltonian matrices.* According to (19_2), the scalar s occurring in the necessary and sufficient conditions (20_1), (20_2) is, for every fixed t, independent of the position Y in the phase-space. This was a consequence of the lemma of § 3. Now it is easy to see that the pair of conditions (20_1), (20_2) implies that the scalar s must be independent not only of Y but of t as well. In fact, it will be shown that *a transformation* (12) *is a canonical transformation if and only*

if there exists an $s \neq 0$ *which is independent both of* t *and* Y *and is such that the Jacobian matrix* $C = C(Y; t)$ *of* (12) *is, for every* Y *and* t, *a Hamiltonian matrix which has, in the sense of* § 2, *the constant* s *as multiplier.*[12]

In fact, (12) is canonical if and only if (20_1) and (20_2) can be satisfied by suitably chosen functions (19_1) and (19_2). Now (20_1) is, according to § 6, satisfied by some function (19_1) if and only if $C'GC$ is independent of t, which means in view of (20_2) and (19_2) that s is a constant. This clearly proves the statement, since, the Jacobian being non-singular, $s \neq 0$ in view of (20_2).

The criterion thus proved may be formulated as follows: A transformation (12) is a canonical transformation if and only if its Jacobian matrix $C = C(Y; t)$ determines a linear substitution under which the bilinear form in cogradiant variables which belongs to the skew-symmetric matrix (1) is relative invariant with a multiplier $s \neq 0$ which is independent both of Y and t.

8. *The distortion of a canonical transformation.* If one calls the determinant of $C = C(Y; t)$ the distortion of (12), it is seen from (7) and from the integrability condition $s = \text{const.}$ found in § 6 that the distortion of a canonical transformation is independent both of Y and t.

It is also seen that if $s = 1$, then the distortion is 1, so that a canonical transformation with the multiplier $s = 1$ not only is volume-preserving, as shown by (4), but orientation-preserving as well, as shown by (7).

These results do not assume that t does not occur in (12) and are, therefore, more general than when formulated for the particular case of completely canonical transformations. In fact, $s = 1$ for every completely canonical transformation, while $s = 1$ does not imply that the transformation is completely canonical. This is clear from the relation (21) which shows that a canonical transformation is completely canonical if and only if the multiplier s is 1 and the function (19_1) is independent of Y.

Incidentally, the result $\det C = 1$ is new in the completely canonical subcase of the case $s = 1$ also, since the usual treatment of the completely canonical case gives[13] $\det C = \pm 1$. The proof of this weaker result is usually based on an application of the Lagrangian brackets or, what is the same thing, on the reciprocal matrix of the brackets of Poisson.

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[12] Since the invariant matrix G is real and skew-symmetric, the matrix iG is Hermitian. Thus one might attempt to treat the question of relative invariance with respect to G by starting with the group under which the fundamental surface of an Hermitian space is invariant. This approach would necessitate discussions of reality which are analogous to those treated in the calculus of spinors.

[13] Cf. G. Prange, *loc. cit.*, p. 769.

The usual characterization[14] of the completely canonical transformation by means of the brackets of Lagrange or Poisson follows from (20_2) and (1) without any calculation, since $s = 1$ in the completely canonical case. Furthermore, it is not necessary to assume that the transformation is completely canonical but merely that it is canonical and has the multiplier $s = 1$.

As pointed out in § 1, the usual treatment starts with the implicit *assumption* that $s = 1$, an assumption which implies, by (7), that $\det C = 1$.

9. *The composition rule of canonical transformations.* It is clear from the definition of a canonical transformation that the set of canonical transformations defined in a common $(2n + 1)$-dimensional region forms a group. The composition rule of the Jacobian matrices and of the multipliers of canonical transformations is clear from (20_2) and § 2. As far as the composition rule of the function (19_1) is concerned, it is easily verified that if $r_1 = r_1(Y; t)$ belongs to a canonical transformation which is substituted into another canonical transformation, and if $r_2 = r_2(Y; t)$ and s_2 belong to the latter transformation, then the function r belonging to the composite transformation is $r = s_2 r_1 + r_2$. In particular, if C, r, s belong to a canonical transformation, then C^{-1}, $-r/s$, $1/s$ belong to the inverse transformation.

Needless to say, the multiplier s is, according to (20_2), uniquely determined by the canonical transformation, while the function (19_1) remains, according to (20_1), undetermined up to an additive term which is a function of t alone.

On uniting a canonical transformation of a $2n_1$-dimensional phase-space with a canonical transformation of a $2n_2$-dimensional phase-space, it is often stated that one obtains a canonical transformation of a $(2n_1 + 2n_2)$-dimensional phase-space. Actually, while this is true under the traditional hypothesis $s = 1$, it is false in the general case. In fact, it is clear from the above rules that the united transformation is canonical if and only if $s_1 = s_2$, where s_1 and s_2 denote the multipliers of the two partial transformations.

10. *Another form of the criterion of* § 7. Let it here be mentioned for later application that if (12) satisfies (20_1) for every Y and t and (20_2) for a fixed t and every Y, then (12) is a canonical transformation. In fact, if (20_1) is satisfied for every Y and t, then $C'GC$ is, according to § 6, independent of t, so that (20_2) holds for every Y and t, if it holds for every Y and for a single t.

In order to formulate a converse of the fact thus proved, let a trans-

[14] Cf. G. Prange, *loc. cit.*, pp. 768-769.

formation (12) be termed a conservative transformation if it does not contain t. Now if (12) is a canonical transformation and t_0 denotes any fixed value of t, then the conservative transformation defined by $Y = Y(X; t_0)$ is a canonical transformation. In fact, any conservative transformation satisfies (20_1) by choosing $r \equiv 0$, and it satisfies (20_2) either for every t or for no t.

It may be mentioned that (12) need not be a canonical transformation even if the conservative transformation $Y = Y(X; t_0)$ is a canonical transformation for *every* fixed $t = t_0$. This is clear from the criterion of § 7.

11. *An existence and uniqueness theorem for the initial value problem of canonical transformations.* So far it has been assumed that there is given a transformation (12) and the question was whether or not there exist a function $r = r(Y; t)$ and a constant $s \neq 0$ such that (12) satisfies (20_1) and (20_2). In what follows, the converse problem will be discussed. It will be shown that if there are given a fixed value t_0 of t, a conservative canonical transformation $Y = Y_0(X)$, finally a function $r = r(Y; t)$ which has continuous partial derivatives of the third order, then there exists exactly one canonical transformation (12) which reduces for $t = t_0$ to the given transformation $Y = Y_0(X)$ and belongs to the given scalar function $r = r(Y; t)$. According to § 7, the multiplier of the solution $Y = Y(X; t)$ of this problem is the multiplier of the given initial transformation. It is understood that t is to be restricted to a sufficiently small vicinity of t_0.

The proof proceeds as follows. First, one can write (20_1) with the use of (2) in the form

$$Y_t = - G \operatorname{grad}_Y r(Y; t), \tag{22}$$

where $r = r(Y; t)$ is the given function and $Y = Y(X; t)$ the unknown. Now consider (22) as a system (13) of ordinary differential equations with the independent variable t and assign for these ordinary differential equations the initial values $Y(X; t_0) = Y_0(X)$, where the point X of the phase-space is fixed. Let $Y = Y(X; t)$ be the corresponding solution of (22). According to the existence theorem of ordinary differential equations depending on parameters, the solution $Y = Y(X; t)$ exists in a portion of the space of the initial values X and in a sufficiently small interval $t_0 - b < t < t_0 + b$, where b can be chosen independent of X. Furthermore, the conditions of regularity made with regard to the given functions r, Y_0 imply that the solution $Y = Y(X; t)$ has continuous partial derivatives of the second order. Since the Jacobian of $Y(X; t)$ with respect to X is at $t = t_0$ the Jacobian of $Y_0(X)$, hence distinct from zero, it is clear from reasons of continuity that the Jacobian of $Y(X; t)$ does not vanish, if t is near enough to t_0. Thus (12) defines a

transformation. Now this transformation is, in view of § 10, a canonical transformation.

12. *Canonical integration constants.* Let $Y = Y(X; t)$ be the general solution of the canonical system (13), the $2n$ components of X being the $2n$ integration constants. Let it be assumed that these integration constants are combinations of initial values in such a way that the general solution $Y = Y(X; t)$ has continuous partial derivatives of the second order and that the Jacobian is nowhere zero. The integration constants represented by X are said to be canonical integration constants if the transformation (12) defined by means of the general solution of (13) is a canonical transformation.

Now the integration constants X are canonical integration constants if and only if the conservative transformation $Y = Y(X; t_0)$, where t_0 is one particular value of t, is a canonical transformation. This is clear without any calculation from the criterion of § 10, since (20_1) may be written in the form (22), while (22) is the same thing as (13), the vector X being a vector of integration constants.

It follows, in particular, that if $Y = Y(X; t)$ is the general solution of (13) and the integration constants represented by X are chosen to be the initial values of Y which belong to $t = t_0$, then (12) is a canonical transformation with the multiplier $s = 1$. In fact, (12) reduces then for $t = t_0$ to the identical transformation, hence to a canonical transformation with the multiplier $s = 1$.

If $Y = Y(X; t)$ is the general solution of

$$G\dot{Y} = \operatorname{grad}_Y \bar{h} \tag{23}$$

with canonical integration constants X, then (12) transforms every canonical system (11) into a canonical system (13), where

$$h^* = sh + \bar{h}, \tag{24}$$

s being the multiplier of (12). This is clear from (21), since $r = \bar{h}$ in view of (23) and (20_1). The rule (24) only means (cf. § 9) that the inverse transformation (14) transforms the Hamiltonian function $\bar{h}$ into the Hamiltonian function which is identically zero.

Remark. It may be mentioned that *every* canonical transformation (12) may be considered as one defined by means of some canonical integration constants X of a suitably chosen canonical system (13). This is clear from § 10, if one writes (20_1) in the form (13).

13. *Verification of the standard Pfaffian criterion.* Corresponding to (10), let in (12)

$$y_i = v_i, \qquad y_{i+n} = u_i, \qquad (i = 1, \cdots, n). \tag{25}$$

Then the usual criterion for a canonical transformation,[15] when adjusted to the fact that the constant $s \neq 0$ need not be 1 (cf. § 1), may be expressed by saying that (12) is a canonical transformation if and only if there exists a function r for which the Pfaffian

$$s \sum_{i=1}^{n} p_i dq_i - \sum_{i=1}^{n} u_i dv_i + r dt \tag{26}$$

becomes a complete differential in virtue of (12). This Pfaffian condition contains differentials of some of the $2n$ old variables and of some of the $2n$ new variables and is therefore, in contrast with the criterion of § 7, an unsymmetric criterion. On the other hand, it is easy to deduce this criterion from the theory developed above. In the verification use will be made of the fact that the alternating derivative (Stokesian) of a covariant vector is a tensor.

For a given transformation (12) which need not be canonical, consider the Pfaffian

$$\tfrac{1}{2} s X'GdX - \tfrac{1}{2} Y'GdY + r dt, \tag{27}$$

where $Z'AW$ denotes the bilinear form belonging to the square matrix A and to the vectors Z, W, so that $Z'AW$ is considered in the usual manner as a product of three matrices. It is easily verified from (1) that, whether the transformation (12) is canonical or not, the difference of the two Pfaffians (26), (27) is, in virtue of (1), (10) and (25), a complete differential, namely that of the function

$$s \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} u_i v_i.$$

Now (27) is a complete differential in virtue of (12) if and only if (12) is a canonical transformation. In fact, (27) becomes in virtue of (12) the Pfaffian

$$\tfrac{1}{2}(sX'G - Y'GC) dX + (r - \tfrac{1}{2} Y'GY_t) dt, \tag{28}$$

where Y and Y_t are considered as functions of X and t. It is clear that the Pfaffian (28) is a complete differential of a function of $2n + 1$ variables represented by X and t if and only if (i) the partial derivative of the covariant vector

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[15] G. Prange, *loc. cit.*, p. 758.

(29) $$\tfrac{1}{2}(sX'G - Y'GC)$$

with respect to t is the (covariant) gradient of the scalar

(30) $$r - \tfrac{1}{2}Y'GY_t$$

with respect to the vector X and (ii) the alternating derivative of the covariant vector (29) with respect to the vector X is identically zero. Now the conditions (i), (ii) are equivalent to the conditions (20_1), (20_2) respectively.

In fact, condition (i) is satisfied if and only if

$$-\tfrac{1}{2}(Y'GC)_t = (\mathrm{grad}_X r)' - \tfrac{1}{2}[\mathrm{grad}_X(Y'GY_t)]',$$

a relation which, when transposed, may be written as

$$-\tfrac{1}{2}(C'G'Y)_t = \mathrm{grad}_X r - \tfrac{1}{2}\,\mathrm{grad}_X(Y'GY_t),$$

or, on using (1), (16) and the definition of C, also as

$$\tfrac{1}{2}C'_t GY + \tfrac{1}{2}C'GY_t = C'\,\mathrm{grad}_Y r - \tfrac{1}{2}C'GY_t + \tfrac{1}{2}C'_t GY,$$

i. e., as

$$C'GY_t = C'\,\mathrm{grad}_Y r.$$

Since this may be written in the form (20_1), it follows that condition (i) is equivalent to (20_1).

On the other hand, since an alternating derivative is transformed by any transformation (12) as a tensor, and since the alternating derivative of $X'G$ with respect to X is G in view of (1), it follows that the alternating derivative of $Y'GC$ with respect to X is $C'GC$ and that condition (ii) is satisfied if and only if the matrix

$$\tfrac{1}{2}(sG - C'GC)$$

is identically zero. Hence condition (ii) is equivalent to (20_2).

Remark. While the above considerations were based on the transformations of a $2n$-dimensional phase-space which depend on t, it is possible to develop the above theory in a $(2n+2)$-dimensional space which is obtained by adjoining to the $2n$-dimensional phase-space the function (19_1) and the time t as a pair of canonically conjugate variables. This approach is particularly convenient when the method is applied to the reduction of the degree of freedom by means of canonical transformations based on known first integrals in involution.

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NOTE ON A CERTAIN BILINEAR FORM THAT OCCURS IN STATISTICS.[1]

By Allen T. Craig.

It is the purpose of this note to present some of the properties of the distribution of a real symmetric bilinear form of $2n$ normally but independently distributed variables.

Let x any y be independently distributed in accord with

$$f(x) = (1/\sqrt{2\pi})\exp[-(x^2/2)] \quad \text{and} \quad g(y) = (1/\sqrt{2\pi})\exp[-(y^2/2)].$$

Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be n random pairs of values of x and y and let $A = \| a_{uv} \|$ $(u, v = 1, 2, \cdots, n)$, be a real symmetric matrix of rank r. If B denote the bilinear form $\sum_{u,v=1}^{n} a_{uv}x_u y_v$, the characteristic function $\phi(t)$ of the distribution of B is given by

$$\phi(t) = \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(itB - \tfrac{1}{2}Q_1 - \tfrac{1}{2}Q_2)\, dy_n dx_n \cdots dy_1 dx_1$$

where $Q_1 = \sum_1^n x_u^2$ and $Q_2 = \sum_1^n y_u^2$. Since A is a real symmetric matrix, there exists a real orthogonal matrix $L = \| l_{uv} \|$ such that

$$L'AL = \left\| \begin{array}{cccccc} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & \cdots & \lambda_r & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{array} \right\|$$

where $\lambda_1, \cdots, \lambda_r$ are the r real, non-zero roots of the characteristic equation

[1] Presented to the American Mathematical Society on September 12, 1935.

of A, and L' is the conjugate of L. If we subject the x's and y's to the same linear homogeneous transformation with matrix L, then B becomes $\sum_1^r \lambda_u x'_u y'_u$ while Q_1 and Q_2 remain unchanged. Thus, upon dropping the primes, $\phi(t)$ becomes

$$\phi(t) = \left(\frac{1}{2\pi}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(it \sum_1^r \lambda_u x_u y_u - \tfrac{1}{2}Q_1 - \tfrac{1}{2}Q_2)\, dy_n dx_n \cdots dy_1 dx_1$$
$$= [(1+\lambda_1^2 t^2)(1+\lambda_2^2 t^2)\cdots(1+\lambda_r^2 t^2)]^{-\frac{1}{2}}.$$

If ω_s represents the s-th semi-invariant of the distribution of B, it follows from

$$i^s \omega_s = \left. \frac{d^s \log \phi(t)}{dt^s} \right|_{t=0}$$

that

$$\omega_{2s+1} = 0 \quad \text{and} \quad \omega_{2s} = (2s-1)!\sum_1^r \lambda_u^{2s}.$$

In terms of the semi-invariants, the distribution function $F(B)$ of B is then given formally by

$$\exp\left[\sum_1^\infty \frac{(it)^s}{(s)!}\omega_s\right] = \int_{-\infty}^{\infty} \exp(itB) F(B)\, dB.$$

Thus,

$$F(B) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-it\mathrm{B} + \sum_1^\infty [(it)^s/(s)!]\omega_s\}\, dt$$
$$= \psi(B) + A_2\psi^{(2)}(B) + A_4\psi^{(4)}(B) + \cdots,$$

where

$$\psi(B) = \frac{1}{\sigma\sqrt{2\pi}}\exp[-(B^2/2\sigma^2)], \qquad \sigma^2 = \sum_1^r \lambda_u^2, \qquad \psi^{(s)}(B) = \frac{d^s\psi(B)}{dB^s},$$

and the A's are the well known functions of the semi-invariants.[2]

Of particular interest is the case in which $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 1$ and the rank r is even, say $r = 2k$. For example, B may be n times the sample covariance,

$$B = \sum (x_j - \bar{x})(y_j - \bar{y}) = \sum_{u,v} a_{uv} x_u y_v$$

where $n\bar{x} = \Sigma x_u$, $n\bar{y} = \Sigma y_v$ and $a_{uv} = (n-1)/n$ or $-1/n$ according as $u = v$ or $u \neq v$ respectively. The n roots of the characteristic equation of the matrix $\| a_{uv} \|$ are $0, 1, 1, \cdots, 1$ and, if n is odd, the rank of the matrix is $r = n - 1 = 2k$. Under these conditions we have [3]

[2] Cf. T. N. Thiele, *Theory of Observations* (1903), pp. 33-35.

[3] Cf. J. Wishart and M. S. Bartlett, "The distribution of second order moment statistics in a normal system," *Proceedings of the Cambridge Philosophical Society*, vol. 28 (1931-32), p. 458.

$$\phi(t) = (1 + t^2)^{-k}$$

and

$$F(B) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-itB)}{(1+t^2)^k}\,dt.$$

By a suitably chosen contour for integration in the complex plane, it may be shown that

$$F(B) = \exp(-\mid B \mid)[a_0 + a_1 \mid B \mid + a_2B^2 + a_3 \mid B^3 \mid + \cdots + a_{k-1} \mid B^{k-1} \mid]$$

where

$$a_s = \frac{(2k-s-2)!}{2^{2k-s-1}(k-s-1)!\,s!\,(k-1)!}, \qquad (s = 0, 1, \cdots, k-1).$$

It is interesting to observe [4] that this distribution function is identical with that of the sum of k independent values of x drawn at random from a population characterized by $f(x) = \frac{1}{2}\exp(-\mid x \mid)$.

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[4] Cf. K. Mayr, "Wahrscheinlichkeitsfunktionen und ihre Anwendungen," *Monatshefte für Mathematik und Physik*, Bd. 30 (1920), p. 25.

THE APPLICATION OF THE THEORY OF ADMISSIBLE NUMBERS TO TIME SERIES WITH VARIABLE PROBABILITY.[1]

By Francis Regan.

I. *Introduction.* This paper is concerned with the consistency of the statistical theory of probability as applied to time series. In testing this consistency, we shall employ the theory of admissible numbers. A time series is a sequence of occurrences distributed independently in such a manner that there is a definite probability of an occurrence in any given interval of time. A time series may be represented by a sequence of points on the positive time axis. If E is any (Lebesgue measurable) set of points on this axis, then there is a definite probability $f(\alpha, E)$ that there will be α points of the time series in the set E. We shall assume that this function possesses a certain periodicity which is defined in terms of the following transformation. Let $T_y(x) = x + y$, i.e., the transformation $T_y(x)$ translates a point x into a point $x' = x + y$. This operation also transforms a set of points E into a set E' and this transformation will be denoted by the equation $E' = T_y(E)$. The function $f(\alpha, E)$ is said to possess a period A provided

$$f[\alpha, T_A(E)] = f(\alpha, E)$$

where A is independent of α. If all numbers A are periods of this function, the time series is said to possess a constant probability. Otherwise the probability will be said to be variable.

The mathematical idealization of a time series is an increasing sequence of positive numbers $s_1, s_2, \cdots$ whose properties are defined in terms of the expression $x(\alpha, E, A)$ where

$$x(\alpha, E, A) = x^{(1)}, x^{(2)}, x^{(3)}, \cdots, x^{(k)}, \cdots$$

$$x^{(k)} = \begin{cases} 1 & \text{if there are exactly } \alpha \text{ points of the series in } T_{(k-1)A}(E) \\ 0 & \text{otherwise.} \end{cases}$$

The set E is contained in the interval $0 < y \leq A$ and A is a period of $f(\alpha, E)$. Thus $x(\alpha, E, A)$ represents an event which succeeds on its k-th trial if and only if there are exactly α points of the time series belonging to the set

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[1] This paper was presented to the Society June 22, 1933.

$T_{(k-1)A}(E)$. The success ratio for the first n trials of the event $x(\alpha, E, A)$ is denoted by $p_n[x(\alpha, E, A)]$ and is given by the equation

$$p_n[x(\alpha, E, A)] = \sum_{k=1}^{n} x^{(k)}/n.$$

The probability $p[x(\alpha, E, A)]$ of this event is the limit of the success ratio, i. e.,

$$p[x(\alpha, E, A)] = \lim_{n\to\infty} p_n[x(\alpha, E, A)].$$

Thus the time series must satisfy the condition $p[x(\alpha, E, A)] = f(\alpha, E)$. We shall make the further demand that $x(\alpha, E, A)$ is an element of the set $A[f(\alpha, E)]$, where $A[f(\alpha, E)]$ is the set of all admissible numbers associated with the probability $f(\alpha, E)$. We shall however restrict ourselves in Art. II to the consideration of sets E consisting of finite sums of intervals and to the function of variable probability while in § 4 of Art. III, we shall consider for constant probability sets E which are not of the form already discussed by the author.[2]

It is now clear that the assumption of periodicity of $f(\alpha, E)$ is necessary in order that the probability can have a meaning in terms of the statistical theory. It can be proved that

$$f(\alpha, E) = [K(E)]^{\alpha} e^{-K(E)}/\alpha!$$

where $K(E)$ is a non-negative absolutely additive, absolutely continuous function of point sets.[3]

II. *At least one point or exactly α points lying in an interval.*

1. In this section we shall concern ourselves with the restricted case in which E is the interval of length η, beginning at time s. The function $f(\alpha, E)$ shall be designated by $f(\alpha, \eta, s)$ and $K(E)$ shall be denoted by $K(\eta, s)$. The function $f(\alpha, \eta, s)$ is periodic in s, where the period depends upon the nature of the physical event; for example, let us assume that the probability of an

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[2] See Regan, "The application of the theory of admissible numbers to time series with constant probability," *Transactions of the American Mathematical Society*, vol. 36 (July, 1934), pp. 511-529.

[3] Fry, *Probability and Its Engineering Uses*, pp. 216-227 and pp. 232-235, shows that under suitable restrictions the function $f(\alpha, E)$ must be expressible in the above form where $K(E)$ is a Riemann integral of a continuous function. In a paper written in conjunction with A. H. Copeland, entitled "A postutational treatment of the Poisson law," *Annals of Mathematics*, vol. 37, no. 2, April, 1936, pp. 357-362, the above equation under a more general set of hypotheses has been derived by the authors.

event happening at 9.00 a.m. Monday is the same as that at 9.00 a.m. Tuesday, etc. Here the primitive period is the day. Hence $f(\alpha, \eta, s)$ is periodic in s for a certain η. Let this period be m, then $f(\alpha, \eta, s) = f(\alpha, \eta, s + mn)$, $(n = 1, 2, \cdots)$. If this period is not one it may be made so by a linear transformation. Therefore there will be no loss in generality, if we confine our work to the case where the period is one.

We shall first consider the case of at least one point lying in an interval of length η, beginning at time s. A time series representing this phenomenon may be represented by a set of points $s_1 < s_2 < s_3 < \cdots < s_i < \cdots$ distributed along the positive s-axis. The probability of at least one point lying in any interval η of the time axis, beginning at time s, is $[1 - f(0, \eta, s)]$.

The set of points s_i may be obtained by a certain transformation applied to a time series $\tau_1, \tau_2, \tau_3, \cdots$ which has a constant probability, and which is such that if

$$\sim x_0(0, \tau, t, \Lambda) = x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(k)}, \cdots$$

where

$x_0^{(k)} = 1$ if there is at least one point τ_i in

$$I_k\colon\ t + (k-1)\Lambda < h \leqq t + \tau + (k-1)\Lambda$$

otherwise

$$x_0^{(k)} = 0,$$

where $\Lambda = \delta \cdot \rho \cdot 2^{-\sigma+1}$, $t = \delta \cdot r \cdot 2^{-\sigma+1}$, $\tau = \delta \cdot m \cdot 2^{-\sigma+1}$; ρ, σ, r and m being integers such that $r + m \leqq \rho$, and $\delta = K(1, 0)$, then $\sim x_0(0, \tau, t, \Lambda)$ is an element of $A[1 - e^{-\tau}]$.[4]

For our purpose here we will only consider those values of $\rho \cdot 2^{-\sigma+1}$ which are integers. Hence $\Lambda = \delta \cdot M$, where M is an integer.

It is now our problem to show how this series may be transformed into one which has a variable probability and which is consistent with the frequency theory.

Let $I_{\eta,s}$ be the interval $s < \xi \leqq s + \eta$. Let $K(\eta, 0) = T(\eta) = \lambda$. It is assumed that $T(\eta + 1) = T(\eta) + \delta$. From the nature of the function $K(\eta, s)$, we know that the inverse $\eta = T^{-1}(\lambda)$ exists.[5] Then $T^{-1}(\lambda + \delta) = T^{-1}(\lambda) + 1$.

[4] A time series of this type has been constructed by the author in a paper entitled "The application of the theory of admissible numbers to time series with constant probability," *Transactions of the American Mathematical Society*, vol. 36 (July, 1934), no. 3, pp. 511-529. It should be noted that the constant factor δ that has been used here is nothing more than the k (page 512) which we chose as one, but for our purpose here k is chosen to be $\delta = K(1, 0)$.

[5] In the paper mentioned above written by Copeland and Regan, they show that if I_x is the interval $0 < y \leqq x$ and $t = K(I_k) = T(x)$, then this transformation has a unique inverse $x = T^{-1}(t)$.

We shall now construct a time series $s_1 < s_2 < \cdots < s_i < \cdots$ such that $s_i = T^{-1}(\tau_i)$. Let

$$\sim x(0, \eta, s, M) = x^{(1)}, x^{(2)}, \cdots, x^{(k)}, \cdots$$

where

$x^{(k)} = 1$ if there is at least one point in

$$I_k \colon s + (k-1)M < h \leq s + \eta + (k-1)M$$

otherwise

$$x^{(k)} = 0,$$

where $s = T^{-1}(t)$, $s + \eta = T^{-1}(t + \tau)$, t and τ being equal to $\delta \cdot r \cdot 2^{-\sigma+1}$ and $\delta \cdot m \cdot 2^{-\sigma+1}$ respectively. Since

$$T^{-1}[t + (k-1)\Lambda] = T^{-1}(t) + (k-1)M = s + (k-1)M$$

and

$$T^{-1}[t + \tau + (k-1)\Lambda] = T^{-1}(t + \tau) + (k-1)M = s + \eta + (k-1)M$$

it follows that $x^{(k)} = x_0{}^{(k)}$ and hence $\sim x(0, \eta, s, M)$ is an element of $A[1 - e^{-\tau}]$, but $t = T(s)$ and $t + \tau = T(s + \eta)$ or

$$\tau = T(s + \eta) - T(s) = K(\eta, s).$$

Therefore $\sim x(0, \eta, s, M)$ is an element of $A[1 - e^{-K(\eta, s)}]$ and

$$s_1 < s_2 < \cdots < s_i < \cdots$$

is a time series with the desired properties.

2. We shall investigate the case when within the interval I_k of length $T^{-1}(\tau = \delta \cdot m \cdot 2^{-\sigma+1})$, sub-intervals of the form $T^{-1}(\delta \cdot 2^{-\sigma+1})$ are omitted from consideration.

In the case of constant probability we have shown that if

$$\sim x(0, \tau', t, \Lambda) = x_1{}^{(1)}, x_1{}^{(2)}, \cdots, x_1{}^{(k)}, \cdots$$

where $x_1{}^{(k)}$ is one if there is at least one point in the n intervals of I_k, where the sub-intervals of I_k are:

$$t + t_i + (k-1)\Lambda < h \leq t + t_i + \delta \cdot 2^{-\sigma+1} + (k-1)\Lambda, \qquad (i = 1, 2, \cdots, n),$$

then $\sim x(0, \tau', t, \Lambda)$ is an element of $A[1 - e^{-\tau'}]$, where

$$\tau' = \tau'_1 + \tau'_2 + \cdots + \tau'_n, \qquad \tau'_i = \delta \cdot 2^{-\sigma+1},$$

beginning at time t_i. The numbers Λ, t, τ' and τ are $\delta \cdot M$, $\delta \cdot r \cdot 2^{-\sigma+1}$, $\delta \cdot n \cdot 2^{-\sigma+1}$ and $\delta \cdot m \cdot 2^{-\sigma+1}$ respectively where ρ, σ, r, M, m and n are integers with $n \leq m$ and the sub-intervals begin at $t_i = \delta \cdot \rho_i \cdot 2^{-\sigma+1}$, $(i = 1, 2, \cdots, n)$, where ρ_i is an integer, satisfying, $r \leq \rho_i \leq r + m - 1$.

Now we may use the same transformation as given in § 1. Let

$$\sim x(0, \eta', s, M) = x^{(1)}, x^{(2)}, \cdots, x^{(k)}, \cdots [6]$$

where $x^{(k)}$ is one if there is at least one point in the n intervals of I_k, where the sub-intervals are:

$$s + t'_i + (k-1)M < h \leq s + t'_i + T^{-1}(\delta 2^{-\sigma+1}) + (k-1)M, \qquad (i = 1, 2, \cdots, n)$$

and zero otherwise, where $s = T^{-1}(t)$, $(t = \delta \cdot r \cdot 2^{-\sigma+1})$, $t'_i = T^{-1}(\delta \cdot \rho_i \cdot 2^{-\sigma+1})$, $s + \eta = T^{-1}(t + \tau)$, where $s \leq t'_i \leq s + \eta - T^{-1}(\delta \cdot 2^{-\sigma+1})$. Since

$$\begin{aligned} &T^{-1}[t + t_i + (k-1)\Lambda] \\ &\qquad = T^{-1}(t) + T^{-1}(t_i) + (k-1)M = s + t'_i + (k-1)M \end{aligned}$$

and

$$\begin{aligned} &T^{-1}[t + t_i + \delta \cdot 2^{-\sigma+1} + (k-1)\Lambda] \\ &\qquad = T^{-1}(t) + T^{-1}(t_i) + T^{-1}(\delta \cdot 2^{-\sigma+1}) + (k-1)M \\ &\qquad = s + t'_i + T^{-1}(\delta \cdot 2^{-\sigma+1}) + (k-1)M, \end{aligned}$$

it follows that $x^{(k)} = x_1^{(k)}$ and hence $\sim x(0, \eta', s, M)$ is an element of $A[1 - e^{-\tau'}]$, where $\tau' = \tau'_1 + \tau'_2 + \cdots + \tau'_n$, with each of the $\tau_i = \delta \cdot 2^{-\sigma+1}$ beginning at time t_i. Therefore, since $t_i = T(t'_i)$ and

$$t_i + \delta 2^{-\sigma+1} = T[t'_i + T^{-1}(\delta 2^{-\sigma+1})],$$

we have

$$\delta \cdot 2^{-\sigma+1} = T[t'_i + T^{-1}(\delta \cdot 2^{-\sigma+1}) - T(t'_i)] = K[T^{-1}(\delta 2^{-\sigma+1}), t'_i].$$

Therefore, $\sim x(0, \eta', s, M)$ is an element of

$$A[1 - \exp\{-\sum_{i=1}^{n} K[T^{-1}(\delta 2^{-\sigma+1}), t'_i]\}].$$

3. When we consider α points in an interval of length η, or the case

[6] The symbol η' represents the sum of the n sub-intervals $T^{-1}(\delta \cdot 2^{-\sigma+1})$ under consideration.

where there are α points in an interval where sub-intervals of the form $T^{-1}(\delta \cdot 2^{-\sigma+1})$ have been omitted from consideration, we need only apply §§ 1 and 2 to the corresponding time series of constant probability.

III. *Intervals not of the form* $(r + m) \cdot 2^{-\sigma+1}$.

4. We shall discuss for constant probability the case in which the interval becomes irrational or rational and not of the form $(r + m) \cdot 2^{-\sigma+1}$. It may be such that t and τ are both irrational, both rational and not of the forms $r \cdot 2^{-\sigma+1}$, $m \cdot 2^{-\sigma+1}$, or one may be rational and the other irrational.

Let us choose r_1, r_2, m_1, m_2, and Λ so that $r_1 \cdot 2^{-\sigma+1} < t < r_2 \cdot 2^{-\sigma+1}$, $(r_2 + m_2) \cdot 2^{-\sigma+1} < t + \tau < (r_1 + m_1) \cdot 2^{-\sigma+1}$, and $\Lambda = \rho \cdot 2^{-\sigma+1}$, where $r_1 + m_1 \leq \rho$ and $r_2 + m_2 \leq \rho$. Let $\tau_1 = m_1 \cdot 2^{-\sigma+1}$, $\tau_2 = m_2 \cdot 2^{-\sigma+1}$, $t_1 = r_1 \cdot 2^{-\sigma+1}$, $t_2 = r_2 \cdot 2^{-\sigma+1}$.

Given any positive number, ϵ, we can select r_1, r_2, ρ_1, ρ_2, σ such that

$$\text{(a)} \quad \{1 - f(0, \tau_1, t_1)\}^k - \epsilon/2 \leq \{1 - f(0, \tau, t)\}^k \leq \{1 - f(0, \tau_2, t_2)\}^k + \epsilon/2.$$

It follows from Theorem 2 [7] that the numbers N_s can be chosen so that for every m, r, σ and Λ such that $m + r \leq \Lambda 2^{\sigma-1}$ that the number $\sim x(0, \tau_1, t_1, \Lambda)$ is a member of the set $A[1 - f(0, \tau_1, t_1)]$. For the same numbers N_s, then the number $\sim x(0, \tau_2, t_2, \Lambda)$ is a member of the set $A[1 - f(0, \tau_2, t_2)]$. Hence we can select a number N_0 such that

$$\text{(b)} \quad \{1 - f(0, \tau_2, t_2)\}^k - \epsilon/2 \leq p_N[\prod_{i=1}^{k} (r_i/n) \sim x(0, \tau_2, t_2, \Lambda) \cdot],$$

and

$$\text{(c)} \quad p_N[\prod_{i=1}^{k} (r_i/n) \sim x(0, \tau_1, t_1, \Lambda) \cdot] \leq \{1 - f(0, \tau_1, t_1)\}^k + \epsilon/2$$

whenever $N > N_0$.

Now let us define a number $\sim x(0, \tau, t, \Lambda)$ such that its k-th digit is one if there exists at least one point of the time series in the interval I_k: $t + (k-1)\Lambda < h \leq t + \tau + (k-1)\Lambda$, where t and τ have been restricted as above and $\Lambda = \rho \cdot 2^{-\sigma+1}$. Then we have

$$\text{(d)} \quad p_N[\prod_{i=1}^{k} (r_i/n) \sim x(0, \tau_2, t_2, \Lambda) \cdot] \leq p_N[\prod_{i=1}^{k} (r_i/n) \sim x(0, \tau, t, \Lambda) \cdot] \leq p_N[\prod_{i=1}^{k} (r_i/n) \sim x(0, \tau_1, t_1, \Lambda) \cdot].$$

[7] See author's memoir, *loc. cit.*, p. 522.

Subtracting ϵ from the second inequality of (a), we have

$$\{1-f(0,\tau,t)\}^k-\epsilon \leq \{1-f(0,\tau_2,t_2)\}^k-\epsilon/2.$$

Combining with (b) we get

$$\text{(e)} \qquad \{1-f(0,\tau,t)\}^k-\epsilon \leqq p_N[\prod_{i=1}^{k}(r_i/n)\sim x(0,\tau_2,t_2,\Lambda)\cdot].$$

Adding ϵ to the first inequality of (a), we have

$$\{1-f(0,\tau_1,t_1)\}^k+\epsilon/2 \leq \{1-f(0,\tau,t)\}^k+\epsilon.$$

Combining with (c) we get

$$\text{(f)} \qquad p_N[\prod_{i=1}^{k}(r_i/n)\sim x(0,\tau_1,t_1,\Lambda)\cdot] \leq \{1-f(0,\tau,t)\}^k+\epsilon.$$

Using (e) and (f) with (d), we get

$$\{1-f(0,\tau,t)\}^k-\epsilon < p_N[\prod_{i=1}^{k}(r_i/n)\sim x(0,\tau,t,\Lambda)\cdot] \leq \{1-f(0,\tau,t)\}^k+\epsilon$$

whenever $N > N_0$. Therefore, we have

$$p[\prod_{i=1}^{k}(r_i/n)\sim x(0,\tau,t,\Lambda)\cdot] = \{1-f(0,\tau,t)\}^k.$$

Hence, we see that for the same numbers N_s as found from Theorem 2, that $\sim x(0,\tau,t,\Lambda)$ is a member of the set $A[1-f(0,\tau,t)]$, where t and τ are not of the forms $r\cdot 2^{-\sigma+1}$ and $\rho\cdot 2^{-\sigma+1}$ respectively but $t+\tau<\Lambda$.

From the principles of admissible numbers, it follows that $x(0,\tau,t,\Lambda)$ is an element of the set $A[f(0,\tau,t)]$ for every t and τ, such that $t+\tau<\Lambda$.

We have now proved the following theorem:

THEOREM. *If the hypothesis* (H_1) *of Theorem 2 is satisfied, then for the same numbers* N_s *which may be obtained for this case of Theorem 2, it is true for every* t *and* τ, *such that* $t+\tau<\Lambda$, *that the corresponding number* $x(0,\tau,t,\Lambda)$ *is an element of* $A[f(0,\tau,t)]$.

By the transformation of § 1, we can show that the series for variable probability is consistent for intervals not of the form $(r+m)2^{-\sigma+1}$.

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CERTAIN RATIONAL r-DIMENSIONAL VARIETIES OF ORDER N IN HYPERSPACE WITH A RATIONAL PENCIL OF $(r-1)$-DIMENSIONAL VARIETIES OF LOWER ORDER.

By B. C. Wong.

1. *Introduction.* Let $M_{r-k}^{m^k}$ denote the $(r-k)$-dimensional manifold, of order m^k, of complete intersection of k given general hypersurfaces of order m in an r-space, S_r. The hypersurfaces of order n passing through $M_{r-k}^{m^k}$ form a linear system, $|V|$, of dimension [1]

$$
\begin{aligned}
(1)\quad \rho(r,n,m,k) = -1 &+ \binom{k}{1}\binom{n-m+r}{r} - \binom{k}{2}\binom{n-2m+r}{r} \\
&+ \binom{k}{3}\binom{n-3m+r}{r} - \cdots - (-)^{k-1}\binom{k}{k-1}\binom{n-km+m+r}{r} \\
&- (-1)^k\binom{k}{k}\binom{n-km+r}{r},
\end{aligned}
$$

where the second factor in any term is to be taken equal to zero if its upper number is less than r. It is implied that $n \geqq m$. If $n = m$, then $\rho(r, m, m, k) = k - 1$ and we have the ∞^{k-1}-system determined by the k given hypersurfaces intersecting in $M_{r-k}^{m^k}$. We shall exclude this case from our consideration and shall, in what follows, assume $n > m$.

In this paper we are concerned with the rational r-dimensional variety, denoted by $\Phi_r^{N(r,n,m,k)}$ or just Φ_r, of order $N(r, n, m, k)$ in a $\rho(r, n, m, k)$-space, $\Sigma_{\rho(r,n,m,k)}$, which is representable upon S_r by the hypersurfaces of $|V|$. The results obtained will be, in some measure, generalizations of those already obtained by D. W. Babbage,[2] B. C. Wong,[3] and W. L. Edge.[4] We shall, in §§ 2, 3, 4, describe, in general terms, the properties of the representation and then, in §§ 5-8, describe, in some detail, the case $k = 2$, and, in particular, the case $n = m + 1 \geqq 3$. In conclusion, we shall mention, in § 9, some other special cases of interest.

2. *The determination of* $N(r, n, m, k)$. To determine the order $N(r, n, m, k)$ of the variety Φ_r, we determine the number of points of free

[1] For the derivation of this formula see Bertini, *Projektive Geometrie Mehrdimensionaler Räume*, 1924, Kapitel XII.

[2] "A series of rational loci with one apparent double point," *Proceedings of the Cambridge Philosophical Society*, vol. 27 (1931), pp. 399-403.

[3] "On a certain rational V_n^{2n+1} in S_{2n+1}," *American Journal of Mathematics*, vol. 56 (1934), pp. 219-224.

[4] "The number of apparent double points of certain loci," *Proceedings of the Cambridge Philosophical Society*, vol. 28 (1932), pp. 285-299.

intersection of r variable hypersurfaces of $|V|$. We may consider, without loss of generality for our purpose, r composite hypersurfaces of $|V|$ each composed of a hypersurface U of order m containing $M^{m^k}_{r-k}$ and a general hypersurface W of order $n-m$ which is in no way related to $M^{m^k}_{r-k}$. The $N(r,n,m,k)$ points of intersection of these r composite V's can now be calculated without difficulty. First, we have the $(n-m)^r$ points common to all the r W's. Then, we see that the U belonging to any one of the r V's and the W's belonging to the remaining $r-1$ V's intersect in $m(n-m)^{r-1}$ points. There are in all $\binom{r}{1}m(n-m)^{r-1}$ such points. Again, the U's of any two of the r V's and the W's of the other $r-2$ V's intersect in $m^2(n-m)^{r-2}$ points. The total number of such points is $\binom{r}{2}m^2(n-m)^{r-2}$. If we continue in this manner, we shall soon arrive at the final step where we find that the U's of any $k-1$ of the r V's and the W's of the remaining $r-k+1$ V's meet in $m^{k-1}(n-m)^{r-k+1}$ points. We have in all $\binom{r}{k-1}m^{k-1}(n-m)^{r-k+1}$ such points. Since there are no other points common to the r composite V's outside of the basic variety, we have, adding,

$$(2) \qquad N(r,n,m,k)=\sum_{i=0}^{k-1}\binom{r}{i}m^i(n-m)^{r-i}$$
$$=n^r-\sum_{i=0}^{r-k}\binom{r}{i}m^{r-i}(n-m)^i$$

for the order of the variety Φ_r.

In a similar manner, we find that the order of the $(r-q)$-dimensional variety V_{r-q} of intersection of q variable hypersurfaces of $|V|$ is $N(q,n,m,k)$ whose value is obtainable from (2) by replacing r by q. If q is increased by unity, we have a V_{r-q-1} of intersection of $q+1$ hypersurfaces of $|V|$, of order $N(q+1,n,m,k)$. We now find the intersection of V_{r-q} and the base manifold $M^{m^k}_{r-k}$. For this purpose, we notice that V_{r-q} and another hypersurface of $|V|$ intersect completely in a composite $(r-q-1)$-dimensional variety of order $n\cdot N(q,n,m,k)$. One of the components is the V_{r-q-1} just mentioned and the other, of order $n\cdot N(q,n,m,k)-N(q+1,n,m,k)$, lies on $M^{m^k}_{r-k}$ and is therefore the intersection of V_{r-q} and $M^{m^k}_{r-k}$.

3. *Varieties corresponding to sub-spaces of S_r.* A general $(k-t)$-space, where $1\leqq t\leqq k$, in S_r has no point in common with $M^{n^k}_{r-k}$ and therefore meets $|V|$ in a system of $(k-t-1)$-dimensional varieties of order n having no base manifold. The dimenison of this system is, then,

$$\rho(k-t,n,m,k)=\binom{n+k-t}{k-t}-1.$$

Therefore, the transform of S_{k-t} is a $\Phi^{n^{k-t}}_{k-t}$ of order n^{k-t} on Φ_r, contained in a space of $\binom{n+k-t}{k-t}-1$ dimensions. Thus, for $t=k$, a point in S_r is the image

of a point on Φ_r; for $t=k-1$, a line of S_r is the image of a curve of order n contained in an n-space; and so on.

If we consider a general $(k+h)$-space S_{k+h}, where $0 \leqq h \leqq r-k$, in S_r, we see that it meets M^{mk}_{r-k} in a manifold $M_h{}^{mk}$ and meets $|V|$ in a $\rho(k+h, n, m, k)$-dimensional system of $(k+h-1)$-dimensional varieties of order n having $M_h{}^{mk}$ for base manifold. The transform of S_{k+h} is a Φ_{k+h} of order $N(k+h, n, m, k)$ on Φ_r, contained in a $\rho(k+h, n, m, k)$-space. Thus, for $h=0$, an S_k has for transform a Φ_k of order $N(k, n, m, k) = n^k - m^k$; for $h=1$, an S_{k+1} has for transform a Φ_{k+1} of order

$$N(k+1, n, m, k) = n^{k+1} - m^{k+1} - (k+1)m^k(n-m);\ \text{etc.}$$

From the fact that a general S_k meets M^{mk}_{r-k} in points we see that a point of M^{mk}_{r-k} is the image of a $(k-1)$-space on Φ_r. Then, the $M_h{}^{mk}$ common to S_{k+h} and M^{mk}_{r-k} is the image of a $(k+h-1)$-dimensional manifold μ_{k+h-1} of order $\nu(k+h, nm, k)$, a locus of ∞^h $(k-1)$-spaces. We now calculate $\nu(k+h, n, m, k)$. Consider an S_{k+h-1} of S_{k+h}. Its transform on Φ_r is a Φ_{k+h-1} of order $N(k+h-1, n, m, k)$. Therefore, a $V^n{}_{k+h-1}$ of order n in S_{k+h} will have for transform a variety of order $n \cdot N(k+h-1, n, m, k)$. Now let $V^n{}_{k+h-1}$ pass through $M_h{}^{mk}$, and the transform is now of order

$$n \cdot N(k+h-1, n, m, k) - \nu(k+h, n, m, k).$$

But this order is equal to $N(k+h, n, m, k)$ which is that of the section of Φ_{k+h} by a $\Sigma_{\rho(k+h,n,m,k)-1}$ of $\Sigma_{\rho(k+h,n,m,k)}$. Then, we have

$$\begin{aligned}\nu(k+h, n, m, k) &= n \cdot N(k+h-1, n, m, k) - N(k+h, n, m, k)\\ &= \tbinom{k+h-1}{h} m^k(n-m)^h.\end{aligned}$$

Thus, for $h=r-k$, the transform of M^{mk}_{r-k} is a μ_{r-1} of order

$$\nu(r, n, m, k) = \tbinom{r-1}{r-k} m^k(n-m)^{r-k}.$$

If $h=1$, we see that the curve $M_1{}^{mk}$ in which an S_{k+1} meets M^{mk}_{r-k} is the image of a μ_k of order $km^k(n-m)$; and, if $h=0$, each of the m^k points common to an S_k and M^{mk}_{r-k} is the image of a $(k-1)$-space.

4. *Rational systems of loci on* Φ_r. A hyperplane, S_{r-1}, of S_r, as was seen above, goes into a variety of order $N(r-1, n, m, k)$, and hence an n'-ic hypersurface, where $m \leqq n' \leqq n$, goes into one of order $n' \cdot N(r-1, n, m, k)$. Consider a $V^{n'}_{r-1}$ passing through M^{mk}_{r-k}, and there are $\infty^{\rho(r,n',m,k)}$ such hypersurfaces. Now the transform of this $V^{n'}_{r-1}$ is a $\Phi^{N'}_{r-1}$ of order

$$N' = n' \cdot N(r-1, n, m, k) - \tbinom{r-1}{r-k} m^k(n-m)^{r-k}.$$

For the various values of n' from m to n, there are, then, various rational $\infty^{\rho(r,n',m,k)}$-systems of $(r-1)$-dimensional varieties of order N'. If, in particular, we put $n'=m$, $k=2$, we see that Φ_r contains a rational pencil of varieties of dimension $r-1$ and order

$$N'=m\cdot N(r-1,n,m,k)-(r-1)m^2(n-m)^{r-2}=m(n-m)^{r-1}.$$

5. *The case* $k=2$, $n=m+1\geqq 3$. In this interesting case we have a system $|V|$, of dimension[5] $\rho=2r+1$, of hypersurfaces of order n passing through the complete intersection $M_{r-2}^{(n-1)^2}$ of two general hypersurfaces of order $n-1$, in S_r. The equation of a general member of the system is of the form

$$(3) \qquad F_0\cdot\sum_{i=0}^{r}A_ix_i+F_1\cdot\sum_{i=0}^{r}B_ix_i=0,$$

where $F_0=0$, $F_1=0$ are the equations, of degree $n-1$ in $x_0,x_1,\cdots,x_r$, representing the two hypersurfaces intersecting in $M_{r-2}^{(n-1)^2}$ and the A's and B's are arbitrary constants. Setting

$$(4) \qquad X_{00}:X_{01}:\cdots:X_{0r}:X_{10}:\cdots:X_{1r} \\ =F_0x_0:F_0x_1:\cdots:F_0x_r:F_1x_0:\cdots:F_1x_r,$$

where X_{ij} $[i=0,1;\ j=0,1,\cdots,r]$ are the coördinates of a point in a $(2r+1)$-space Σ_{2r+1}, we have the equations of the variety Φ_r^N, where $N=rn-r+1$, represented upon S_r by the hypersurfaces of $|V|$.

A hyperplane in S_r, say $x_r=0$, is transformed by (4) into a $\Phi_{r-1}^{N'}$ of order $N'=rn-n-r+2$, which is of the same nature as Φ_r^N for r diminished by unity. Then, the transform of a hypersurface of order $n-1$ is a variety of order $N'(n-1)$. If the hypersurface is one, say V_{r-1}^{n-1}, of the pencil determined by $F_0=0$, $F_1=0$, the corresponding variety degenerates into the transform μ_{r-1} of $M_{r-2}^{(n-1)^2}$, of order $(r-1)(n-1)^2$, and a Θ_{r-1}^{n-1} of order $n-1$, which is the proper transform of V_{r-1}^{n-1}. Thus, Φ_r^N contains a rational pencil of $(r-1)$-dimensional varieties of order $n-1$ corresponding to the pencil $F_0+\lambda F_1=0$ in S_r.

6. *Different varieties on* Φ_r^N. Each of the ∞^1 Θ_{r-1}^{n-1}'s on Φ_r^N is contained in an r-space and the ∞^1 containing r-spaces form a rational locus, Ω_{r+1}^{r+1}, of order $r+1$ whose equations are

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[5] For $k=2$, $n=2$, $m=1$, the system of quadric hypersurfaces in r-space having a fixed $(r-2)$-space in common is of dimension $2r$ and not $2r+1$. This system can best be regarded as a special case of the linear system of hypersurfaces of order n passing through a given $(r-2)$-space $n-1$ times.

D. W. Babbage, in the paper cited in footnote [2], has already studied the case $n=3$.

$$\left\| \begin{matrix} X_{00} \, X_{01} & \cdots & X_{0r} \\ X_{10} \, X_{11} & \cdots & X_{1r} \end{matrix} \right\| = 0.$$

A general $(r+2)$-space of Σ_{2r+1} meets Φ_r^N in a curve Γ^N and Ω_{r+1}^{r+1} in a ruled surface of order $r+1$ whose rulings are all $(n-1)$-secant lines of the curve.

Now Φ_r^N contains $\infty^{(t+1)(r-t)}$ Φ_t's of order $tn - t + 1$ each contained in a $(2t+1)$-space. They correspond to the $\infty^{(t+1)(r-t)}$ t-spaces of S_r and are of the same nature as Φ_r^N for $r = t$. The $(2t+1)$-space of a Φ_t meets Ω_{r+1}^{r+1} in an Ω_{t+1}^{t+1}. This Ω_{t+1}^{t+1} is the locus of ∞^1 t-spaces and each of these t-spaces meets a Θ_{r-1}^{n-1} in a Θ_{t-1}^{n-1}. Therefore, Φ_t contains ∞^1 Θ_{t-1}^{n-1}'s.

Consider an $(r+t+1)$-space Σ_{r+t+1} passing through the $(2t+1)$-space Σ_{2t+1} of a Φ_t. It meets Ω_{r+1}^{r+1} in a $(t+1)$-dimensional variety of order $r+1$. Since Σ_{2t+1} already meets Ω_{r+1}^{r+1} in an Ω_{t+1}^{t+1}, the variety just mentioned is composed of Ω_{t+1}^{t+1} and $r - t$ $(t+1)$-spaces. Note that each of these $(t+1)$-spaces contains a Θ_t^{n-1}.

Now a general $(r+t+1)$-space meets Φ_r^N in a t-dimensional variety of the same order. If the $(r+t+1)$-space passes through Σ_{2t+1}, it contains Φ_t which is of order $tn - t + 1$. Then, its intersection with Φ_r^N is composed of Φ_t and $r - t$ Θ_t^{n-1}'s.

We may make use of the preceding result to find the genus of the curve Γ^N in which an $(r+2)$-space meets Φ_r^N. Putting $t = 1$, we have a composite curve whose components are a Φ_1^n which is a rational curve of order n and $r - 1$ Θ_1^{n-1}'s which are plane curves of order $n - 1$. Each Θ_1^{n-1} has $n - 1$ points in common with Φ_1^n and therefore has $(n-1)^2$ apparent intersections with it. There are $(r-1)(n-1)^2$ apparent intersections between Φ_1^n and the $r - 1$ Θ_1^{n-1}'s; there are also $\frac{1}{2}(r-1)(r-2)(n-1)^2$ apparent intersections of the $r - 1$ Θ_1^{n-1}'s two by two. Φ_1^n itself has $\frac{1}{2}(n-1)(n-2)$ apparent double points. Therefore, we say that the total curve Γ^N has

$$\begin{aligned} (r-1)(n-1)^2 + \tfrac{1}{2}(r-1)(r-2)(n-1)^2 + \tfrac{1}{2}(n-1)(n-2) \\ = \tfrac{1}{2}(n-1)[(r^2 - r + 1)n - (r^2 - r + 2)] \end{aligned}$$

apparent double points. Denote this number by b_{r-1}. We note that b_{r-1} is also the order of the double variety, $\Delta_{r-1}^{b_{r-1}}$, on the projection of Φ_r^N upon an $(r+1)$-space. The genus of Γ^N is, then,

$$\tfrac{1}{2}(N-1)(N-2) - b_{r-1} = \tfrac{1}{2}(r-1)(n-1)(n-2).$$

On Φ_r^N we have ∞^{2r-2} curves of order n and they are rational curves and correspond to the ∞^{2r-2} lines of S_r. Each of them is contained in a 3-space and this 3-space meets Ω_{r+1}^{r+1} in a quadratic regulus Ω_2^2. Of the ∞^{2r-2} such

3-spaces one and only one passes through a general given point P of the $(2r+1)$-space Σ_{2r+1} containing Φ_r^N. Therefore, through P we can construct $\frac{1}{2}(n-1)(n-2)$ lines meeting in two points the rational curve of order n of Φ_r^N contained in the 3-space through it, and these are the only lines through P bisecant to Φ_r^N. Hence, we say that Φ_r^N has $\frac{1}{2}(n-1)(n-2)$ apparent double points and that its projection upon a $(2r)$-space of Σ_{2r+1} has $\frac{1}{2}(n-1)(n-2)$ improper double points.

7. *Projections and sections of* Φ_r^N. Let us project Φ_r^N from an $(r-1)$-space, denoted by Z_{r-1}, of Σ_{2r+1} upon an $(r+1)$-space Σ_{r+1}. Denote the projection by ${}_{r-1}\Phi_r^N$. We have already remarked, in § 6, that the double variety $\Delta_{r-1}^{b_{r-1}}$ on ${}_{r-1}\Phi_r^N$ is of order $b_{r-1}=\frac{1}{2}(n-1)[(r^2-r+1)n-(r^2-r+2)]$. Now we verify this fact. Suppose the center of projection, Z_{r-1}, is contained in the Σ_{2r-1} containing a $\Phi_{r-1}^{rn-n-r+2}$ [see § 6]. This Σ_{2r-1} meets Ω_{r+1}^{r+1} in an Ω_r^r which is in turn met by Z_{r-1} in r points. Designate these points by $A^{(1)}, A^{(2)}, \cdots, A^{(r)}$. Through $A^{(i)}$ passes an r-space $\Sigma_r^{(i)}$ containing a Θ_{r-1}^{n-1}. Now the projection ${}_{r-1}\Phi_r^N$ in Σ_{r+1} contains an $(rn-n-r+2)$-fold $(r-1)$-space which is the projection of $\Phi_{r-1}^{rn-n-r+2}$ in Σ_{2r-1} and r $(n-1)$-fold $(r-1)$-spaces which are the projections of the Θ_{r-1}^{n-1}'s in the Σ_r's through $A^{(1)}, \cdots, A^{(r)}$. The $(rn-n-r+2)$-fold $(r-1)$-space is equivalent to a double variety of order $\frac{1}{2}(rn-n-r+r)(rn-n-r+1)$ and each of the r $(n-1)$-fold $(r-1)$-spaces is equivalent to a double variety of order $\frac{1}{2}(n-1)(n-2)$. Therefore, the total double variety is of order

$$b_{r-1}=\tfrac{1}{2}(rn-n-r+2)(rn-n-r+1)+\tfrac{1}{2}r(n-1)(n-2)$$

which is reduced to the value already given.

We now find the pinch variety, $J_{r-2}^{c_{r-2}}$, on the projection ${}_{r-1}\Phi_r^N$. We remark that the $\Phi_{r-1}^{rn-n-r+2}$ in the Σ_{2r-1} containing Z_{r-1} has an apparent double variety of order $b_{r-2}=\frac{1}{2}(n-1)[(r^2-3r+3)n-(r^2-3r+4)]$ and is therefore of rank

$$a=(rn-n-r+2)(rn-n-r+1)-2b_{r-2}=(n-1)(rn-2n+2).$$

Then, on the $(rn-n-r+2)$-fold $(r-1)$-space, the projection of $\Phi_r^{rn-n-r+2}$, is a pinch variety of order a. Now each of the r Θ_{r-1}^{n-1}'s is of rank $(n-1)(n-2)$ and, therefore, in each of the r $(n-1)$-fold $(r-1)$-spaces, the projections of the Θ_{r-1}^{n-1}'s, is a pinch variety of order $(n-1)(n-2)$. Then, the total locus of pinch points is of order $c_{r-2}=a+r(n-1)(n-2)=2(r-1)(n-1)^2$.

Consider a general projection of Φ_r^N upon a Σ_{r+1}, that is, one from a general center of projection upon a general $(r+1)$-space. Denote it also by ${}_{r-1}\Phi_r^N$. There are ∞^1 r-spaces in Σ_{r+1} each meeting ${}_{r-1}\Phi_r^N$ in a composite

section composed of a variety of order $rn - n - r + 2$ and one of order $n - 1$. These ∞^1 r-spaces envelop a curve of order $r + 1$. Let a 3-space meet ${}_{r-1}\Phi_r^N$ in a surface F^N. The ∞^1 planes containing composite sections envelop a curve of order $3(r - 1)$, rank $2r$, class $r + 1$, with $4(r - 2)$ cusps.

8. *Other projections of* Φ_r^N. Suppose we now project Φ_r^N from an $(r-2)$-space, Z_{r-2}, upon an $(r+2)$-space Σ_{r+2}. The projection, ${}_{r-2}\Phi_r^N$, has a locus, $\Delta_{r-2}^{b_{r-2}}$, of double points, of order b_{r-2}, and a locus, $J_{r-3}^{c_{r-3}}$, of pinch points, of order c_{r-3}. A 4-space section of the projection is a surface with b_{r-2} improper nodes. By making use of a relation, which is known concerning surfaces in 4-space,[6] namely, $2b_{r-1} - 2b_{r-2} = c_{r-2}$, we find that

$$\begin{aligned} b_{r-2} &= b_{r-1} - \tfrac{1}{2}c_{r-2} \\ &= \tfrac{1}{2}(n-1)[(r^2 - r + 1)n - (r^2 - r + 2)] - (r-1)(n-1)^2 \\ &= \tfrac{1}{2}(n-1)[(r^2 - 3r + 3)n - (r^2 - 3r + 4)]. \end{aligned}$$

This value is identical with that found in § 7 for the double variety $\Delta_{r-2}^{b_{r-2}}$ on the projection of $\Phi_{r-1}^{rn-n-r+2}$ upon an r-space. To verify this fact, put Z_{r-2} in the Σ_{2r-1} containing $\Phi_{r-1}^{rn-n-r+2}$. Then, the $\Delta_{r-2}^{b_{r-2}}$ on the projection ${}_{r-2}\Phi_r^N$ is the same as that on the projection of $\Phi_{r-1}^{rn-n-r+2}$. From this last fact we derive the result that the order of $J_{r-3}^{c_{r-3}}$ is $c_{r-3} = 2(r-2)(n-1)^2$.

Reasoning in the same manner, we see that the projection ${}_{r-t}\Phi_r^N$ from a Z_{r-t} of Σ_{2r+1} upon an $(r+t)$-space has a double locus $\Delta_{r-t}^{b_{r-t}}$ of order $b_{r-t} = \tfrac{1}{2}(n-1)[(r^2 - 2rt + t^2 + r - t + 1)(n-1) - 1]$ and a pinch locus $J_{r-t-1}^{c_{r-t-1}}$ of order $c_{r-t-1} = 2(r-t)(n-1)^2$. For $t = r$, the projection ${}_0\Phi_r^N$ upon Σ_{2r} has $b_0 = \tfrac{1}{2}(n-1)(n-2)$ improper double points, as was found in § 6.

9. *Various special cases.* It is of interest to mention that, if $r = n$, we have a $\Phi_n^{n^2-n+1}$ in a Σ_{2n+1}. It is ruled, being the locus of ∞^{n-1} lines, which correspond to the $(n-1)$-secant lines of $M_{n-2}^{(n-1)^2}$ in S_n. The number of lines passing through each point of the variety is $(n-1)!$ What has just been said holds true for $n = 2$, except that the ruled cubic surface Φ_2^3 is in a 4-space and not in a 5-space. If $r \geqq n$, then Φ_r^N contains ∞^{2r-n-1} lines of which ∞^{r-n} pass through each point. If $r \leqq n$, Φ_r^N may be regarded as a locus of ∞^{r-1} curves of order $n - r + 1$. Through a general point pass $(r-1)!\binom{n-1}{r-1}^2$ of them.

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[6] Severi, "Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti," *Rendiconti di Palermo*, vol. 15 (1901), pp. 33-51.